DIFFERENTIABLE CONJUGACY NEAR COMPACT INVARIANT MANIFOLDS

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0. Introduction

In this paper [1], we show how the differentiable linearization of a diffeomorphism near a hyperbolic fixed point (a la Sternberg [11]) can be adapted to a neighborhood of a compact invariant submanifold. There are two parts of the standard proof. The first part says that if two diffeomorphisms have all their derivatives equal at a hyperbolic fixed point, then they are C^∞ conjugate to one another in a neighborhood. This result is true in a neighborhood of a compact invariant submanifold with little change in the statement or proof. See Theorem 1. The second part says that if a diffeomorphism f satisfies eigenvalue conditions at a hyperbolic fixed point, then there is a C^∞ diffeomorphism h such that all the derivatives of $g = h^{-1} f\, h$ at the fixed pint are equal to the derivatives of the linear part of f. Near an invariant submanifold, there is no general condition that replaces the eigenvalue condition, so we got only a very much weakened result in this direction. See Theorem 2. However, Theorem 2 does imply that under some conditions the strong stable manifolds of points vary differentiably. See Corollaries 3 and 4.

We were aware that Theorem 1 was true before reading the recent paper of Takens [12]. However, his proof is the easiest to adapt to our setting and also save one more derivative than some other proofs. We could just say that Theorem 1 follows from the proof in [12], however for clarity, we repeat the proof with the necessary modifications. The only essential changes are in the definitions of $\eta(\delta)$ and $\mathscr{O}$. All other changes are a matter of style.

To prove Theorem 2, we adapt the type of proof used for Theorem 1. At a hyperbolic fixed point, this can be solved much more directly by solving for coefficients of polynomials using eigenvalue conditions. See [9], [11], or [12].

1. Statement of the theorems

For $h : M \to M$, let

$$j^r h(x) = (x, h(x), Dh(x), \ldots, D^r h(x)).$$

This is called the r-jet of h at x in local coordinates on the domain. (It is possible to define these without local coordinates, but it really changes none of the ideas in our proofs. See [4].)

Key words and phrases. dynamical systems, differentiable conjugacy, normally hyperbolic manifold.

[1] This paper originally appeared in the Bolletim da Sociedade Brasileira de Matemática, 2 (1971). We have made slight changes in wording in a few places. Also, we have added footnotes to explain certain points.

Let V be a compact submanifold of M. Give M a Riemannian metric. Let ρ be the distance between point of M induced by the metric. Let $p : TM \to M$ be the usual projection. Let $T_xM = p^{-1}(x)$ and $T_VM = p^{-1}(V)$. A diffeomorphism $f : M \to M$ is called hyperbolic along V if $f(V) = V$, there is a splitting $T_VM = TV \oplus E^u \oplus E^s$ as Whitney sum of subbundles, and there is an integer n such that

$$\mu_x = \|Df^n(x)|E^s_x\| < 1 \qquad \text{and}$$
$$\lambda_x = \|Df^{-n}(x)|E^u_x\| < 1$$

for all $x \in V$, where $E^s_x = E^s \cap T_xM$ and $E^u_x = E^u \cap T_xM$.

For $h : M \to M$ and $x \in V$, let

$$D_1h(x) = Dh(x)|T_xV,$$
$$D_2h(x) = Dh(x)|E^u_x, \qquad \text{and}$$
$$D_3h(x) = Dh(x)|E^s_x.$$

A diffeomorphism f is called *r-normally hyperbolic along V* if

$$\lambda_x\,\|D_1f^n(x)\|^k < 1 \qquad \text{and}$$
$$\mu_x\,\|D_1f^{-n}(x)\|^k < 1$$

for all $x \in V$ and all $0 \le k \le r$. This says that f more contracting (resp. expanding) normally to V than any contraction (resp. expansion) along V.

Let

$$W^s(V,f) = \{\, x \in M : \rho(f^j(x), V) \to 0 \text{ as } j \to \infty \,\} \quad \text{and}$$
$$W^u(V,f) = \{\, x \in M : \rho(f^{-j}(x), V) \to 0 \text{ as } j \to \infty \,\}.$$

These are called the *stable* and *unstable manifolds of V for f*. For $x \in V$, let

$$W^{ss}(x,f) = \{\, y \in M : \text{there exits a constant } c_y \text{ such that}$$
$$\rho(f^{jn}(x), f^{jn}(y)) \le c_y \mu_x \cdots \mu_{f^{(j-1)n}(x)} \text{ for } j \ge 0 \,\} \quad \text{and}$$
$$W^{uu}(x,f) = \{\, y \in M : \text{there exits a constant } c_y \text{ such that}$$
$$\rho(f^{-jn}(x), f^{-jn}(y)) \le c_y \lambda_x \cdots \lambda_{f^{(-j+1)n}(x)} \text{ for } j \ge 0 \,\}.$$

These are called the *strong stable* and *strong unstable manifolds of x for f*.

If the diffeomorphism f is C^r and r-normally hyperbolic, then the papers [6] and [7] show that $W^s(V,f)$, $W^u(V,f)$, V, $W^{ss}(x,f)$, and are C^r, and

$$W^s(V,f) = \bigcup \{W^{ss}(x,f) : x \in V\} \qquad \text{and}$$
$$W^u(V,f) = \bigcup \{W^{uu}(x,f) : x \in V\}.$$

Also,

$$T_V(W^s(V,f)) = TV \oplus E^s \qquad \text{and}$$
$$T_V(W^u(V,f)) = TV \oplus E^u.$$

A more general theorem of this kind is contained in [8].

Now we define the loss of derivatives that occurs in the conjugation of Theorem 1. Given α, let $\beta = \beta(f,\alpha) \le \alpha$ be the largest integer such that

$$\|Df^{-n}(f^n(x))\| \cdot \|Df^n(x)\|^{\beta} \cdot \|D_3f^n(x)\|^{\alpha-\beta} < 1 \quad \text{for all } x \in V.$$

Next, let $\gamma = \gamma(f, \beta) \leq \beta$ be the largest integer such that

$$\|Df^n(f^{-n}(x))\| \cdot \|Df^{-n}(x)\|^{\gamma} \cdot \|D_2 f^{-n}(x)\|^{\beta-\gamma} < 1 \quad \text{for all } x \in V.$$

Theorem 1. *Assume $f, g : M \to M$ are C^α diffeomorphisms, and $V \subset M$ is a compact C^1 invariant submanifold such that both f and g are 1-normally hyperbolic along V with $j^\alpha f(x) = j^\alpha g(x)$ for all $x \in V$. Let β and γ be defined as above with $\alpha \geq \beta \geq \gamma \geq 1$. Assume $W^s(V, f)$ is a C^β submanifold near V.*

Then there exist a neighborhood U of V and a C^β diffeomorphism $h : U \to M$ such that $k = h^{-1}gh$ has $j^\beta k(x) = j^\beta f(x)$ for $x \in W^s(V, f) \cap U$. Also there exists a C^γ diffeomorphism $h' : U \to M$ such that $(h')^{-1}gh'(x) = f(x)$ for $x \in U$. Further, $h|V = id$ and $h'|V = id$.

The proof is contained in §3.

Theorem 2. *Let $f : M \to M$ be a C^α diffeomorphism, and $V \subset M$ a compact invariant C^α submanifold. Assume f contracts along V, i.e., E^u is the zero section in the definition of f being hyperbolic along V. Assume that for $1 \leq \beta \leq \alpha$*

$$\|D_1 f^{-1}(f(x))\| \cdot \|Df(x)\|^{\beta-1} \cdot \|D_3 f(x)\| < 1 \qquad \text{for all } x \in V.$$

Then there exists a neighborhood U of V and a C^β diffeomorphism $h : U \to M$ such that $h|V = id$ and $g = h^{-1}fh$ has $D_3^j(\mathrm{pr} \circ g)(x) = 0$ for $1 \leq j \leq \beta$ where $\mathrm{pr} : U \to V$ is a differentiable normal bundle projection. Thus infinitesimally g preserves the fibers of $\mathrm{pr} : U \to V$.

The proof is contained in §4.

Corollary 3. *Let f be a C^α diffeomorphism contracting along V. Assume*

$$\|D_1 f^{-1}(f(x))\| \cdot \|Df(x)\|^{\alpha-1} \cdot \|D_3 f(x)\| < 1$$

for all $x \in V$. Let p and r be integers such that

$$\|D_3 f^{-1}(f(x))\| \cdot \|D_3 f(x)\|^p \leq 1$$

$$\|D_1 f(x)\|^{\alpha-p} \left(\frac{\|D_3 f(x)\|}{\|D_1 f(x)\|} \right)^{r+1} < 1.$$

Let $\beta = \alpha - 1 - p - r$.

Then there exists a neighborhood U of V and a C^β diffeomorphism $h : U \to M$ such that $g = h^{-1}fh$ preserves the fibers of $\mathrm{pr} : U \to V$. Actually, h has all derivatives $D^j D_2^k h(x)$ for $x \in U$, $0 \leq j \leq \beta$, and $0 \leq j + k \leq \alpha$. In particular, the set of $W^{ss}(x, f)$ for $x \in V$ form a foliation of $W^s(V, f) \cap U$ such that each leaf is C^α and they vary C^β.

Proof. By applying Theorem 2, we can assume $D_2^j(\mathrm{pr} \circ f)(x) = 0$ for $1 \leq j \leq \alpha$ and $x \in V$. Define $g_1 : U \to V$ by $g_1(x) = f_1(\mathrm{pr}\, x)$. In vector bundle charts of $\mathrm{pr} : U \to V$ define $g_2(x) = f_2(x)$ Use bump functions to define $g = (g_1, g_2) : U \to U$. Then $g_1(x) = g_1(\mathrm{pr}\, x) = f_1(\mathrm{pr}\, x)$ for $x \in M$ and $j^\alpha f(x) = j^\alpha g(x)$ for $x \in V$. Theorem 1 gives the C^β conjugacy of f and g where $\beta = \alpha - 1 - p - r$ since

$$\begin{aligned} \|D_3 f^{-1}\| \cdot \|D_1 f\|^{\alpha-1-p-r} \cdot \|D_3 f\|^{1+p+r} &\leq \|D_1 f\|^{\alpha-1-p-r} \cdot \|D_3 f\|^{1+r} \\ &\leq \|D_1 f\|^{\alpha-p} \left(\frac{\|D_3 f\|}{\|D_1 f\|} \right)^{1+r} < 1. \end{aligned}$$

The extra derivatives of h exist as remarked in the proof of Theorem 1. □

Using the methods of the proof of Theorem 1 differently, we can get a stronger statement about the differentiability of the foliation $\{ W^{ss}(x,f) : x \in V \}$ of $W^s(V,f)$.

Corollary 4. [2] *Let f be a C^α diffeomorphism that is* 1*-normally hyperbolic along V and $W^s(V,f)$ is $\mathcal{C}^\alpha$. Assume that $1 \le \beta \le \alpha - 1$ satisfies*

$$\|D_1 f^{-1}(f(x))\| \cdot \|D_1 f(x)\|^\beta \cdot \|D_3 f(x)\| < 1 \quad \textit{for all } x \in V.$$

Then there is a neighborhood U of V such that the set of $W^{ss}(x,f)$ for $x \in V$ form a C^β foliation of $W^s(V,f) \cap U$.

The proof is contained in §5.

Using the estimates in [9], the proofs of the above theorems should go over to flows. However, beware of the proof of linearization given in [9]. "By induction" does not works since the variation equation does not satisfy a global Lipschitz constant.

We would like to discuss how the above theorems relate to some of the results in [5], [6], and [10]. The condition of [6] and [7] that f is r-normally hyperbolic is similar but different than the condition we require in Theorem 2 and Corollaries 3 and 4. If f is r-normally hyperbolic, then $W^s(V,f)$, $W^u(V,f)$, and V are C^r manifolds. See [6]. Also, for each $x \in V$, $W^{ss}(x,f)$ and $W^u(x,f)$ are C^r and they vary continuously in the C^r topology. Corollaries 3 and 4 give that they vary differentiably.

[10] show that if f is 1-normally hyperbolic, then f is C^0 conjugate to a map g that preserves the fibers of $\mathrm{pr} : U \subset M \to V$ and such that g is linear on fibers of $\mathrm{pr} : U \subset M \to V$. Corollary 3 gives a differentiable conjugacy in the contracting case to a fiber preserving map g, but g is not necessarily linear on fibers.

If V is replaced by an expanding attractor, then 6.4 in [5] gives conditions under which the stable manifolds of points form a C^1 foliation of a neighborhood. Corollary 4 possibly could be adapted to this setting to give the same answer. The result in [5] only applies to stable manifolds of points,

$$W^s(x,f) = \{ y \in M : \rho(f^j(x), f^j(y)) \to 0 \text{ as } j \to \infty \},$$

and not the strong stable manifolds of points. [3] Thus, when a submanifold V is an attractor, the results are different. Also, we give a condition that insure higher differentiability.

Added in proof: M. Shub points out to me that 6.4 in [5] and the C^r section theorem prove Corollary 4.

2. Notation and definitions

Since we are only interested in a conjugacy of diffeomorphisms in a neighborhood of V, we can take a tubular neighborhood of V. Thus, we can consider M as a vector bundle over V, $\mathrm{pr} : M \to V$. Let $p : TM \to M$ be the projection of the tangent bundle of M to M. Denote a norm induced by a Riemannian metric on TM by $|\cdot|$. Let ρ be the distance between points of M induced by $|\cdot|$.

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[2] The original paper only stated this theorem for the case when f is contracting along V.
[3] $W^s(x,f)$ could include some directions within V.

Let $L^r_s(T_xM, T_yM)$ be the (linear) space of all symmetric r-linear maps from T_xM to T_yM.[4] Let

$$J^0(M,M) = M \times M \qquad \text{and}$$
$$J^r(M,M) = \bigcup\{\, (x,y) \times L^1_s(T_xM,T_xM) \times \cdots \times L^r_s(T_xM,T_xM) : x,y \in M \,\}.$$

If $h : M \to M$ is C^r, let

$$j^r\, h(x) = (x, h(x), Dh(x), \ldots, D^r h(x)) \in J^r(M,M).$$

This is called the *r-jet of h at x*. Let $\pi_0 : J^0(M,M) \to M$ be the projection on the first factor, and

$$\pi_r : J^r(M,M) \to J^{r-1}(M,M)$$

be the natural projection for $r \geq 1$. Let

$$\psi_r = \pi_0 \circ \cdots \pi_r : J^r(M,M) \to M.$$

All of these projections are fiber bundles, and $\psi^r : J^r(M,M) \to M$ is called the *r-jet bundle*. Define a distance on $J^0(M,M)$ by

$$\rho_0((x_1,x_2),(y_1,y_2)) = \max\{\, \rho(x_i,y_i) : i = 1,2 \,\}.$$

Let the distance on each fiber of $\pi_r : J^r(M,M) \to J^{r-1}(M,M)$ be the usual one induced by $|\cdot|$ pm TM,

$$\begin{aligned}\rho_r\,(\,(x,y,A_0,\ldots,A_r),\,(x,y,A_0,\ldots,A_{r-1},B_r)\,) &= \|A_r - B_r\| \\ &= \sup\{\, |(A_r - B_r)(v_1,\ldots,v_r)| : v_i \in T_xM \text{ and } |v_i| = 1 \text{ for all } i \,\}.\end{aligned}$$

By using the distance on the base space, there is an induced (noncanonical) distance on $J^r(M,M)$. Given a subset $U \subset M$, let

$$J^r(\,(M,U),\, M\,) = \psi_r^{-1}(U).$$

Let $\Gamma^r(\,(M,U),\, M\,)$ be the space of continuous section of $\psi_r : J^r(\,(M,U),\, M) \to U$.

3. Proof of Theorem 1

I. First we prove that the conjugacy exists along $W^s(V,f)$. We use the notation given in §1 and §2. By the assumptions of Theorem 1, there exists an integer n and a $0 < \mu < 1$ such that

$$\|Df^{-n}(f^n(x))\| \cdot \|Df^n(x)\|^\beta \cdot \|Df^n(x)|E^s_x\|^{\alpha-\beta} < \mu \quad \text{for all } x \in V.$$

Below we construct a conjugacy h between f^n and g^n. Because of its special form, $f \sim \lim_{j\to\infty} g^{-nj} f^{nj}$, h is also a conjugacy between f and g. Thus for convenience, we take $n = 1$. The reader can check the details for $n > 1$. The constant μ is fixed during the proof.

[4]Note: Higher derivatives are only defined in terms of local coordinates. Therefore, cover a neighborhood of V with a finite number of coordinate charts and define the jets and norms in terms of these coordinate charts.

We define the following numbers

$$
\begin{aligned}
a_x &= \|Dg^{-1}(x)\| && \text{for } x \in M \\
A_x &= \begin{cases} \rho(f(x), V)\, \rho(x, V)^{-1} \\ \lim\{\, A_y : y \in W^s(V, v) \setminus V \ \text{ and } \ y \to x \,\} \end{cases} && \begin{matrix} \text{for } x \in W^s(V, f) \setminus V \\ \text{for } x \in V \end{matrix} \\
b_x &= \|Df(x)\| && \text{for } x \in W^s(V, f).
\end{aligned}
$$

Note for $x \in V$,

$$
\begin{aligned}
&A_x \leq \|Df(x)|E^s_x\| < 1, \\
&a^{-1}_{f(x)} \leq A_x < 1 \leq B_x, \qquad \text{and} \\
&a_{f(x)}\, B^{\beta}_x\, A^{\alpha-\beta}_x < \mu < 1.
\end{aligned}
$$

There exist neighborhoods

$$
\begin{aligned}
&\eta(\delta) = \{\, x \in W^s(V, f) : \rho(x, V) < \delta \,\} \qquad \text{and} \\
&\mathscr{O} \ \text{ of } \ \{\, (m, m) : m \in V \,\} \ \text{ in } \ M \times M
\end{aligned}
$$

such that (i) $f(\eta(\delta)) \subset \eta(\delta)$ and (ii) if $x \in \eta(\delta)$ and $(f(x), y) \in \mathscr{O}$, then $a_y\, B^{\beta}_x\, A^{\alpha-\beta}_x < \mu$.

For simplicity of notation, let

$$
\begin{aligned}
&J^r = J^r((M, \eta(\delta)), M) = \psi^{-1}_r(\eta(\delta)) \qquad \text{and} \\
&\Gamma J^r \ \text{ be the continuous sections of the bundle } \ \psi_r : J^r \to \eta(\delta).
\end{aligned}
$$

We define a second norm on the fibers of $\pi_r : J^r \to J^{r-1}$ (possibly infinite) as follows: [5] for $\pi_r c^1 = \pi_r c^2$,

$$
\sigma_r(c^1, c^2) = \sup\{\, \rho_r(c^1(x), c^2(x))\, \rho(x, V)^{-(\alpha - r)} : x \in \eta(\delta) \setminus V \,\}.
$$

If $c \in \Gamma J^0$, then we can identify it with the map $c_0 : \eta(\delta) \to M$ such that $c(x) = (x, c_0(x))$. Let

$$
\begin{aligned}
&\Phi_0 : \Gamma J^0 \to \Gamma J^0 \ \text{ be defined by} \\
&\Phi_0(c)(x) = (x, g^{-1}(c_0(f(x)))) = j^0(g^{-1} c_0 f)(x).
\end{aligned}
$$

Let

$$
\begin{aligned}
&\Phi_r : \Gamma J^r \to \Gamma J^r \ \text{ be defined by} \\
&\Phi_r(c)(x) = j^r(g^{-1}(c_0(f(x)))) = j^r(g^{-1} h f)(x) \quad \text{where } \ j^r h(f(x)) = c(f(x)).
\end{aligned}
$$

First we prove Φ_r contracts along fibers of π_r.

Lemma 1. *Let* $0 \leq r \leq \beta$, c^1, $c^2 \in \Gamma J^r$ *with* $\pi_r c^1 = \pi_r c^2$, $\sigma_r(c^1, c^2) < \infty$, *and*

$$
\pi_1 \circ \cdots \circ \pi_r c^i(f(x)) \in \mathscr{O} \quad \textit{for all } \ x \in \eta(\delta), \ \ i = 1, 2.
$$

Then,

$$
\sigma_r(\Phi_r(c^1), \Phi_r(c^2)) \leq \mu\, \sigma_r(c^1, c^2).
$$

---

[5] The map on sections is not a contraction in the usual metric. It needs a factor related to moving closer to the invariant manifold.

Proof. Assume $r \geq 1$.

$$\begin{aligned}\sigma_r(\Phi_r c^1,\, \Phi_r c^2) &= \sup\{\, \rho_r(\Phi_r c^1(x),\, \Phi_r c^2(x))\, \rho(x,V)^{-(\alpha-r)} : \\ &\qquad\qquad x \in \eta(\delta) \smallsetminus V \,\} \\ &\leq \sup\{\, \rho_r(c^1(f(x)), c^2(f(x)))\, a_y B_x^r\, \rho(x,V)^{-(\alpha-r)} : \\ &\qquad x \in \eta(\delta) \smallsetminus V \text{ and } \pi_1 \circ \cdots \circ \pi_r c^i(f(x)) = (f(x), y) \,\}.\end{aligned}$$

This last inequality is true using the formula for higher derivatives of a composition of functions and the fact that $\pi_r c^1 = \pi_r c^2$. Then, this is

$$\begin{aligned}&\leq \sup\{\, \rho_r(c^1(f(x)), c^2(f(x)))\, \mu\, \rho(f(x),V)^{-(\alpha-r)} : \\ &\qquad\qquad x \in \eta(\delta) \smallsetminus V \,\} \\ &\leq \mu\, \sigma_r(c^2, c^2).\end{aligned}$$

When $r = 0$, $\rho(x,V)^{-\alpha} \leq \mu\, \rho(f(x),V)^{-\alpha}$. The details are left to the reader. □

Let $I_r \in \Gamma J^r$ be defined by $I_r(x) = j^r(id)(x) = (x, x, id_x, 0, \ldots, 0)$ where $id : M \to M$ is the identity map and $id_x : T_xM \to T_xM$ is the identity map. Let $C_0 = \sigma_0(\Phi_0 I_0, I_0)$. C_0 is finite because $j^\alpha f(x) = j^\alpha g(x)$ for all $x \in V$ and V is compact. Let $D_0 = C_0(1-\mu)^{-1}$. Let 0_r be the zero section of $\pi_r : J^r \to J^{r-1}$. Let

$$\begin{aligned}\mathscr{F}_0 &= \{\, c \in \Gamma J^0 : \sigma_0(c, I_0) \leq D_0\} \quad \text{ and} \\ \mathscr{F}_r &= \{\, c \in \Gamma J^r : \pi_r c \in \mathscr{F}_{r-1} \text{ and } \sigma_r(c, 0_r\pi_r c) \leq D_0\} \quad \text{for } r \geq 1.\end{aligned}$$

Since $\sigma_0(c, I_0) \leq D_0$ for $c \in \mathscr{F}_0$, there exists a $\delta > 0$ smaller than above if necessary, such that for $c \in \mathscr{F}_0$ and $x \in \eta(\delta)$, then $c(f(x)) \in \mathscr{O}$.

Lemma 2. *Let $0 \leq r \leq \beta$. Then $\Phi_r : \Gamma J^r \to \Gamma J^r$ maps $\mathscr{F}_r$ into itself.*

Proof. We prove the lemma by induction. $\mathscr{F}_{-1} = \emptyset$ is invariant by Φ_{-1}. Assume $\mathscr{F}_{r-1}$ is invariant by Φ_{r-1}. Let $c \in \mathscr{F}_r$. Then

$$\begin{aligned}\sigma_r(\Phi_r c,\, 0_r\pi_r\Phi_r c) &\leq \sigma_r(\Phi_r c,\, \Phi_r 0_r \pi_r c) + \sigma_r(\Phi_r 0_r \pi_r c,\, 0_r\pi_r\Phi_r c) \\ &\leq \mu\, \sigma_r(c,\, 0_r\pi_r c) + \sigma_r(\Phi_r 0_r\pi_r c,\, 0_r\pi_r\Phi_r c).\end{aligned}$$

For $r = 0$, this last term is $\leq \mu\, D_0 + C_0 \leq D_0$. For $r > 0$, it is $< \infty$. □

Lemma 3. *Let $0 \leq r \leq \beta$. Then $\Phi_r : \mathscr{F}_r \to \mathscr{F}_r$ is continuous in terms of σ_r.*

Proof. We use the chain rule for higher derivatives of a composition.

$$\begin{aligned}\sigma_r(\Phi_r c^1,\, \Phi_r c^2) &= \sup\{\rho_r(\Phi_r c^1, \Phi_r c^2)\, \rho(x,V)^{-(\alpha-r)} : x \in \eta(\delta) \smallsetminus V\,\} \\ &\leq (\text{constant})\ \sup\{\, \|D^i g^{-1}(y_2)\|\, \rho_{j_1}(c^1(f(x)), c^2(f(x))) \cdots \\ &\qquad \rho_{j_r}(c^1(f(x)), c^2(f(x)))\, \|D^{k_1} f(x)\| \cdots \|D^{k_j} f(x)\|\, \rho(x,V)^{-(\alpha-r)} : \\ &\qquad\quad x \in \eta(\delta) \smallsetminus V,\ \pi_1 \circ \cdots \pi_r c^2(f(x)) = (f(x), y_2),\ 1 \leq i \leq r, \\ &\qquad\quad j = j_1 + \cdots + j_r,\ k_1 + \cdots + k_j = r\,\} \\ &\quad + (\text{constant})\ \sup\{\, \rho_j(g^{-1}(y_1), g^{-1}(y_2))\, \|D^{j_1} c^1(f(x))\| \cdots \|D^{j_i} c^1(f(x))\| \cdot \\ &\qquad\quad \|D^{k_1} f(x)\| \cdots \|D^{k_j} f(x)\|\, \rho(x,V)^{-\alpha-r)} : \\ &\qquad\qquad\qquad \pi_1 \circ \cdots \pi_r c^1(f(x)) = (f(x), y_1)\,\}.\end{aligned}$$

Here the constants depend only on the binomial coefficients. We look at the first supremum and leave the second to the reader. It is

$$\begin{aligned}&\le (\text{constant})\ \sup\{\, \sigma_{j_1}(c^1,c^2)\cdots\sigma_{j_i}(c^1,c^2)\cdot\rho(f(x),V)^{(i\alpha-j)}\,\rho(x,V)^{-(\alpha-r)} :\\ &\qquad\qquad 1\le i\le r,\ 1\le j\le r\,\}\\ &\le (\text{constant})\,\sigma_r(c^1,c^2)^r.\end{aligned}$$

These last two constants include the supremum of derivatives of f and g^{-1}. From this it follows that Φ_r is continuous. □

By Lemma 1, $\Phi_0 : \mathscr{F}_0 \to \mathscr{F}_0$ is a contraction in terms of σ_0. Thus, there is a unique attractive fixed point, c^0. Attractive means that for each $c \in \mathscr{F}_0$, $\sigma_0(c^0, \Phi_0^j(c)) \to 0$ as $j \to \infty$. Assume that $\mathscr{F}_{r-1}$ has an attractive fixed point for $1 \le r \le \beta$. By Lemma 3, Φ_r is continuous. By Lemma 2, $\Phi_r : \mathscr{F}_r \to \mathscr{F}_r$ contracts along fibers of $\pi_r : \mathscr{F}_r \to \mathscr{F}_{r-1}$ by a factor of μ. By the fiber contraction theorem (Theorem 1.2 in [5]), Φ_r has a unique fixed point in $\mathscr{F}_r$ and it is attractive.

Let $id : M \to M$ be the identity diffeomorphism and $I_\beta(x) = j^\beta(id)(x)$. Then $\Phi_\beta(I_\beta)$ converges (in the uniform topology of sections of $\psi_\beta : J^\beta \to \eta(\delta)$) to a section $c \in \Gamma J^\beta$. Let $c(x) = (x, c_0(x), \ldots, c_\beta(x))$ with $c_i(x) \in L_s^i(T_xM, T_{c_0(x)}M)$ for $1 \le j \le \beta$. By the uniform convergence, it follows that $c_i : \eta(\delta) \to \bigcup L_s^i(T_xM, T_yM) : x \in \eta(\delta),\ y \in M\,\}$ is $C^{\beta-i}$. Thus, the conditions of the Whitney Extension Theorem are satisfied. See [1] for a statement of the theorem. There exists a C^β function $h : M \to M$ such that for $x \in \eta(\delta)$, $j^\beta h(x) = c(x)$, $Dh(x) = id_x$ for $x \in V$ so h is a local diffeomorphism in a neighborhood of V. Thus, $h^{-1}gh$ is defined in a neighborhood of V in M and $j^\beta(h^{-1}gh)(x) = j^\beta f(x)$ for $x \in \eta(\delta) \subset W^s(V,f)$. This completes the conjugacy of f and g along $W^s(V,f)$.

Remark. In Corollary 3, we noted that more derivatives of the conjugacy existed along the fibers. In that setting, we have C^α diffeomorphisms such that $D_2^j(\mathrm{pr}\circ g)(x) = 0$ for $1 \le j \le \alpha$ and $x \in V$ and $f(\mathrm{pr}\,x) = \mathrm{pr}\circ f(x)$. Here $\mathrm{pr} : M \to V$ is a normal bundle. For $\alpha \ge r \ge \beta$, let $J^{\beta,r}$ be the bundle of maps with all derivatives $D^jD_2^kh(x)$ for $0 \le j \le \beta$ and $0 \le j+k \le r$. Let $\rho_{\beta,r}$ be the associated norm. Let $\pi_r : J^{\beta,r} \to J^{\beta,r-1}$ be as before. For $\pi_rc^1 = \pi_rc^2$, let

$$\sigma_{\beta,r}(c^1,c^2) = \sup\{\, \rho_{\beta,r}(c^1(x), c^2(c))\,\rho(x,V)^{-(\alpha-r)} : x \in \eta(\delta) \setminus V\,\}.$$

If $c^1, c^2 \in J^{\beta,r}$ and $\pi_rc^1 = \pi_rc^2$, then

$$\begin{aligned}&\rho_{\beta,r}(\Phi_{\beta,r}c^1(x), \Phi_{\beta,r}c^2(x))\,\rho(x,V)^{-(\alpha-r)}\\ &\qquad\qquad \le \rho_{\beta,r}(c^1(f(x)),\, c^2(f(x)))\,a_y\,B_x^\alpha\,A_x^{r-\beta}\,\rho(x,V)^{-(\alpha-r)}.\end{aligned}$$

A little checking is necessary to show this depends only on $\rho_{\beta,r}$ and not ρ_r. (f preserves fibers.) Then this is $\le \mu\,\rho_{\beta,r}(c^1(f(x)),\, c^2(f(x)))$. Lemma 1 follows. The other details are left to the reader.

II. Now we can assume f and g are C^β and $j^\beta f(x) = j^\beta g(x)$ for all all $x \in W^s(V,f) = W^s$ and x near V. For $x \in M$, define the following numbers:

$$a_x = \|Df^{-1}(x)\|$$
$$b_x = \begin{cases} \rho(f^{-1}(x),\, W^s)\, \rho(x, W^s)^{-1} & \text{for } x \notin W^s \\ \lim\{\, b_y : y \notin W^s \text{ and } y \to x \,\} & \text{for } x \in W^s, \end{cases}$$
$$B_x = \|Dg(x)\|.$$

For $x \in V$,

$$a_x^{-1} < 1 < b_x^{-1} \leq B_{f^{-1}(x)} \qquad \text{and}$$
$$B_{f^{-1}(x)}\, a_x^\gamma\, b_x^{\beta-\gamma} < \mu < 1.$$

By using a bump function, we can make $g(x) = f(x)$ at points x such that $\rho(x,V) \geq \delta$. (g is then defined on all of M.) Also, g can be left unchanged at points x with $\rho(x,V) \leq \delta/2$. Let

$$\eta(\delta) = \{\, x \in M : \rho(x, W^s) < \delta \,\} \qquad \text{and}$$
$$\eta'(\delta) = \{\, x \in M : \rho(x, V) < \delta \,\}.$$

By taking δ smaller if necessary and $\mathscr{O}$ to be a small neighborhood of $\{\,(x,x) : m \in W^s\,\}$ in $M \times M$, we can insure that for $x \in \eta'(\delta)$ and $(f^{-1}(x), y) \in \mathscr{O}$, it follows that $B_y\, a_x^\gamma\, b_x^{\beta-\gamma} < \mu$.

Let Φ_r be induces by $h \mapsto g \circ h \circ f^{-1}$. That is, in the earlier definition replace f by f^{-1} and g^{-1} by g. Continue as before taking sections c of $\psi_r : J^r(\eta(\delta), M) \to \eta(\delta)$ such that $c(x) = j^r\, id(x)$ for $x \in \eta(\delta) \smallsetminus \eta'(\delta)$. Lemmas 1, 2, and 3 apply to these sections. The $\lim_{j\to\infty} \Phi_\gamma^j(I_\gamma)$ gives the γ-jet of the conjugacy h on $\eta(\delta)$.

4. Proof of Theorem 2

In this section, we assume $T_V M = TV \oplus E^s$. The bundle $F^1 = TV$ is differentiable. Since we do not assume the bundles are invariant, we can approximate E^s by F^3 that is differentiable. Write $D_i h(z) = Dh(z)|F^i$. We assume in the theorem that

$$\|D_1 f^{-1}(f(z))\| \cdot \|Df(z)\|^{\beta-1} \cdot \|D_3 f(x)\| < \mu < 1 \qquad \text{for all } z \in V.$$

Denote a normal bundle projection by $\mathrm{pr} : M \to V$. For $c \in J^r((M,V),V)$, we write $c(z) = (z, c_0(z), \ldots, c_r(z))$ with $c_k(z) \in L_s^k(T_z M, F^1_{c_0(z)})$.

Let $\mathscr{F}_r$ be the set of sections c of $J^r((M,V),V)$ such that, for each $z \in V$ there is a C^r function $h : M \to V$ such that $h|V = id$ and $c(z) = j^r h(z)$. This is equivalent to assuming that for each $z \in V$, (i) $\pi_1 \circ \cdots \circ \pi_r\, c(z) = (z,z)$ and (ii) $c^k(z)|(F^1 \times \cdots \times F^1) = D^k(id)(z)$ where $id : V \to V$ is the identity function.

Let $f_V = f|V : V \to V$ and $f_V^{-1} = (f|V)^{-1} : V \to V$. Define $\Phi_r : \mathscr{F}_r \to \mathscr{F}_r$ by

$$\Phi_r\, c(z) = j^r\,(f_V^{-1} h f)(z) \quad \text{where } j^r\, h(f(z)) = c(f(z)).$$

By abuse of notation, $\Phi_r\, c(z) = j^r\,(f_V^{-1} c f)(z)$.

Lemma 4. *Let $0 \leq r \leq \beta$ and c^1, $c^2 \in \mathscr{F}_r$ be such that $\pi_r\, c^1 = \pi_r\, c^2$. Then $\rho_r(\Phi_r\, c^1,\, \Phi_r\, c^2) \leq \mu\, \rho_r(x^1, c^2)$.*

Proof.

$$\begin{aligned}&\rho_r(\Phi_r\, c^1,\ \Phi_r\, c^2)\\ &\qquad \le \sup\{\, \|D_1 f_V^{-1}(f(z))\| \cdot \|(c^1(f(z)) - c^2(f(z))(Df(z))^r\| : z \in V\,\}\\ &\qquad \le \sup\{\, \|D_1 f_1^{-1}(f(z))\|\, \rho_r(c^1, c^2)\, \|D_3 f(z)\| \cdot \|Df(z)\|^{r-1} : z \in V\,\}\end{aligned}$$

since $c^1_r|(F^1 \times \cdots \times F^1) = c^2_r|(F^1 \times \cdots \times F^1)$. Then

$$\rho_r(\Phi_r\, c^1,\ \Phi_r\, c^2) \le \mu\, \rho_r(c^1, c^2).$$

□

As in the proof of Theorem 1, we can apply the fiber contraction principle to find $c \in \mathscr{F}_\beta$ such that $\Phi_r\,(c) = c$. Let $s \in \Gamma J^\beta(\,(M,V), M)$ be given by $s(z) = (c(z), j^\beta(\mathrm{pr}_3\, z))$, i.e., the components of s in F^3 in the range is like the jet of the identity function on fibers zero derivatives in the directions along V. (This has meaning at the jet level but not as maps.) By the uniform convergence of $\Phi^k_\beta(j^\beta\ \mathrm{pr})$ to c, it follows that s satisfies the conditions of the Whitney Extension Theorem. There exists a C^β h such that $j^\beta\, h(z) = s(z)$ for $z \in V$. The map h is a diffeomorphism on a neighborhood of V because of the form of the derivatives at points of V.

Let $g = hfh^{-1}$ and $g_1 = \mathrm{pr}\, g$. At the level of jets for $z \in V$, $j^\beta(f_V^{-1} h_1 f)(z) = j^\beta(h_1)(z)$ and $j^\beta(\mathrm{pr}\, g \circ h)(z) = j^\beta(h_1 \circ f)(z)$, so $j^\beta(g_1)(z) = j^\beta(\mathrm{pr}\, g)(z) = j^\beta(f_V\, \mathrm{pr})(z)$ has the derivatives zero in the direction of F^3 as claimed. [6] This completes the proof of Theorem 2.

5. Proof of Corollary 4

Proof. Since $W^s(V,f)$ is C^α, we can restrict the map to this space and assume f is contracting along V. By applying Theorem 2, we can assume $D_3^j(\mathrm{pr} \circ f)(x) = 0$ for $1 \le j \le \beta$ and $x \in V$. Define $g_1 : U \to V$ by $g_1(x) = f_1 \circ \mathrm{pr}(x)$. In the proof of Theorem 1, replace $a_x = \|Dg^{-1}(x)\|$ by $a_x = \|Dg_1^{-1}(x)\|$ where $g_1^{-1} = (f|V)^{-1} : V \to V$. Next consider jets in $J^r = J^r(\eta(\delta), V)$ instead of $J^r(\eta(\delta), M)$. Define $\Phi_r : \Gamma J^r \to \Gamma J^r$ by

$$\Phi_r(c)(x) = j^r(g^{-1} h f)(x) \qquad \text{where} \quad j^r h(f(x)) = c(f(x)).$$

As in the earlier proof, we can find a c such that $\Phi_\beta c = c$ and c satisfies the conditions of the Whitney Extension Theorem. There exists a $C^{\beta-1}$ function $h : M \to V$ such that $g^{-1}hf = h$. The map h is a projection onto V and defines a C^β foliation. Since $hf = g_1 h$, it follows that f preserves the foliation. Since the foliation is tangent to E^s, it follows it is $W^{ss}(x,f)$. □

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