NONSYMMETRIC LORENZ ATTRACTORS FROM A HOMOCLINIC BIFURCATION

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ABSTRACT. We consider a bifurcation of a flow in three dimensions from a double homoclinic connection to a fixed point satisfying a resonance condition between the eigenvalues. For correctly chosen parameters in the unfolding, we prove that there is a transitive attractor of Lorenz type. In particular we show the existence of a bifurcation to an attractor of Lorenz type which is semiorientable, i.e., orientable on one half and nonorientable on the other half. We do not assume any symmetry condition, so we need to discuss nonsymmetric one dimensional Poincaré maps with one discontinuity and absolute value of the derivative always greater than one. We also apply these results to a specific set of degree four polynomial differential equations. The results do not apply to the actual Lorenz equations because they do not have enough parameters to adjust to make them satisfy the hypothesis.

1. INTRODUCTION

In previous papers, [10] and [11], we proved that there is a bifurcation for differential equations in three dimensions with a symmetry from a double homoclinic connection for a fixed point to an attractor of Lorenz type. This attractor could either be untwisted or twisted on both sides. In this paper we consider the situation without a symmetry: in particular, we show that there can be a bifurcation from a double homoclinic connection to an attractor which is twisted on one side but untwisted on the other side. We given basic assumptions which are sufficient for this to take place. We also verify that specific polynomial differential equations in three dimensions can realize this bifurcation.

A transversality assumption and the dominance of the strong stable eigenvalue are used with standard stable manifold theory to reduce the problem to a one dimensional map just in the previous papers. The problem of the unfolding of the bifurcation is thus reduced to a question of understanding the unfolding of a certain type of one dimensional maps. In all the cases of the homoclinic bifurcation of the three dimensional flow satisfying a set of assumptions, the resulting one dimensional map can be shown to have a transitive invariant set for correctly chosen parameter values.

The standard symmetric untwisted situation leads to a symmetric one dimension problem with is monotonically increasing on both sides. In this paper, we consider one dimensional maps which are not symmetric; in one case the map is increasing on one side and decreasing on the other side. We present the results of the thesis of M. Byers [2] which show how to carry through the result of Williams to show that the one dimensional map is transitive in these nonsymmetric cases when the absolute value of the derivative is greater than square root of two. We also refer to the recent result of Morales and Pujals [7], a previous work of Li and Yorke [6], and the thesis of Choi [3] which show that if the absolute value of the derivative is greater than one then the map has a transitive invariant set which is not always the whole original interval. One of these transitive invariant sets has a stable set which forms a dense open subset of a neighborhood, but it does not always have a trapping region. We are interesting in verifying that the corresponding flow on $\mathbb{R}^3$ does have a trapping region so we give some conditions in Section 2 which implies its existence.

A *trapping region* for a map f is a nonempty open set U such that $\mathrm{cl}(f(U)) \subset \mathrm{int}(U)$. A set Λ is called an *attracting set* provided there is a trapping region U such that $\Lambda = \bigcap_{k\geq 0} f^k(U)$. A set Λ is called an

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attractor provided it is an attracting set and $f|\Lambda$ is chain transitive. These definitions follow those given in [9]. There are other definitions of attractors including Milnor's which only requires that there a set B of positive measure such that the ω-limit sets of points in B are contained in Λ, i.e., B is contained in the stable set of Λ. In this paper we consider a definition between the two above: a set Λ is called a *weak attractor* provided (i) there is a neighborhood U of Λ and a dense open subset U' of U such that for all $x \in U'$ the ω-limit set of x is contained in Λ and (ii) $f|\Lambda$ is chain transitive.

A weak attractor can have a 1-cycle in the terminology of Palis, i.e., there can be points $x_0 \in U \setminus \Lambda$ which are on both the stable and unstable set of Λ, i.e., $\omega(x_0) \subset \Lambda$ and there is some choice of preimages $\{x_i\}_{i \le 0}$ with $f(x_{i-1}) = x_i$ for $i \le 0$ and the distance from x_i to Λ goes to zero as i goes to $-\infty$. (If f is one to one, then it has a 1-cycle provided there is a point $x_0 \in U \setminus \Lambda$ for which $\omega(x_0) \subset \Lambda$ and $\alpha(x_0) \subset \Lambda$.) An example of such a weak attractor with a 1-cycle is $x = 0$ for

$$f(x) = 1 + \frac{1}{2}x^2(1-x)^2 \qquad \text{for } x \text{ mod } 1.$$

For any point $x_0 \in (0,1)$, $\alpha(x_0) = 0$ and $\omega(x_0) = 1 = 0$ mod 1. In Section 2, we given another type of example of a map with a weak attractor but not an attractor.

In this paper as in [10] and [11], we consider a homoclinic bifurcation from the situation where there is a resonance between the eigenvalues together with transversality conditions. There are two other results by Rychlik [14] and Dumortier, Kokubu, and Oka [4] which give different homoclinic bifurcations to Lorenz attractors than the one we analyze. These other authors assume there is no resonance of the eigenvalues, but each also assumes that there is a type of nontransversality along the homoclinic orbit (which is different in the two papers) while we assume there is transversality.

In Section 2, we present the results on the one dimensional maps. The main theorem about the homoclinic bifurcation of flows is given in Section 3 together with the assumptions that are needed for this result. Section 4 contains the proof of the homoclinic bifurcation theorem. Section 5 contains some further comments about the unfolding of the bifurcation. Finally, Section 6 proves that the assumptions for the bifurcation can be satisfied for specific polynomial differential equations in $\mathbb{R}^3$.

2. One dimensional results

We are interested in conditions which imply that a one dimensional map with a single discontinuity is topologically transitive.

We consider a map $f : J \to \mathbb{R}$ where $J \subset \mathbb{R}$ is an open interval and which we assume satisfies the following conditions:

(a) The map f has a discontinuity at a single point $c \in J$.
(b) The map f is continuously differentiable on $J \setminus \{c\}$, with

$$\lambda = \inf_{x \in J \setminus \{c\}} |f'(x)| > 1.$$

(c) The right and left limits of f exist at c: let

$$a^+ = \lim_{x \to c+} f(x) \quad \text{and} \quad a^- = \lim_{x \to c-} f(x).$$

Often we act as if f is not defined at c, but we could always take $f(c) = a^+$, $f(c) = a^-$, or $f(c) = c$.

We state the last two assumptions separately for the cases when f has the same monotonicity for x less than c and x greater than c. First, we consider the case when f is either monotonically increasing on both sides of c or monotonically deceasing on both sides.

(d1) Let $a = \max\{a^-, a^+\}$ and $b = \min\{a^-, a^+\}$. We assume that $b < c < a$, so that c is in the interior of the interval $[b, a]$.
(e1) Finally, we assume that $b < f(a),\ f(b) < a$, so that the interval $[b, a]$ is invariant, $f([b, a]) \subset [b, a]$.

Next, we consider the case when f is monotonically increasing on one side of c and monotonically decreasing on the other side.

(d2) Let $a = \max\{a^-, a^+\}$ and $b = f(a)$. We assume that $b = f(a) < c < a$, so that c is in the interior of the interval $[b, a]$. (If $a = \min\{a^-, a^+\} < c$ and $b = f(a) > c$, then a reversal of orientation changes this case into the one considered here.)

(e2) Finally, we assume that $b < f(b) < a$, so that the interval $[b, a]$ is invariant.

It is not very hard to check that if f satisfies assumptions (a-e), then there is a small $\epsilon > 0$ such that the slightly larger interval $[b - \epsilon, a + \epsilon]$ is a trapping region.

According to a theorem of Williams, [15], if a map f satisfies conditions (a-e), has the same monotonicities on both the subintervals $[b, c)$ and $(c, a]$, and $\lambda > \sqrt{2}$, then f is topologically transitive on $[a, b]$. Theorem 2.2 below gives a generalization of this result to other cases when f is increasing on one of the subintervals and is decreasing on the other.

There are other results which extend the results to the case of a map which satisfies conditions (a-e) for any $\lambda > 1$. Li and Yorke, [6], proved that such maps have an ergodic measure whose support can be a subset of the original interval. More recently, Morales and Pujals [7] proved a different generalization of the result of Williams: they proved that that if the map f satisfies conditions (a-e) for any $\lambda > 1$ then there is a closed subset $L_f \subset [b, a]$ which contains c in its interior such that f is topologically transitive on L_f and a dense open subset of points of $[b, a]$ have forward orbits which eventually are contained in L_f (the stable manifold of L_f is dense and open in $[b, a]$.) In fact, L_f contains an interval I with c in its interior and L_f is the forward orbit of I. In general, the set L_f is the support of the measure found earlier by Li and Yorke.

In [3], Y. Choi has made more explicit the properties of L_f. In particular, (i) L_f is the finite union of closed intervals; (ii) the maximal invariant set in $\mathrm{cl}(J \setminus L_f)$ is a hyperbolic repeller R_f; (iii) L_f is always a weak attractor as defined in the introduction, but [3] gives an example where there is no trapping region for L_f so L_f is not an attractor in our strong sense of the term. The repeller R_f can be a set of periodic orbits and their preimages. (There are cases when R_f contains wandering points which have α-limit set in one periodic orbit in R_f and ω-limit set in another periodic orbit in R_f.) It is also possible for R_f to be a subshift of finite type as an example below shows. Choi has also shown that there are examples for which the set L_f does not have a trapping region (so L_f is not an attractor); such examples have a repelling periodic point on the boundary of L_f, i.e., a periodic orbit in $R_f \cap L_f$. For this example, the set L_f has a 1-cycle of the type discussed in the introduction. We give a different example below for which L_f does not have a trapping region, but without a 1-cycle. She also shows that the map can always be perturbed to a new map g without periodic points on the boundary of L_g, so L_g has a trapping region and so is an attractor for the new map g. We give a different example where L_h is not an attractor below.

We summarize these results in the following theorem.

Theorem 2.1. *Assume that $f : J \to \mathbb{R}_f$ satisfies the assumptions (a-e) above with $\lambda > 1$.*

(a) (Morales and Pujals) There is a $\delta_f > 0$ such that f is topologically transitive on

$$L_f \equiv \mathrm{cl}\{\mathcal{O}^+((c - \delta_f, c + \delta_f), f)\},$$

and

$$W^s(L_f, f) \equiv \{x \in J : f^i(x) \in L_f \textit{ for some } i \geq 0\}$$

is dense and open in J.

(b) (Choi) (i) The set L_f is the finite union of closed intervals, $\bigcup_{i=1}^n [x_i, y_i]$ and the endpoints

$$\{x_i, y_i\}_{i=1}^n \subset \mathcal{O}^+(a^+, f) \cup \mathcal{O}^+(a^-, f).$$

(ii) The maximal invariant set in $\mathrm{cl}(J \setminus L_f)$ *is a closed hyperbolic repelling set* R_f*. (Some of the points in* R_f *can be wandering.) (iii) The set* L_f *has a trapping region for* f *if and only if*

$$R_f \cap L_f = \emptyset, \qquad \textit{i.e.,} \qquad \mathrm{Per}(f) \cap \partial(L_f) = \emptyset.$$

(iv) Assume $\mathrm{Per}(f) \cap \partial(L_f) \neq \emptyset$*. If it is possible to perturb* f *to* g *such that* $\mathrm{Per}(g) \cap \partial(L_g) = \emptyset$*, then* L_g *will have a trapping region for* g*. More specifically, if* $z_0 \in \mathrm{Per}(f) \cap \partial(L_f)$ *with* $f^k(a^\sigma) = z_0$ *where* σ *is either* $+$ *or* $-$*, then we need to be able to perturb* f *to* g *such that* $g^k(a_g^\sigma)$ *is not in the perturbed periodic orbit for* g *(corresponding to* z_0 *for* f*).*

Example 1. A simple example of a function f for which L_f is not the whole interval $[a, b]$ is given by

$$f(x) = \begin{cases} \dfrac{4}{3}\,x + 10 & \text{for } -6 \le x \le 0, \\ -1.3x + 10 & \text{for } 0 \le x \le 11. \end{cases}$$

Note that $f(-3) = 6$, $f(0) = 10$, $f(2.2) = 7.14 > 6$, $f(6) = 2.2$, and $f(10) = -3$. Therefore $[a, b] = [-3, 10]$, and the transitive set $L_f = [-3, 2.2] \cup [6, 10]$.

Example 2. An example of a function g for which R_g is a subshift of finite type is given by

$$g(x) = \begin{cases} \dfrac{4}{3}\,x + 18 & \text{for } -21 \le x \le 0, \\ -\dfrac{7}{6}\,x + 18 & \text{for } 0 \le x \le 6, \\ -5\,(x-6) + 11 & \text{for } 6 \le x \le 10, \\ -\dfrac{9}{8}\,(x-10) - 9 & \text{for } 10 \le x \le 20. \end{cases}$$

The transitive set $L_g = [-18, -9] \cup [-6, 6] \cup [10, 18]$, since $g(-18) = -6$, $g(-9) = 6$, $g(-6) = 10$, $g(0) = 18$, $g(6) = 11 > 10$, $g(10) = -9$, and $g(18) = -18$. The repeller R_g is determined by the images of of the gaps and is a subshift of finite type: $g([-9, -6]) = [6, 10]$, and $g([6, 10]) = [-9, 11] \supset [-9, -6] \cup [6, 10]$.

Example 3. If we change the function g above so that $h(6) = 10$ but keep h piecewise linear with images of -18, -9, -6, 0, 10, and 18 unchanged, then $h^3(-9) = h^2(6) = h(10) = -9$ is a period three orbit which lies on the intersection of L_h and R_h. The set L_h does not have a trapping region for h since it is accumulated on by points in R_h outside of L_h. The stable set of L_h will include $[-19, 20] \setminus R_h$, which is dense and open in $[-19, 20]$ but is not a neighborhood of L_h.

In the rest of this section, we give conditions from [2] for various cases which imply that L_f is the whole interval $[b, a]$ as is the case which Williams considered.

Rather than prove directly that the map f is topologically transitive, we verify another condition called weakly locally eventually onto; Williams called a map $f : [b, a] \to [b, a]$ locally eventually onto provided for any nonempty open subinterval K there is an $n > 0$ such that $f^n(K) = [b, a]$. A map $f : [b, a] \to [b, a]$ is said to be *weakly locally eventually onto* (hereafter abbreviated *wleo*) provided for any nonempty open interval $K \subset [b, a]$ there are an $n > 0$ and a finite set of points A such that $\bigcup_{i=0}^{n} f^i(K) = [b, a] \setminus A$, i.e., the forward orbit of K misses at most a finite set of points. It is easier to verify that a map is wleo than locally eventually onto and it still implies that the map is topologically transitive on $[b, a]$ by the Birkhoff Transitivity Theorem.

In proving that these maps f are wleo, there are several cases depending on whether f is increasing or decreasing on the two subintervals $[b, c)$ and $(c, a]$.

Case (i): (The original Lorenz map) The map f is increasing on both subintervals $[b, c)$ and $(c, a]$, $a = a^- > c$, $b = a^+ < c$, $b \le f(b)$, and $f(a) \le a$.

Case (ii): (The twisted Lorenz map) The map f is decreasing on both subintervals $[b, c)$ and $(c, a]$, $a = a^+ > c$, $b = a^- < c$, $f(b) \le a$, and $b \le f(a)$.

Case (iii): (Variation of case (ii): the left end point is not a^- but is the image of a^+.) The function f is decreasing on both subintervals $[b, c)$ and $(c, a]$, $a = a^+ > c$, $b = f(a) < c$, $b < a^-$, and $f(b) < a$.

Case (iv): The function f is increasing on $[b, c)$ and decreasing on $(c, a]$, $a = a^+ \ge a^- > c$, $b = f(a) < c$, and $b \le f(b)$.

Case (v): The function f is increasing for $[b, c)$ and decreasing on $(c, a]$, $a = a^- > a^+ > c$, $b = f(a) < c$, and $b \le f(b)$.

There are other cases with $a^+, a^- < c$ which are equivalent to cases (iv) and (v) by a change of orientation which we do not list.

The proof of Williams shows that in cases (i - ii), if $\lambda > \sqrt{2}$ then f is leo and transitive on all of $[b, a]$. As was shown in [2], for cases (iv) and (v) this is not true: f is not always transitive on all of $[b, a]$ even when $\lambda > \sqrt{2}$. We state this in the following theorem.

Theorem 2.2. *(Byers) Assume that $f : J \to \mathbb{R}$ satisfies the assumptions (a-e) above.*

In case (iv) above, there is a fixed point $p \in (c, a)$. Assume that $f(x) < p$ for all $x \in [b, c)$. Then f is not transitive on $[b, a]$.

In case (v) above, there is a orbit of period two, $\{q^-, q^+\}$ with $q^- \in (b, c)$ and $q^+ \in (c, a)$. Assume that $c < a^+ < q^+$ and $q^- = f(q^+) < f(b) = f^2(a^-)$. Then f is not topologically transitive on $[b, a]$.

Idea of the proof. Case iv: Since $f'(x) > 1$ and $f[c, a] = [f(a), a]$, it follows that $f(a) < c$ and $f[c, a] \supset [c, a]$. Because the interval covers itself, there is a fixed point $p \in (c, a)$. (Notice that the fixed point can not be either of the end points.)

This fixed point p must be repelling because $f'(x) > 1$ everywhere. There is an interval K about p which covers itself but is not in the image of any other points in $[b, a] \setminus K$. Therefore K is not contained in the transitive attractor L and f is not topologically transitive on all of $[b, a]$. See [2] for details.

An example of such a function given in [2] is

$$f(x) = \begin{cases} 1.6x + 0.35 & \text{for } -0.5 \le x < 0, \\ -1.5x + 1 & \text{for } 0 \le x \le 1. \end{cases}$$

Case v: Let $\ell(K)$ be the length of an interval K. If $f(b) \ge c$, then $f[b, c] = [f(b), a] \subset [c, a]$ and $\ell(f[b, c]) \ge \lambda \ell([b, c])$. Then $f^2[b, c] = [b, f^2(b)]$ and $\ell(f^2[b, c]) \ge \lambda \ell(f[b, c]) \ge \lambda^2 \ell([b, c])$, so this interval covers itself, $f^2[b, c] \supset [b, c]$. Thus there is a point of period two with $q^- \in [b, c]$, $q^+ = f(q^-) \in f[b, c] \subset [c, a]$. This proves the existence of a point of period two under the assumption that $f(b) \ge c$.

Otherwise, $f(b) < c$. We also have that $a^- > a^+ > c$. Then $f[b, c] \supset [c, a]$ and $f^2[b, c] \supset f[c, a] = [b, a^+] \supset [b, c]$. Again, there is a point of period two as desired.

With the assumptions of the theorem for f in case (v), $q^+ > a^+$ and $q^- < f(b)$. Therefore there is a neighborhood K of $\{q^-, q^+\}$ made up of two intervals, which covers itself but is not in the image of any other points in $[b, a] \setminus K$. Therefore K is not contained in the transitive attractor L and f is not topologically transitive on all of $[b, a]$. See [2] for details.

An example of such a function given in [2] is

$$f(x) = \begin{cases} 1.415\,x + 1 & \text{for } -0.815 \le x \le 0, \\ -1.415\,x + 0.6 & \text{for } 0 \le x \le 1. \end{cases}$$

□

Notice that the examples given of the above theorem satisfy $\lambda > \sqrt{2}$ and are still not wleo or topologically transitive. Therefore it is necessary to add further assumptions in order to insure that the map f is topologically transitive.

We now combine the various results in [2] into a single theorem.

Theorem 2.3. *(Byers) Assume f satisfies assumptions (a-e) above and $\lambda > \sqrt{2}$. With the following added assumptions in each of the cases, $f|[a, b]$ is wleo and so topologically transitive.*

In cases (i) or (ii) no added assumption is needed.

In case (iii), further assume that $f(a^-) \geq p$, where $p \in (c, a)$ is the fixed point.

In case (iv) where $a^+ \geq a^-$, further assume that $a^- \geq p$, where $p \in (c, a)$ is the fixed point.

In case (v) where $a^- > a^+$, further assume that $a^+ \geq q^+$, where $q^+ \in (c, a]$ is the point of period two.

Remark 2.4. The M. Byers proved in [2] that in case (iv) it is sufficient to assume that $(1 + \sqrt{2})a^- \geq a^+$: this condition implies that $a^+ \geq a^-$. Similarly in case (v) it is sufficient to assume that $(3 - \sqrt{2})a^+ \geq a^-$: this condition implies that $a^+ \geq q^+$.

The proofs for all of the cases use the same basic construction due to Williams. Given an open interval $K \subset [b, a]$, we define inductively a sequence of intervals $K_i \subset [b, a]$ for $i \geq 0$. Define K_0 to be the longer component of $K \setminus \{c\}$. (Note that if $c \notin K$ then $K_0 = K$.) If K_j is defined for $0 \leq j < i$, then let K_i be longer component of $f(K_{i-1}) \setminus \{c\}$. Since $f(K_{i-1})$ is an open interval at each stage, it follows that all the K_i are open intervals.

Let $\ell(K)$ be the length of an open interval K.

Lemma 2.5. *If $\lambda > \sqrt{2}$, then there exists an $n > 0$ such that $c \in f(K_{n-1})$ and $c \in f(K_n)$, so $c \in \partial(K_n) \cap f(K_n)$.*

Proof. If $c \notin f(K_i)$ then $\ell(K_{i+1}) \geq \lambda \ell(K_i)$. On the other hand, if $c \in f(K_i)$ then $c \in \partial(K_{i+1})$ and $\ell(K_{i+1}) \geq \dfrac{\lambda}{2}\ell(K_i)$. So if $c \notin f(K_{i-1}) \cap f(K_i)$ we get that

$$\ell(K_{i+1}) \geq \frac{\lambda^2}{2}\,\ell(K_{i-1}).$$

Since $\dfrac{\lambda^2}{2} > 1$, this can not go on indefinitely, and there must be an $n > 0$ such that $c \in f(K_{n-1}) \cap f(K_n)$. □

In the proofs below, we take n as given in the above lemma for which $c \in \partial(K_n) \cap f(K_n)$.

Proof of Theorem for case (i). This is the case considered by Williams in [15]. We do not assume that $f(b) < c$ or $f(a) > c$. However, by modifying the argument in [15] or [9] in ways similar to the cases below, it still follows that f is wleo. □

Proof of Theorem for case (ii). Because f expands lengths by a factor of $\lambda > 1$, it follows that $f(b) > c$ and $f(a) < c$. Therefore the proof is exactly as given before. □

Proof of Theorem for case (iii). If $K_n \subset (c, a]$, then $c \in \partial(K_n) \cap f(K_n)$ implies that

$$\begin{aligned} f(K_n) &\supset [c, a) \qquad \text{and} \\ f^2(K_n) &\supset (b, a). \end{aligned}$$

On the other hand, if $K_n \subset [b, c)$, then

$$\begin{aligned} f(K_n) &\supset (a^-, c], \\ f^2(K_n) &\supset (a^-, p] \supset [c, p], \\ f^3(K_n) &\supset [p, a^+) = [p, a), \qquad \text{and} \\ f^4(K_n) &\supset (b, p]. \end{aligned}$$

Therefore $f^3(K_n) \cup f^4(K_n) \supset (b, p] \cup [p, a) = (b, a)$. This completes the proof of this case. □

Proof of Theorem for case (iv). This case is very similar to case (iii). We leave the details to the reader. Also see [2] □

Proof of Theorem for case (v). If $K_n \subset [b,c)$, then

$$f(K_n) \supset [c,a^-) = [c,a), \qquad \text{and}$$
$$f^2(K_n) \supset (b,a^+) \supset (b,c].$$

Therefore $f(K_n) \cup f^2(K_n) \supset (b,a)$.

On the other hand, suppose $K_n \subset (c,a]$. Then

$$f(K_n) \supset [c,a^+) \supset (c,q^+),$$
$$f^2(K_n) \supset (f(a^+),a^+) \supset (q^-,c),$$
$$f^3(K_n) \supset (q^+,a), \qquad \text{and}$$
$$f^4(K_n) \supset (b,q^-).$$

Therefore

$$f(K_n) \cup f^2(K_n) \cup f^3(K_n) \cup f^4(K_n) \supset (b,a) \setminus \{q^-,q^+\}.$$

This completes the proof of this case and the theorem. □

3. Statement of results for a homoclinic bifurcation

In this section we give the assumptions on flows in three dimensions which insure that a homoclinic bifurcation to a Lorenz attractor can take place. The first six assumptions, (A1)-(A6), on the parameterized differential equations concern the properties at the bifurcation value, η_0. The last assumption (A7) is on the unfolding of the parameter η which insures that there are parameter values which posses an attractor. The parameter space needs to be big enough to verify the assumptions of the one dimensional map given in the last section.

(A1) We consider a C^2 vector field X_η on $\mathbb{R}^3$ which depend on the parameter η and which has a fixed point $\mathbf{Q}_\eta$ for all parameter values near η_0. We assume that the eigenvalues of $DX_\eta(\mathbf{Q}_\eta)$ are all real with $\lambda_{ss}(\eta) < \lambda_s(\eta) < 0 < \lambda_u(\eta)$, and with respective eigenvectors $\mathbf{v}^{ss}$, $\mathbf{v}^s$, and $\mathbf{v}^u$,

With this assumption, there are several invariant manifolds for the fixed point at the origin. We denote the one-dimensional unstable manifold tangent to $\mathbf{v}^u$ by $W^u(\mathbf{Q}_\eta,\eta)$, and the two-dimensional stable manifold tangent to the $\mathbf{v}^s$ and $\mathbf{v}^{ss}$ by $W^s(\mathbf{Q}_\eta,\eta)$. Next, there is a one-dimensional strong stable manifold tangent to $\mathbf{v}^{ss}$ which we denote by $W^{ss}(\mathbf{Q}_\eta,\eta)$. This latter manifold is made up of points which converge to $\mathbf{Q}_\eta$ at an asymptotic rate determined by the eigenvalue λ_{ss}. All of these manifolds are C^r if the vector field is C^r, and are even real analytic if the vector field is real analytic. Finally, there is a two-dimensional manifold tangent to the two most expanding directions, $\mathbf{v}^u$ and $\mathbf{v}^s$, which we denote by $W^{eu}(\mathbf{Q}_\eta,\eta)$. This manifold is local in the stable direction but can be extended along the unstable manifold by flowing forward in time. We call this the *extended unstable manifold* even though it is not expanding in all directions. (Some people call this the center unstable manifold.) This manifold is at least C^1 (and C^2 with assumption (A2) on the dominance of the contraction toward $W^{eu}(\mathbf{Q}_\eta,\eta)$ given by $e^{\lambda_{ss}}$ in comparison with the greatest contraction within $W^{eu}(\mathbf{Q}_\eta,\eta)$ given by e^{λ_s}.) With this notation we can make the second assumption about the existence of a homoclinic orbit. Without a symmetry assumption on the differential equations it is a codimension two condition to have a double homoclinic connection.

(A2) For the bifurcation value η_0, there is a *double homoclinic connection* with the unstable manifold of $\mathbf{Q}_{\eta_0}$ contained in the stable manifold but outside the strong stable manifold,

$$\Gamma \equiv W^u(\mathbf{Q}_{\eta_0},\eta_0) \subset W^s(\mathbf{Q}_{\eta_0},\eta_0) \setminus W^{ss}(\mathbf{Q}_{\eta_0},\eta_0).$$

(The fact that Γ misses the strong stable manifold can be expressed as a transversality condition by stating that $W^u(\mathbf{Q}_{\eta_0}, \eta_0)$ is transverse to $W^{ss}(\mathbf{Q}_{\eta_0}, \eta_0)$.) In fact, we assume that the two branches $\Gamma^{\pm}$ of $\Gamma \setminus \{\mathbf{Q}_{\eta_0}\}$ are contained in the same component of $W^s(\mathbf{Q}_{\eta_0}, \eta_0) \setminus W^{ss}(\mathbf{Q}_{\eta_0}, \eta_0)$: $\Gamma = \{\mathbf{Q}_{\eta_0}\} \cup \Gamma^+ \cup \Gamma^-$.

(A3) For η_0, the two-dimensional extended unstable manifold $W^{eu}(\mathbf{Q}_{\eta_0}, \eta_0)$ is transverse to the two-dimensional stable manifold $W^s(\mathbf{Q}_{\eta_0}, \eta_0)$ along Γ.

The transversality condition in (A3) is generically satisfied and so does not add a codimension to the bifurcation. Let

$$P(\mathbf{q}) \equiv T_{\mathbf{q}} W^{eu}(\mathbf{Q}_{\eta_0}, \eta_0) \qquad \text{for } \mathbf{q} \in \Gamma.$$

The transversality condition in (A3) together with the condition that $W^u(\mathbf{Q}_{\eta_0}, \eta_0) \cap W^{ss}(\mathbf{Q}_{\eta_0}, \eta_0) = \emptyset$ in Assumption (A2) implies that $P(\mathbf{q})$ converges to $P(\mathbf{Q}_{\eta_0})$ as $\mathbf{q}$ converges to $\mathbf{Q}_{\eta_0}$ along Γ by the Inclination Lemma (Lambda Lemma). Therefore $\{P(\mathbf{q}) : \mathbf{q} \in \Gamma\}$ is a continuous bundle over Γ. Considering one half of the homoclinic connection $\Gamma^+ \cup \mathbf{Q}_{\eta_0}$, let $\nu^+ = 1$ if the bundle $\{P(\mathbf{q}) : \mathbf{q} \in \Gamma^+ \cup \mathbf{Q}_{\eta_0}\}$ is orientable (not twisted) and $\nu^+ = -1$ if this bundle is nonorientable (twisted). In the same way considering the other half of the homoclinic connection $\Gamma^- \cup \mathbf{Q}_{\eta_0}$, let $\nu^- = \pm 1$ whenever the bundle $\{P(\mathbf{q}) : \mathbf{q} \in \Gamma^- \cup \mathbf{Q}_{\eta_0}\}$ is orientable or nonorientable respectively. If the bundle is orientable, then the resulting one-dimensional map (which is discussed in the next section) is increasing on the corresponding subinterval; if the bundle is nonorientable then the resulting one-dimensional map is decreasing on the corresponding subinterval.

(A4) We assume that for η_0 the strong stable eigenvalue dominates the other two eigenvalues in the sense that

$$\lambda_{ss}(\eta_0) + [\lambda_u(\eta_0) - \lambda_s(\eta_0)] < 0 \qquad \text{and}$$
$$\lambda_{ss}(\eta_0) < 2\,\lambda_s(\eta_0).$$

This is an open condition and so does not add a codimension to the bifurcation. The second inequality in (A4) is what assures that the manifold $W^{eu}(\mathbf{Q}_{\eta_0}, \eta_0)$ is C^2. It is also redundant with the following resonance assumption (but sometimes we want to assume (A4) but not necessarily assume (A6).) These conditions are used to prove that the one dimensional Poincaré map is differentiable.

The next assumption on the equations is a restriction on the total change in area within the $P(q)$ directions ("within the attractor directions") when a solution travels the whole length of one of the loops Γ^+ or Γ^-.

(A5) Let $\mathbf{q}^{\pm}(t)$ be a parameterization of the solution along $\Gamma^{\pm}$. Let $\mathrm{div}_2(\mathbf{q}^{\pm}(t))$ be the infinitesimal change of area within the two dimensional planes $P(\mathbf{q}^{\pm}(t))$ as the solution $\mathbf{q}^{\pm}(t)$ moves along Γ, i.e., the "two dimensional divergence in $P(\mathbf{q})$" along Γ. Define $C^{\pm}_{\eta_0}$ by

$$C^{\pm}_{\eta_0} = \exp\Big(\int_{-\infty}^{\infty} \mathrm{div}_2(\mathbf{q}^{\pm}(t))\,dt\Big).$$

We assume that $0 < C^{\pm}_{\eta_0} < 1$. The quantity $C^{\pm}_{\eta_0}$ is the change in area within the planes $P(\mathbf{q})$ along the whole length of $\Gamma^{\pm}$.

Assumption (A5) is an open condition. If $C^{+}_{\eta_0} \approx C^{-}_{\eta_0}$ then we can take the interval in a symmetric fashion and we only need $C^{\pm}_{\eta_0} < 2$.

Lemma 4.1 in the next section shows that $C^{\pm}_{\eta_0}$ has meaning in terms of a one-dimensional Poincaré map, f_{η_0}, as the coefficient of the lowest order nonconstant term. Therefore, in a limiting sense that $f'_{\eta_0}(c_{\eta_0}\pm) = \nu^{\pm}\, C^{\pm}_{\eta_0}$. The fact that $C^{\pm}_{\eta_0} < 2$ means that f_{η_0} stretches lengths by a factor less than two and there is a hope that η near η_0 for f_η to map the appropriate interval $[b_\eta, a_\eta]$ inside itself (since there is one discontinuity). We restrict to $C^{\pm}_{\eta_0} < 1$ because the in the proof this gives $E_\eta < 1$. The fact that $C^{\pm}_{\eta} > 0$ means that it is possible to make the derivative of the one dimensional map to have derivative with absolute value greater than one. Lemma 4.2 gives conditions on unfolding parameters a^+_η, a^-_η, and $e_\eta = 1 - E_\eta = 1 - |\lambda_s(\eta)|/\lambda_u(\eta)$ which insures that this interval is invariant and absolute value of the derivative is always bigger than one.

If $\lambda_u(\eta_0) + \lambda_s(\eta_0) \neq 0$, then $\mathrm{div}_2(\mathbf{q}^{\pm}(t)) \neq 0$ for $|t|$ large, the integral in Assumption (A5) would be $\pm\infty$, $C^{\pm}_{\eta_0}$ would be ∞ or 0, and the total change of area along $\Gamma^{\pm}$ would be ∞ or 0. Therefore, the final resonance assumption for η_0 is one which makes Assumption (A5) possible. This resonance condition is a codimension one condition; in total, the conditions of η_0 are codimension three. (Two codimensions are from the double homoclinic connection and resonance condition gives the third and final codimension.)

(A6) There is a one-to-one resonance between the unstable and weak stable eigenvalue for η_0:

$$\lambda_u(\eta_0) + \lambda_s(\eta_0) = 0.$$

Letting $E_\eta = |\lambda_s(\eta)|/\lambda_u(\eta)$ and $e_\eta = 1 - E_\eta$, this condition can be expressed by saying that $E_{\eta_0} = 1$ or $e_{\eta_0} = 0$.

The final assumption relates to the unfolding of the bifurcation.

(A7) We need to assume that the parameter space is big enough so that a^+_η, a^-_η, and $E_\eta = |\lambda_s(\eta)|/\lambda_u(\eta)$ can be varied independently for η near η_0. (If the equation have a symmetry as was the case in [10] and [11], then we need only assume that a^+_η and E_η can be varied independently for η near η_0.)

It is now possible to state the main theorem.

Theorem 3.1. *Assume that vector field in $\mathbb{R}^3$, depending on a parameter η is C^2 and satisfies assumptions (A1)–(A7). Let $\mathcal{N}$ be a small neighborhood of η_0 in parameter space. Then, there exists a subset $\mathcal{N}' \subset \mathcal{N}$ with nonempty interior such that $\eta_0 \in \mathrm{cl}(\mathcal{N}')$, and such that for $\eta \in \mathcal{N}'$ the flow for η has a topologically transitive weak attractor which contains the fixed point $\mathbf{Q}_\eta$. In fact the weak attractor is determined by a one-dimensional Poincaré map f_η which is wleo on a finite union of closed intervals L_η containing a single point of discontinuity in its interior. The values of $\nu^{\pm}$ determine whether the attractor is orientable or not on the two branches. If the vector field is C^3 then the resulting one-dimensional Poincaré map f_η for $\eta \in \mathcal{N}'$ has an ergodic invariant measure with support equal to the whole invariant set L_η and which is equivalent to Lebesgue on L_η.*

The proof of the theorem is contained in the next section.

Remark 3.2. The fact that the flows satisfies Assumptions (A1)-(A4) means that standard stable manifold theory applies to show that that the problem can be reduced to a one dimensional Poincaré maps. Thus with the given assumptions, the proof of the theorem reduces to analyzing the unfolding of the one dimensional map and showing that we can get the situation discussed in Theorem 2.1. The three unfolding parameters of the one dimensional map are e_η and the two constant terms $a^{\pm}_\eta$ which are defined in Lemma 4.1. The proof indicates more fully what part of the parameter space yields an attractor. This is discussed more fully in Section 5.

Remark 3.3. Although we call these Lorenz attractors for the differential equations, if the equations are very nonsymmetric (C^+_η and C^-_η have very different values) then the one dimensional Poincaré map will be transitive on a set made up of a finite number of intervals and not just one. In other words, the results of Morales/Pujals and Choi apply rather than Byers' extension of the result of Williams. Therefore all we verify is that the invariant set is a weak attractor. We believe that for a dense and open set of values $\eta \in \mathcal{N}'$, the invariant set is an attractor and not just a weak attractor. To prove this would require showing that changing the parameters e_η and $a^{\pm}_\eta$ it is possible to realize the type of perturbations of the one dimensional map f_η indicated in Theorem 2.1b(iv).

Remark 3.4. If the equations are nearly symmetric in the sense that

$$\sqrt{2} - 1 < \frac{C^+_{\eta_0}}{C^-_{\eta_0}} < \frac{1}{\sqrt{2} - 1},$$

then it is possible to insure that the derivative is greater than $\sqrt{2}$ in absolute value. For these parameters the equations have an attractor and the one dimensional map is topologically transitive on a single interval I_η.

Remark 3.5. The existence of an ergodic invariant measure follows as in [11] using the result of Keller [5].

4. Proof of the Theorem from the Assumptions

We begin the proof by discussing the construction of the Poincaré map from the homoclinic connection and its form as given in [11].

Let Σ be a transversal to both $\Gamma^{\pm}$ out a short distance along the local stable manifold of $\mathbf{Q}_{\eta_0}$. There is a neighborhood $V \subset \Sigma$ of $\Gamma \cap \Sigma$ such that points in $V \setminus W^s(\mathbf{Q}_\eta)$ return to Σ, defining a Poincaré map

$$F_\eta : V \setminus W^s(\mathbf{Q}_\eta, \eta) \subset \Sigma \to \Sigma.$$

In [11], it was shown that assumption (A4) implies that the flow has an invariant continuous bundle of strong stable directions over Γ, $\{E^{ss}(\mathbf{q}) : \mathbf{q} \in \Gamma\}$, with $E^{ss}(\mathbf{Q}_{\eta_0}, \eta_0) = \langle \mathbf{v}^{ss} \rangle$. These conditions are open so this bundle exists not only over Γ for η_0 but also over a neighborhood of Γ for nearby η. Then the Stable Manifold Theory implies that there is a $C^{1+\mu}$ (C^1 plus μ-Hölder for some $\mu > 0$) invariant strong stable foliation in a neighborhood of Γ for η near η_0. If we take the union of these locally along an orbit and then intersect these with Σ, we get a one-dimensional foliation of Σ which is invariant by F_η. The projection along the leaves of the strong stable manifolds of orbits defines a map $\pi_\eta : \Sigma \to \Sigma^1$. By changing the orientation of Σ^1 if necessary, we can insure that we do not have $\nu^- = -1$ and $\nu^+ = 1$. (This last case can be changed into $\nu^- = 1$ and $\nu^+ = -1$.) The projection π_η can be used to define a one-dimensional map

$$f_\eta : V^1 \setminus \{c_\eta\} \subset \Sigma^1 \to \Sigma^1,$$

by $f_\eta(\pi_\eta \mathbf{q}) = \pi_\eta F_\eta(\mathbf{q})$ where $V^1 = \pi_\eta(V)$ and $c_\eta = \pi_\eta(W^s(\mathbf{Q}_\eta, \eta) \cap V)$ is the point of discontinuity.

We need to analyze the one dimensional map well enough to show that for correctly chosen parameter values it has a transitive invariant set containing the point of discontinuity. The next lemma which was proved in [10] and [11] gives an expansion of the map which is used to prove the existence of such a set. First we label the constant terms of the expansion of f_η; let

$$a^{\pm}_\eta = \limsup_{\tau \to c_\eta \pm} f_\eta(\tau).$$

This quantity corresponds to the signed distance of $\Gamma^{\pm}_\eta \subset W^u(\mathbf{Q}_\eta, \eta)$ from $W^s(\mathbf{Q}_\eta, \eta)$ as measured in Σ^1.

Lemma 4.1. *Assume Assumptions (A1)–(A4) are satisfied. Let E_η and $C^{\pm}_{\eta_0}$ be defined as in Assumptions (A6) and (A5). Let $c_\eta = \pi(W^s(\mathbf{Q}_\eta, \eta) \cap V)$. Let $J \subset \Sigma^1$ be a fixed small interval about c_{η_0}. For η in a small neighborhood of η_0, the induced one-dimensional Poincaré map $f_\eta : J \setminus \{c_\eta\} \subset \Sigma^1 \to \Sigma^1$ has continuous derivative on $J \setminus \{c_\eta\}$, and f_η and f'_η have the following form:*

$$f_\eta(\tau) = \begin{cases} a^+_\eta + \nu^+ C^+_\eta |\tau - c_\eta|^{E_\eta} + o(|\tau - c_\eta|^{E_\eta}) & \text{for } \tau > c_\eta, \\ a^-_\eta - \nu^- C^-_\eta |\tau - c_\eta|^{E_\eta} + o(|\tau - c_\eta|^{E_\eta}) & \text{for } \tau < c_\eta, \end{cases}$$

$$f'_\eta(\tau) = \begin{cases} \nu^+ E_\eta C^+_\eta |\tau - c_\eta|^{E_\eta - 1} + o(|\tau - c_\eta|^{E_\eta - 1}) & \text{for } \tau > c_\eta, \\ \nu^- E_\eta C^-_\eta |\tau - c_\eta|^{E_\eta - 1} + o(|\tau - c_\eta|^{E_\eta - 1}) & \text{for } \tau < c_\eta \end{cases}$$

The constants $C^{\pm}_\eta$ depends continuously on η.

See [11] for its proof. The proof uses either linearization near the fixed point or the analysis of the follow in terms of some normal form.

Let $a_\eta = \max\{a^-_\eta, a^+_\eta\}$ and $b_\eta = \min\{a^-_\eta, a^+_\eta, f(a_\eta)\}$. We are interested in parameter values η for which $b_\eta < c_\eta < a_\eta$. In order to have an expanding attractor for these parameter values, we also need f_η to preserve the interval $[b_\eta, a_\eta]$ and the absolute value of the derivative to be greater than one for points in the interval.

The three unfolding parameters which we use are $a^{\pm}_\eta$ and e_η. The parameters $a^{\pm}_\eta$ measure the extent to which the homoclinic connections are broken (and to which sides). The quantity $e_\eta = 1 - E_\eta = 1 - |\lambda_s(\eta)| / \lambda_u(\eta)$ measures the extent to which the two eigenvalues are no longer in resonance.

For the three allowable cases of $\nu^{\pm}$, if we take parameter values η for which $\nu^{+}(a_{\eta}^{+} - c_{\eta}) < 0$ and $\nu^{-}(a_{\eta}^{-} - c_{\eta}) > 0$, then $a_{\eta} > c_{\eta}$.

Lemma 4.2 proves that η_0 can be approximated by parameter values η for which $f_{\eta}([b_{\eta}, c_{\eta})) \subset [b_{\eta}, a_{\eta}]$, $f_{\eta}((c_{\eta}, a_{\eta}]) \subset [b_{\eta}, a_{\eta}]$, $|f_{\eta}'(a_{\eta})| > 1$, and $|f_{\eta}'(b_{\eta})| > 1$. Since $|f_{\eta}'(a_{\eta})| > 1$ and $|f_{\eta}'(b_{\eta})| > 1$. the form of f_{η}' given in Lemma 4.1 implies that there is a $\lambda > 1$ such that $|f_{\eta}'(\tau)| \geq \lambda$ for all $\tau \in [b_{\eta}, a_{\eta}]$. By the result of Morales and Pujals [7] summarized in Theorem 2.1(a) above, this implies that there is a transitive invariant set $L_{f_{\eta}}$ containing c_{η} which is a weak attractor. Because the one dimensional map can be varied by changing the flow, if there is a periodic point on the boundary of $L_{f_{\eta}}$ then it seems likely that it can be perturbed away. If this is indeed the case, then by the results of [3] summarized in Theorem 2.1(b) above, either L_{η} is an attractor for f_{η}, or η can be perturbed to η' for which $L_{\eta'}$ is an attractor for $f_{\eta'}$. Because of the relationship between the flow and the one dimensional Poincaré map, this shows that the flow for η has a transitive weak attractor as claimed in the theorem. Most likely it can be approximated by a parameter η' which has a transitive attractor as discussed in Remark 3.3. The claim about the ergodic measure for the one dimensional map follows just as in [11] using the result of Keller [5]. Thus we only need to prove the following lemma.

Lemma 4.2. *Assume the system satisfies (A1)–(A7). Let $\mathcal{N}$ be a small neighborhood of η_0 in parameter space. Let*

$$\begin{aligned}\mathcal{N}' = \{\eta \in \mathcal{N} : e_{\eta} > 0,\ &\nu^{+}(a_{\eta}^{+} - c_{\eta}) < 0,\ \nu^{-}(a_{\eta}^{-} - c_{\eta}) > 0,\\ &f([b_{\eta}, c_{\eta})) \subset [b_{\eta}, a_{\eta}],\ f((c_{\eta}, a_{\eta}]) \subset [b_{\eta}, a_{\eta}],\\ &|f_{\eta}'(a_{\eta})| > 1,\ |f_{\eta}'(b_{\eta})| > 1\ \}.\end{aligned}$$

Then $\mathcal{N}' \neq \emptyset$ and $\eta_0 \in \mathrm{cl}(\mathcal{N}')$. The conditions on e_{η} and $a_{\eta}^{\pm}$ which insure that $\eta \in \mathcal{N}'$ are given by inequality (4.1) below when $\nu^{+} = \nu^{-} = 1$ and by inequality (4.3) when $\nu^{+} = \nu^{-} = -1$.

Remark 4.3. When $\nu^{+} = \nu^{-} = \pm 1$, the interval found for $\eta \in \mathcal{N}'$ extends from a_{η}^{-} to a_{η}^{+} and satisfies

$$\frac{\log |a_{\eta}^{+} - c_{\eta}|}{\log |a_{\eta}^{-} - c_{\eta}|} \approx \frac{\log(C_{\eta}^{+})}{\log(C_{\eta}^{-})}.$$

Note that for $C_{\eta}^{-} \neq C_{\eta}^{+}$, the interval is not symmetric about c_{η}.

When $\nu^{-} = -\nu^{+} = 1$, the parameters found for $\eta \in \mathcal{N}'$ satisfies $a_{\eta}^{+} \approx a_{\eta}^{-}$. One end of the interval is $a_{\eta} = \max\{a_{\eta}^{+}, a_{\eta}^{-}\}$, and the other end is $b_{\eta} = f_{\eta}(a_{\eta})$. In the proof below we show that

$$\max\{0, \frac{C_{\eta}^{+}}{C_{\eta}^{-}} - 1\} < \frac{c_{\eta} - b_{\eta}}{a_{\eta} - c_{\eta}} < \frac{C_{\eta}^{+}}{C_{\eta}^{-}}.$$

Again, the interval is not symmetric about c_{η} when $C_{\eta}^{-} \neq C_{\eta}^{+}$.

Proof. First consider the case when $\nu^{+} = \nu^{-} = 1$. These maps fall into case (i). To get a transitive attractor, we take parameter values such that $a_{\eta} = a_{\eta}^{-} > c_{\eta}$ and $b_{\eta} = a_{\eta}^{+} < c_{\eta}$. If $C_{\eta}^{+} = C_{\eta}^{-}$ (as in the symmetric case), we can take $a_{\eta}^{+} - c_{\eta} \approx -(a_{\eta}^{-} - c_{\eta})$; this is the situation considered in Lemma 2 of [11]. We need to allow C_{η}^{+} to have a value very different from C_{η}^{-} even though both are in the interval $(0, 2)$. We want the derivative to be greater than one:

$$\begin{aligned}1 <&|f_{\eta}'(a_{\eta})| \approx E_{\eta} C_{\eta}^{+} |a_{\eta}^{-} - c_{\eta}|^{E_{\eta} - 1} \qquad \text{and}\\ 1 <&|f_{\eta}'(b_{\eta})| \approx E_{\eta} C_{\eta}^{-} |a_{\eta}^{+} - c_{\eta}|^{E_{\eta} - 1}.\end{aligned}$$

Also we need the interval $[b_{\eta}, a_{\eta}]$ to be invariant, so

$$\begin{aligned}|a_{\eta}^{-} - c_{\eta}| + |c_{\eta} - a_{\eta}^{+}| &\geq |f_{\eta}(a_{\eta}^{-}) - a_{\eta}^{+}| \approx C_{\eta}^{+} |a_{\eta}^{-} - c_{\eta}|^{E_{\eta}} \qquad \text{and}\\ |a_{\eta}^{-} - c_{\eta}| + |c_{\eta} - a_{\eta}^{+}| &\geq |f_{\eta}(a_{\eta}^{+}) - a_{\eta}^{-}| \approx C_{\eta}^{-} |a_{\eta}^{+} - c_{\eta}|^{E_{\eta}}\end{aligned}$$

Thus the conditions on a_η^+, a_η^-, and e_η are approximately the following:

$$-\log(E_\eta C_\eta^-) < e_\eta \log(|a_\eta^+ - c_\eta|^{-1}) < \log(1 + |\frac{a_\eta^- - c_\eta}{a_\eta^+ - c_\eta}|) - \log(C_\eta^-), \tag{4.1}$$
$$-\log(E_\eta C_\eta^+) < e_\eta \log(|a_\eta^- - c_\eta|^{-1}) < \log(1 + |\frac{a_\eta^+ - c_\eta}{a_\eta^- - c_\eta}|) - \log(C_\eta^+).$$

Since E_η goes to one as η goes to η_0, these conditions can be solved at the same time, with

$$\frac{\log(|a_\eta^+ - c_\eta|^{-1})}{\log(|a_\eta^- - c_\eta|^{-1})} \approx \frac{\log(C_\eta^-)}{\log(C_\eta^+)}. \tag{4.2}$$

When both $C_{\eta_0}^+, C_{\eta_0}^- < 1$, the resulting value of e_η can be made positive. If $C_{\eta_0}^+ \approx C_{\eta_0}^-$ then the interval is nearly symmetric, $\log\left(1 + |\frac{a_\eta^- - c_\eta}{a_\eta^+ - c_\eta}|\right) \approx \log(2)$, and we only need $C_{\eta_0}^\pm < 2$.

Next consider the case when $\nu^+ = \nu^- = -1$. These maps could fall into cases (ii) or (iii) but we just use case (ii) to show that parameter values exist. Again the symmetric case was considered in [11]. In the general (possibly nonsymmetric) case, we choose parameter values so that $a_\eta = a_\eta^+ > c_\eta$ and $b_\eta = a_\eta^- < c_\eta$. We want

$$1 < |f_\eta'(a_\eta)| \approx E_\eta C_\eta^+ |a_\eta^+ - c_\eta|^{E_\eta - 1} \quad \text{and}$$
$$1 < |f_\eta'(b_\eta)| \approx E_\eta C_\eta^- |a_\eta^- - c_\eta|^{E_\eta - 1}.$$

Also we need the interval $[b_\eta, a_\eta]$ to be invariant, so

$$|a_\eta^- - c_\eta| + |c_\eta - a_\eta^+| \geq |f_\eta(a_\eta^+) - a_\eta^+| \approx C_\eta^+ |a_\eta^+ - c_\eta|^{E_\eta} \quad \text{and}$$
$$|a_\eta^- - c_\eta| + |c_\eta - a_\eta^+| \geq |f_\eta(a_\eta^-) - a_\eta^-| \approx C_\eta^- |a_\eta^- - c_\eta|^{E_\eta}$$

Thus the conditions on a_η^+, a_η^-, and e_η are approximately the following:

$$-\log(E_\eta C_\eta^+) < e_\eta \log(|a_\eta^+ - c_\eta|^{-1}) < \log(1 + |\frac{a_\eta^- - c_\eta}{a_\eta^+ - c_\eta}|) - \log(C_\eta^+), \tag{4.3}$$
$$-\log(E_\eta C_\eta^-) < e_\eta \log(|a_\eta^- - c_\eta|^{-1}) < \log(1 + |\frac{a_\eta^+ - c_\eta}{a_\eta^- - c_\eta}|) - \log(C_\eta^-).$$

Again when both $C_\eta^+, C_\eta^- < 1$, these conditions can be solved at the same time with $e_\eta > 0$ and

$$\frac{\log(|a_\eta^+ - c_\eta|^{-1})}{\log(|a_\eta^- - c_\eta|^{-1})} \approx \frac{\log(C_\eta^+)}{\log(C_\eta^-)}. \tag{4.4}$$

We could enlarge set of allowable parameter values to include those which give one dimensional maps which fall into case (iii) as long as $a_\eta^- - f_\eta(a_\eta^+)$ is small enough.

Finally, we consider the case when $\nu^+ = -1$ and $\nu^- = 1$. (The case with $\nu^+ = 1$ and $\nu^- = -1$ can be reduced to this case by reversing orientation of Σ^1.) These maps fall into in cases (iv) or (v). We first take parameter values η so that $a_\eta = a_\eta^- = a_\eta^+ > c_\eta$. After we obtain the result in this case, the interval remains invariant with absolute value of the derivative greater than one for $|a_\eta^+ - a_\eta^-|$ small and $a_\eta = \max\{a_\eta^-, a_\eta^+\}$.

Set $b_\eta = f_\eta(a_\eta)$. Thus $f_\eta[c_\eta, a_\eta] = [b_\eta, a_\eta]$ maps inside the relevant interval. We need to check that $|f_\eta'(a_\eta)| > 1$, $|f_\eta'(b_\eta)| > 1$, and $f_\eta(b_\eta) > b_\eta$. The last condition insures that $f_\eta[b_\eta, a_\eta] \subset [b_\eta, a_\eta]$.

The derivative at a_η must satisfy

$$1 < |f_\eta'(a_\eta)| \approx E_\eta C_\eta^+ |a_\eta - c_\eta|^{-e_\eta},$$

or

$$-\log(C_\eta^+ E_\eta) < e_\eta \log(|a_\eta - c_\eta|^{-1}).$$

Next, we need $f_\eta(b_\eta) > b_\eta$. Since

$$\begin{aligned} f_\eta(b_\eta) - b_\eta &\approx f_\eta(a_\eta - C_\eta^+(a_\eta - c_\eta)^{E_\eta}) - a_\eta + C_\eta^+(a_\eta - c_\eta)^{E_\eta} \\ &\approx a_\eta - C_\eta^-[-a_\eta + C_\eta^+(a_\eta - c_\eta)^{E_\eta} + c_\eta]^{E_\eta} - a_\eta + C_\eta^+(a_\eta - c_\eta)^{E_\eta} \\ &\approx (a_\eta - c_\eta)^{E_\eta}\{C_\eta^+ - C_\eta^-[C_\eta^+(a_\eta - c_\eta)^{-e_\eta} - 1]^{E_\eta}\}, \end{aligned}$$

$f_\eta(b_\eta) - b_\eta > 0$ provided

$$\frac{C_\eta^+}{C_\eta^-} > [C_\eta^+((a_\eta - c_\eta)^{-e_\eta} - 1]^{E_\eta}.$$

Since E_η converges to one, this is approximately the inequality

$$\begin{aligned} \frac{C_\eta^+}{C_\eta^-} &> C_\eta^+(a_\eta - c_\eta)^{-e_\eta} - 1 \qquad \text{or} \\ \frac{1}{C_\eta^-} + \frac{1}{C_\eta^+} &> (a_\eta - c_\eta)^{-e_\eta}. \end{aligned}$$

This is the second inequality we need to satisfy. Thus these two conditions are satisfied provided (approximately)

$$\log\Big(\frac{1}{C_\eta^+ E_\eta}\Big) < e_\eta \log(|a_\eta - c_\eta|^{-1}) < \log\Big(\frac{1}{C_\eta^-} + \frac{1}{C_\eta^+}\Big) \tag{4.5}$$

These two inequalities can be satisfied at the same time.

To check that $|f_\eta'(b_\eta)| > 1$, we need to consider two subcases: (i) $1 > C_{\eta_0}^- > C_{\eta_0}^+ > 0$ and (ii) $1 > C_{\eta_0}^+ \geq C_{\eta_0}^- > 0$. First, define the comparison of the lengths of the two sides of the interval $[b_\eta, a_\eta]$ by

$$\gamma_\eta = \frac{c_\eta - f_\eta(a_\eta)}{a_\eta - c_\eta} = \frac{c_\eta - b_\eta}{a_\eta - c_\eta}.$$

We see below that $0 < \gamma_\eta < 1$ for subcase (i) and $\gamma_\eta > -1 + C_\eta^+/C_\eta^-$ for subcase (ii) which can often be greater than one. Using Lemma 4.1 and the definition of γ_η,

$$\begin{aligned} a_\eta - b_\eta &\approx C_\eta^+(a_\eta - c_\eta)^{E_\eta}, \\ a_\eta - b_\eta &= a_\eta - c_\eta + c_\eta - b_\eta \\ &= (1 + \gamma_\eta)(a_\eta - c_\eta), \quad \text{so} \\ (1 + \gamma_\eta) &\approx C_\eta^+(a_\eta - c_\eta)^{-e_\eta}. \end{aligned}$$

For the first subcase (i) when $1 > C_{\eta_0}^- > C_{\eta_0}^+ > 0$, using (4.5),

$$\begin{aligned} 1 + \gamma_\eta &\approx C_\eta^+(a_\eta - c_\eta)^{-e_\eta}, \\ 1 &< C_\eta^+(a_\eta - c_\eta)^{-e_\eta} < 1 + \frac{C_\eta^+}{C_\eta^-}, \qquad \text{so} \\ 0 &< \gamma_\eta < \frac{C_\eta^+}{C_\eta^-} < 1. \end{aligned}$$

Therefore for parameters satisfying (4.5),

$$\begin{aligned}|f_\eta'(b_\eta)| &\approx E_\eta C_\eta^- |b_\eta - c_\eta|^{-e_\eta}\\ &\approx E_\eta C_\eta^- {\gamma_\eta}^{-e_\eta} |a_\eta - c_\eta|^{-e_\eta}\\ &> \frac{C_\eta^-}{C_\eta^+} {\gamma_\eta}^{-e_\eta}\\ &> \frac{C_\eta^-}{C_\eta^+} > 1,\end{aligned}$$

since $\gamma_\eta^{-1} > 1$. Thus for this subcase all three conditions are satisfied for parameters satisfying (4.5).

Now consider subcase (ii) when $1 > C_{\eta_0}^+ \geq C_{\eta_0}^- > 0$. Again

$$|f_\eta'(b_\eta)| \approx E_\eta C_\eta^- {\gamma_\eta}^{-e_\eta} |a_\eta - c_\eta|^{-e_\eta}.$$

Inequality (4.5) together with the fact that $1+\gamma_\eta \approx C_\eta^+(a_\eta - c_\eta)^{-e_\eta}$ imply that γ_η is bounded as η converges to η_0 in $\mathcal{N}'$. Therefore $E_\eta {\gamma_\eta}^{-e_\eta}$ converges to one as η converges to η_0 in $\mathcal{N}'$. Therefore the inequality $|f_\eta'(b_\eta)| > 1$ is (essentially) equivalent to

$$-\log(C_\eta^-) < e_\eta \log(|a_\eta - c_\eta|^{-1}).$$

Since $-\log(C_\eta^+) < -\log(C_\eta^-)$, this implies that all three conditions are satisfied in this subcase provided

$$\log\big(\frac{1}{C_\eta^-}\big) < e_\eta \log(|a_\eta - c_\eta|^{-1}) < \log\big(\frac{1}{C_\eta^-} + \frac{1}{C_\eta^+}\big) \tag{4.6}$$

Notice that for these parameters

$$\frac{C_\eta^+}{C_\eta^-} < C_\eta^+(a_\eta - c_\eta)^{-e_\eta} \approx 1 + \gamma_\eta < 1 + \frac{C_\eta^+}{C_\eta^-} \qquad \text{so}$$

$$0 \leq \frac{C_\eta^+}{C_\eta^-} - 1 < \gamma_\eta < \frac{C_\eta^+}{C_\eta^-},$$

which can be quite large if the system is very asymmetrical. This completes the proof of the lemma and theorem. □

5. Unfolding of the bifurcation

To make the discussion simpler, we assume that $c_\eta \equiv 0$. We assume that $0 < C_{\eta_0}^{\pm} < 1$ since this is the situation that leads to an attractor of Lorenz type. In fact, the situation we verify for specific equations in this paper and the papers [10], [11], and this paper has $0 < C_{\eta_0}^{\pm} << 1$. We discuss the cases $\nu^+ = \nu^- = \pm 1$ and $\nu^- = -\nu^+$ separately. In each case we take the relationship between a_η^+ and a_η^- found in the proof of Lemma 4.2 so there are only two parameters, $e_\eta = 1 - E_\eta$ and either a_η^+ or a_η^-

First consider $\nu^+ = \nu^- = 1$. By equation (4.2), $a_\eta^+ \approx -|a_\eta^-|^\kappa$ where $\kappa = \log(C_\eta^-)/\log(C_\eta^+)$. So, we can use the two parameters $a_\eta = a_\eta^-$ and $e_\eta = 1 - E_\eta$.

The region of parameters labeled (ii) in Figure 1 is the region $\mathcal{N}'$ found in Theorem 3.1 which corresponds to systems with a attractor of Lorenz type. As a consequence of inequality (4.1) in the proof of Lemma 4.2, the boundary of $\mathcal{N}'$ is contained in $\partial\mathcal{N}$, γ_1, and γ_2, where the latter two are given approximately by

$$\gamma_1 : \quad e_\eta \log(|a_\eta^-|^{-1}) \approx \log\big(\frac{1}{C_\eta^+}\big), \quad a_\eta^- > 0$$

$$\gamma_2 : \quad e_\eta \log(|a_\eta^-|^{-1}) \approx \log\big(\frac{1}{C_\eta^+}\big) + \log(1 + |a_\eta^-|^{\kappa-1}), \quad a_\eta^- > 0.$$

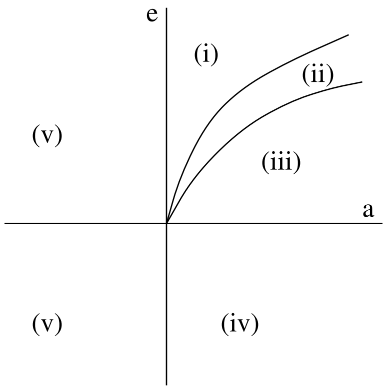

FIGURE 1. Bifurcation diagram

Notice that in the symmetric case $\kappa = 1$ and

$$\gamma_2 : \quad e_\eta \log(|a_\eta^-|^{-1}) \approx \log\big(\frac{2}{C_\eta^+}\big),$$

which is the form of the boundary given in [10] and [11].

Region (iii) in Figure 1: In this case the absolute value of the derivative is not greater than one at all points in $[a_\eta^+, a_\eta^-]$. However for many of these parameters, the map is still eventually expanding and there is a transitive attractor. We do not attempt to analyze these cases more thoroughly.

Region (i) in Figure 1: In this region $a_\eta^- > 0$ and e_η above γ_2. It follows that (a) $|f(a_\eta^-)| > a_\eta^-$, (b) the interval $[a_\eta^+, a_\eta^-]$ is not invariant, and (c) there is a horseshoe which is separated from the fixed point of the flow.

Region (iv) in Figure 1: In this case there is an attracting periodic orbit. See Remark 5.1 below.

Region (v) in Figure 1: Because $a_{\eta_0}^- < 0$, the discontinuity 0 is not in the image of the map; so there is no invariant set that bifurcates off near the homoclinic orbit.

The case for $\nu^- = \nu^+ = -1$ is very similar. By equation (4.4), $a_\eta^- \approx -|a_\eta^+|^\kappa$ where the exponent $\kappa = \log(C_\eta^-)/\log(C_\eta^+)$. So, we can use the two parameters $a_\eta = a_\eta^+$ and $e_\eta = 1 - E_\eta$. The interval is $[a_\eta^-, a_\eta^+]$.

Again the region of parameters labeled (ii) in Figure 1 is the region $\mathcal{N}'$ found in Theorem 3.1 which corresponds to systems with a attractor of Lorenz type. As a consequence of inequality (4.3) in the proof of Lemma 4.2, the boundary of $\mathcal{N}'$ is contained in $\partial\mathcal{N}$, γ_1, and γ_2, where the latter two are given approximately by

$$\gamma_1 : \quad e_\eta \log(|a_\eta^+|^{-1}) \approx \log\big(\frac{1}{C_\eta^+}\big), \quad a_\eta^+ > 0$$

$$\gamma_2 : \quad e_\eta \log(|a_\eta^+|^{-1}) \approx \log\big(\frac{1}{C_\eta^+}\big) + \log(1 + |a_\eta^+|^{\kappa-1}), \quad a_\eta^+ > 0.$$

The other regions are similar to the previous case.

Finally we consider the case when $\nu^- = -\nu^+ = 1$. In this case we take $a_\eta = a^+_\eta = a^-_\eta$. By inequalities (4.5) and (4.6), the two boundary components of region (ii) are now given approximately by

$$\gamma_1 : \quad e_\eta \log(|a^+_\eta|^{-1}) \approx \max\{\log\big(\frac{1}{C^+_\eta}\big),\ \log\big(\frac{1}{C^-_\eta}\big)\}, \quad a_\eta > 0$$

$$\gamma_2 : \quad e_\eta \log(|a^+_\eta|^{-1}) \approx \log\big(\frac{1}{C^+_\eta} + \frac{1}{C^-_\eta}\big) \quad a^+_\eta > 0, \quad a_\eta > 0.$$

The other regions are very similar to the situation of the previous cases.

Remark 5.1. Rovella [13] showed that there are flows with $E_\eta > 1$ and $\nu^\pm = 1$ which have transitive attractors. Such an attractor has a one dimensional Poincaré map whose derivative is zero at the discontinuity. Rovella showed that method of Benedicks and Carleson [1] could be used to show that there is a transitive attractor for a positive set of parameter values. These results should also apply when $\nu^\pm = -1$, but the case for $\nu^- = -\nu^+$ is very different and it is not clear that this argument applies.

These attractors do not occur in our unfolding for $\nu^- = \nu^+ = 1$ because (in the symmetric case) in order for $f_\eta(a^+_\eta) > c_\eta$ it is necessary for $C^+_\eta > 1$ and we consider only $0 < C^+_\eta, C^-_\eta < 1$. On the other hand, if $C^+_{\eta_0} = C^-_{\eta_0} > 1$ and $\nu^+ = \nu^- = \pm 1$, it seems likely that an attractor of the type found by Rovella occurs in the unfolding.

6. Specific Differential Equations Satisfying the Assumptions

In the previously papers [10] and [11] we showed that there were symmetric differential equations satisfying Assumptions (A1)-(A7) with $\nu^+ = \nu^- = 1$ or $\nu^+ = \nu^- = -1$. In this section, we show that there is a polynomial differential equation satisfying (A1)-(A7) with $\nu^+ = -1$ and $\nu^- = 1$. The basic idea of the example is the same are before, but the equations need to be modified so the twisting is different on the two sides. Because of the difference, it is no longer possible to make the equations have a symmetry. Most the the verification of the assumptions is very straight forward. The two things that need to be checked more carefully, is the transversality of Assumption (A3) and the bound on the coefficients $C^\pm_{\eta_0}$ in Assumption (A5).

The equations which we consider are

$$\text{(NSE)} \qquad \begin{aligned} \dot x &= y \\ \dot y &= x - 2\,x^3 - \alpha\, y + \beta\, x^2 y + \epsilon\, x^3 y + xyz \\ \dot z &= -\gamma\, z + \delta\, x^2. \end{aligned}$$

The parameters are $\eta = (\alpha, \beta, \gamma, \delta, \epsilon)$. The changes from the equations considered in the previous papers [10] and [11] is that in the $\dot y$ equation we have added the term $\epsilon\, x^3 y$ and the term xyz replaces one which was yz in [10] and xz in [11].

The fixed point **0** is always the origin. The linearization of the vector field is given by

$$DX(x,y,z) = \begin{pmatrix} 0 & 1 & 0 \\ 1 - 6x^2 + 2\beta xy + 3\epsilon x^2 y + yz & -\alpha + \beta x^2 + \epsilon x^3 + xz & xy \\ 2\delta x & 0 & -\gamma \end{pmatrix}.$$

At the origin, the eigenvalues are $\lambda_{ss} = -\alpha/2 - (1+\alpha^2/4)^{1/2}$, $\lambda_u = -\alpha/2 + (1+\alpha^2/4)^{1/2}$, and $\lambda_s = -\gamma$ giving Assumption (A1).

By picking the parameter $\gamma_0 = \lambda_u = -\alpha_0/2 + (1+\alpha_0^2/4)^{1/2}$ at the bifurcation, we can insure that $\lambda_s(\eta_0) + \lambda_u(\eta_0) = 0$ giving assumption (A6).

To obtain (A4), we need the combination of all three eigenvalues less than zero, $\lambda_{ss}(\eta_0) - \lambda_s(\eta_0) + \lambda_u(\eta_0) < 0$:

$$\begin{aligned} 0 &> [-\alpha_0/2 - (1+\alpha_0^2/4)^{1/2}] + 2[-\alpha_0/2 + (1+\alpha_0^2/4)^{1/2}] \\ &> -3\alpha_0/2 + (1+\alpha_0^2/4)^{1/2} \end{aligned}$$

or

$$\alpha_0 > 2^{-\frac{1}{2}}.$$

Thus to obtain a C^1 foliation, it is not possible to take a small perturbation of the integrable case where $\alpha = \beta = \delta = 0$. Given the resonance condition (A4), the second inequality in (A3) follows from the first.

To verify Assumptions (A2), (A3), and (A5), we start with $\delta_1 = 0$. By adjusting the parameter values β_1 and ϵ_1 we can make a double homoclinic connection with $\delta_1 = 0$. For $\delta_1 = 0$ the (x, y)-plane is invariant and $W^u(\mathbf{0}, \eta_1) \subset W^{ss}(\mathbf{0}, \eta_1)$ so (A2) is not true. Just as in [11], we can perturb δ_1 to $\delta_0 > 0$ and adjust β_1 to β_0 and ϵ_1 to ϵ_0 to keep the double homoclinic connection. Because the $\delta_0 x^2$ terms is positive in $\dot{z}$, the unstable manifold $W^u(\mathbf{0}, \eta_0)$ is pushed upward and $W^{ss}(\mathbf{0}, \eta_0)$ is pushed downward giving Assumption (A2), $W^u(\mathbf{0}, \eta_0) \cap W^{ss}(\mathbf{0}, \eta_0) = \emptyset$. Following the argument in [11], Lemma 6.1 below proves that after this perturbation with $\delta_0 > 0$ but small, the transversality condition of Assumption (A3) is satisfied and that $0 < C_{\eta_0} < 2$ as required in Assumption (A5). It also proves that $\nu^+ = -1$ and $\nu^- = -1$.

The unfolding Assumption (A7) is satisfied because changing γ varies e_η while β and ϵ can adjust $a_\eta^\pm$ independently. Thus all that is left to prove is the following lemma.

Lemma 6.1. *For $\delta_0 > 0$ but small, $W^{eu}(\mathbf{0}, \eta_0)$ is transverse to $W^s(\mathbf{0}, \eta_0)$, $0 < C_{\eta_0}^\pm << 1$, and $\nu^+ = -1$ and $\nu^- = -1$.*

Proof. A normal vector to $W^{eu}(\mathbf{0}, \eta)$, or $P(\mathbf{q}_\eta^\pm(t))$, is a covector and satisfies the adjoint differential equation

$$\dot{\mathbf{p}} = -\mathbf{p}\, DX(\mathbf{q}_\eta^\pm(t)).$$

(Note that in this equation, $\mathbf{p}$ is written as a row vector.) We denote the solution which is perpendicular to $P(\mathbf{q}_\eta^\pm(t))$ by $\mathbf{p}_\eta^\pm(t) = (p_1^\pm(t,\eta), p_2^\pm(t,\eta), p_3^\pm(t,\eta))$ Note that together, $(\mathbf{q}_\eta^\pm(t), \mathbf{p}_\eta^\pm(t))$ lies on the unstable manifold of $(\mathbf{0}, \mathbf{0})$ in the space of positions and covectors, $T^*\mathbb{R}^3$.

We start with $\delta_1 = 0$ and β_1 and ϵ_1 adjusted so there are double homoclinic orbits. As t goes to infinity, we want to show that $p_3^\pm(t, \eta_1)$ goes to $-\infty$ and $\mathbf{p}_{\eta_1}^\pm(t)$ approaches the direction given by $-\mathbf{v}_s^*$ along the negative z-axis. The equations for $\dot{p}_1$ and $\dot{p}_2$ are independent of p_3 and so can be solved independently for a solution $(p_1^\pm(t,\eta_1), p_2^\pm(t,\eta_1))$ that is perpendicular to the homoclinic orbit in the (x, y)-plane, and so it limits on the eigendirection $\mathbf{v}_u^*$ for the eigenvalue $-\lambda_u$. Therefore $(p_1(t,\eta_1), p_2(t,\eta_1)) \to \mathbf{0}$ as t goes to infinity.

We parameterize the homoclinic connections $\mathbf{q}_\eta^\pm(t)$ so that $y^\pm(0, \eta) = 0$, so $x^\pm(t,\eta) y^\pm(t,\eta)$ is positive for $t < 0$ and negative for $t > 0$. Since

$$\begin{aligned} \dot{p}_3^\pm(t,\eta) &= -x^\pm(t,\eta)\, y^\pm(t,\eta)\, p_2^\pm(t,\eta) + \gamma\, p_3^\pm(t,\eta), \\ p_3^\pm(t,\eta) &= e^{\gamma(t-t_0)} p_3^\pm(t_0,\eta) - \int_{t_0}^t x^\pm(s,\eta)\, y^\pm(s,\eta)\, p_2^\pm(s,\eta)\, e^{\gamma(t-s)}\, ds. \end{aligned}$$

As t_0 goes to $-\infty$, since $(\mathbf{q}_\eta^\pm(t_0), \mathbf{p}_\eta^\pm(t_0))$ is in the unstable manifold in $T^*\mathbb{R}^3$, $\mathbf{p}_\eta^\pm(t_0)$ goes to zero exponentially at a rate given by λ_{ss}, $e^{-|\lambda_{ss}||t_0|}$. Since this is faster decay than the growth given by γ, we can let $t_0 \to -\infty$ and obtain

$$p_3^\pm(t,\eta) = -\int_{-\infty}^t x^\pm(s,\eta)\, y^\pm(s,\eta)\, p_2^\pm(s,\eta)\, e^{\gamma(t-s)}\, ds.$$

Because $X(\mathbf{q}_{\eta_1}^\pm(0))$ points in the direction of $(0, 1, 0)$, $p_2^\pm(0, \eta_1) = 0$ and we can take $p_2^\pm(t, \eta_1) > 0$ for $t < 0$ and $p_2^\pm(t, \eta_1) < 0$ for $t > 0$, so $x^\pm(t,\eta_1) y^\pm(t,\eta_1) p_2^\pm(t,\eta_1) \geq 0$ and

$$-p_3(t,\eta_1) > e^{\gamma t} \int_0^t x^\pm(s,\eta_1)\, y^\pm(s,\eta_1)\, p_2(s,\eta_1)\, e^{-\gamma s}\, ds.$$

As t goes to infinity, the integral is positive and $e^{\gamma t}$ goes to infinity, so $-p_3(t, \eta_1)$ goes to infinity. Since we showed above that $(p_1(t,\eta_1), p_2(t,\eta_1)) \to \mathbf{0}, \mathbf{p}^\pm(t, \eta_1)$ has a limiting direction along the negative z axis, i.e., in the direction of the co-eigenvector $-\mathbf{v}_s^*$ for the eigenvalue $-\lambda_s$. Therefore the limiting plane $P(\mathbf{q}_{\eta_1}^\pm(\infty))$

is spanned by the directions $\{\mathbf{v}^u, \mathbf{v}^{ss}\}$ and $W^{eu}(\mathbf{0}, \eta_1)$ is transverse to $W^s(\mathbf{0}, \eta_1)$. Since transversality is an open condition, it remains true for η_0 with $\delta_0 > 0$ small. This given Assumption (A3).

We now want to show that $\nu^+ = -1$ and $\nu^- = 1$. The limiting direction of $\mathbf{p}^{\pm}_{\eta_0}(t)$ as t goes to $-\infty$ is $-\mathbf{v}^*_{ss}$, with negative first coordinate. To understand the behavior as t goes to ∞, notice that for η_1, the limiting direction of $\mathbf{p}^{\pm}_{\eta_1}(t)$ is $-\mathbf{v}^*_s$ which is contained in the space spanned by $\{\mathbf{v}^*_{ss}, \mathbf{v}^*_s\}$. Since it is an open condition not to have a component in the $\mathbf{v}^*_u$ direction for the eigenvalue $-\lambda_u$, it will continue to be true for η_0. (This is the openness of the transverse intersection of Assumption (A3).) For η_0 with $\delta_0 > 0$ but small, there is still a homoclinic connection, but now the homoclinic orbit $\mathbf{q}^{\pm}_{\eta_0}(t)$ approaches $\mathbf{0}$ along the weak stable direction. Since $\mathbf{p}^{\pm}_{\eta_0}(t)$ must remain orthogonal to X, the limiting direction of $\mathbf{p}^{\pm}_{\eta_0}(t)$ is contained in the space spanned by $\{\mathbf{v}^*_{ss}, \mathbf{v}^*_u\}$. Combining the two arguments, the limiting direction must be in the $\pm\mathbf{v}^*_{ss}$ direction. (This shows that the bundle of planes $\{P(\mathbf{q}) : \mathbf{q} \in \Gamma\}$ is continuous, and so is a second way of seeing that Assumption (A3) is true.) Because of the trajectory bends upward and then down asymptotic to the z-axis, the limiting direction of $\mathbf{p}^{+}_{\eta_0}(t)$ is $\mathbf{v}^*_{ss}$ while that of $\mathbf{p}^{-}_{\eta_0}(t)$ is $-\mathbf{v}^*_{ss}$. Therefore $\nu^+ = -1$ and $\nu^- = 1$.

As argued in [10], for η_1 with $\delta_1 = 0$, the integral of Assumption (A5) is $-\infty$ and $C^{\pm}_{\eta_1} = 0$. By the perturbation argument given in [10], for $\delta_0 > 0$ small, the integral is still very negative but finite, so $0 < C^{\pm}_{\eta_0} << 1$. This proves Assumption (A5). Notice that since we do not calculate the integrals, we have no way of knowing whether $C^{+}_{\eta_0}$ is nearly equal to $C^{-}_{\eta_0}$. □

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