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UNIFORM SUBHARMONIC ORBITS FOR SITNIKOV PROBLEM

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Abstract. We highlight the argument in Moser's monograph that the subharmonic periodic orbits for the Sitnikov problem exist uniformly for the eccentricity sufficiently small. We indicate how this relates to the uniformity of subharmonic periodic orbits for a forced Hamiltonian system of one degree of freedom with a symmetry.

1. Introduction and Statement of Result.

The Sitnikov problems concerns the restricted three-body problem in space, where two primaries of mass $\frac{1}{2}$ each move in the (x, y)-plane and the third massless body moves on the z-axes. The radial coordinate of one of the primaries in an elliptical orbit with eccentricity ϵ is

$$r(t) = \frac{1}{2}\left(1 - \epsilon \cos(2\pi t)\right) + O(\epsilon^2). \tag{1}$$

Then the equation for the third body is

$$\frac{d^2 z}{dt^2} = -\frac{z}{(z^2 + r^2(t))^{\frac{3}{2}}}. \tag{2}$$

A good introduction into this problem is given in [4] where it is shown that for $\epsilon > 0$ there exists a horseshoe Λ_ϵ and so many periodic orbits. These orbits have periods that are multiples of the forcing period and so are called *subharmonic orbits*. The minimum of the periods of the orbits in Λ_ϵ goes to infinity as ϵ goes to zero, so this result does not imply that there are periodic orbits for all periods greater than a fixed bound as ϵ goes to zero, i.e., that there uniform subharmonic orbits for the Sitnikov problem. Jaume Llibre pointed out to me that this uniformity was proved in Moser's book [4]. In this paper, I want to explain how this result is contained in [4] and how it relates to a general result about forced Hamiltonian systems with a symmetry and Melnikov functions. Although it is implicit in [4], A. García & Pérez-Chavela [1] showed how to put the Sitnikov problem in the framework of the Melnikov function.

In considering the Sitnikov problem, we compare it with a periodically forced forced Hamiltonian system with one degree of freedom where the original periodic orbit is actually hyperbolic and all the orbits consider lie in a compact part of space. In this situation, the geometry and uniformity is easier to understand. An example

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is the forced Duffing system which we write as an autonomous system in $\mathbb{R}^2 \times S^1$,

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= x - x^3 + \epsilon\, \cos(2\,\pi\,\tau) \\ \dot{\tau} &= 1 \qquad\qquad (\text{mod } 1).\end{aligned} \tag{3}$$

For $\epsilon = 0$, we consider this as a differentiable equation in the (x, y)-space ignoring the τ variable. This system in $\mathbb{R}^2$ has a transversal $\Sigma = \{(x, 0) : x > 0\}$. It also has an invariant energy function $H(x, y) = \dfrac{y^2}{2} - \dfrac{x^2}{2} + \dfrac{x^4}{4}$. The level curve $H^{-1}(0)$ is made up of the hyperbolic fixed point $(0, 0)$ and two homoclinic orbits. Let $\Gamma = \{(x, y) : H(x, y) = 0,\ x > 0\,\}$ be the points on one of these homoclinic orbits and parameterize it by $\mathbf{q}^0(t)$ with $\mathbf{q}^0(0) = (a, 0) \in \Sigma$ where $a = \sqrt{2}$. The region inside Γ is filled with periodic orbits $\{\mathbf{q}^h(t)\}$ that we can parameterize by energy, $H(\mathbf{q}^h(t)) = h < 0$, and choose the initial condition so that $\mathbf{q}^h(0) \in \Sigma$. Letting $\mathcal{R}(x, y) = (x, -y)$ be the linear reflection in the x-axis, 3 has a symmetry under the change of variables $(x, y, \tau, t) \mapsto (x, -y, -\tau, -t) = (\mathcal{R}(x, y), -\tau, -t)$. Note that Σ is the set of fixed points of the reflection.

We next identify the essential features of 3 that we use for the proof of the uniformity of the subharmonics.

We assume that $\mathcal{R}$ is a linear reflection on $\mathbb{R}^2$ with a line of fixed points Σ. We assume that $H(x, y)$ is a real-valued Hamiltonian function that is invariant under $\mathcal{R}$, $H(\mathcal{R}\mathbf{z}) = \mathbf{z}$, with a saddle critical point at $\mathbf{p}_0 \in \Sigma$, $H(\mathbf{p}_0) = 0$ (for notational simplicity), and at least one branch of $H^{-1}(0)$ called Γ is a loop. For simplicity, we assume that $H(\mathbf{z}) < 0$ inside Γ. We assume that $\mathbf{g}(x, y, \tau)$ is period 1 in τ and $g(\mathcal{R}\mathbf{z}, \tau) = g(\mathbf{z}, \tau)$. The *forced Hamiltonian system with symmetry* is given by

$$\begin{aligned}\dot{x} &= \frac{\partial H}{\partial y} + \epsilon g_1(x, y, \tau) \\ \dot{y} &= -\frac{\partial H}{\partial x} + \epsilon g_2(x, y, \tau) \\ \dot{\tau} &= 1 \qquad\qquad (\text{mod } 1).\end{aligned} \tag{4}$$

This system is invariant under the symmetry $(\mathbf{z}, \tau, t) \mapsto (\mathcal{R}\mathbf{z}, -\tau, -t) \equiv \hat{\mathcal{R}}(\mathbf{z}, \tau, t)$. For $\epsilon = 0$, the saddle critical point $\mathbf{p}_0$ of H is a a hyperbolic fixed point, and the loop Γ in $H^{-1}(0)$ is an orbit $\mathbf{q}^0(t)$ that is homoclinic to $\mathbf{p}_0$. We take the parametrization such that $\mathbf{q}^0(0) \in \Sigma$. We also let $\gamma_0 = \{\, (\mathbf{p}_0, \tau) : 0 \le \tau \le 1\,\}$ be the periodic orbit in the 3-dimensional space. For $\epsilon \ne 0$, let γ_ϵ be the nearby hyperbolic periodic orbit with stable and unstable manifolds $W^s_\epsilon(\gamma_\epsilon)$ and $W^u_\epsilon(\gamma_\epsilon)$ in $\mathbb{R}^2 \times S^1$.

For $\epsilon = 0$, inside of Γ, there are also periodic orbits $\mathbf{q}^h$ on level set $H(\mathbf{q}^h(t)) = h < 0$ with $\mathbf{q}^h(0) \in \Sigma$. The points $\mathbf{q}^h(0)$ converges to $\mathbf{q}^0(0)$ as $h < 0$ goes to 0. Let T_h be the period of $\mathbf{q}^h$. Since T_h goes to ∞ as h goes to 0, $\dfrac{\partial T_h}{\partial h} > 0$ for h near enough to 0. For some $m_0 > 0$ and $m \ge m_0$, there is an h_m be such that $T_{h_m} = m$.

There are Melnikov functions for both homoclinic and subharmonic orbits: The *homoclinic Melnikov function* is given by

$$M^0(\tau_0) = \int_{-\infty}^{\infty} \nabla H\big(\mathbf{q}^0(t - \tau_0)\big) \cdot \mathbf{g}(\mathbf{q}^0(t - \tau_0), t)\ dt$$

and the *subharmonic Melnikov functions* for each h_m are given by

$$M^m(\tau_0) = \int_{\tau_0}^{\tau_0+m} \nabla H\big(\mathbf{q}^{h_m}(t-\tau_0)\big)\cdot \mathbf{g}(\mathbf{q}^{h_m}(t-\tau_0),t)\;dt.$$

A *nondegenerate zero* of a real-valued function M is a value τ^* such that $M(\tau^*) = 0$ and $M'(\tau^*) \neq 0$. The following theorem summarizes the standard results about Melnikov functions or easy results.

For a forced Hamiltonian system with symmetry given by 4, the symmetry insures that $M(0) = 0$, but we have to assume that this zero is nondegenerate.

Theorem 1.1. *Comparison of homoclinic and subharmonic Melnikov functions.*

- **a.** $M^m(\tau) \to M^0(\tau)$ *as* $m \to \infty$.
- **b.** *If* τ^* *is a nondegenerate zero of* M^0*, then there is a nondegenerate zero* τ^*_m *for* M^m *as* $m \to \infty$ *(uniformly in* m*) such that* τ^*_m *converges to* τ^* *as* m *goes to infinity.*

Subharmonic Melnikov function:

- **c.** *If* M^m *has a nondegenerate zero* τ^*_m*, then there exists a period-m point* $(\mathbf{p}_m(\epsilon), \tau_m(\epsilon))$ *for* $0 < |\epsilon| \leq \epsilon^*_m$ *such that* $(\mathbf{p}_m(\epsilon), \tau_m(\epsilon))$ *converges to* $(\mathbf{q}^{h_m}(\tau^*_m), \tau^*_m)$ *as* ϵ *converges to* 0.

Homoclinic Melnikov function: Assume that M^0 *has a nondegenerate zero at* τ^*.

- **d.** *There exists a transverse homoclinic point* Q_ϵ *for* $0 < |\epsilon| \leq \epsilon^*$.
- **e.** *There exists a hyperbolic horseshoe,* Λ_ϵ*, for* $0 < |\epsilon| \leq \epsilon^*$.

Part (a) is not hard analysis because subharmonic orbit converges to the homoclinic orbit and $\nabla H(0, 0) = (0, 0)$. For part (b), because the convergence is uniform, the results about the derivatives follows. Part (c) is the standard result about the subharmonic Melnikov function and parts (d) and (e) are the standard results about the homoclinic Melnikov function. See [3] for these results and others.

Remark. *For the proof of part (c) about the subharmonic orbits, it is necessary to ensure that the orbit comes back with both the same energy* H *and the same coordinate along the homoclinic orbit* Γ*. The subharmonic Melnikov function measure the change in energy, so a zero determines a coordinate in the* Γ *direction which returns with the same energy to first order in* ϵ*. Because the period various with changing energy, there is a choice of the* H *coordinate that return with the same same coordinate along the homoclinic orbit* Γ*. Combining gives a periodic orbit. Various people have made this idea rigorous.*

Theorem 1.2. *Assume that there is one of the follow two types of systems:* (**a**) *A force Hamiltonian system with symmetry* (4) *for which* M^0 *has a nondegenerate zero at* $\tau = 0$ *or* (**b**) *the Sitnikov problem* (2)*. Then, there is an* $\epsilon^*_{sub} > 0$ *and an* $m \geq m_0$ *such that there exists a period-m point* $(\mathbf{p}_m(\epsilon), 0)$ *for all* $0 < |\epsilon| \leq \epsilon^*_m$.

Remark. *Note that the statement and the proof of our main theorem only involves the homoclinic Melnikov function.*

We do not attempt to get a uniformity out of the convergence given in Theorem 1.1(a): we would need to reverse the order of taking limits in ϵ *and* m.

The hyperbolic horseshoe Λ_ϵ *given by Theorem 1.1(e) has a periodic point of period* 1 *and periodic points of periods* j *for all* $j \geq J_\epsilon$*, where* $J_\epsilon \sim \ln\left(\frac{1}{|\epsilon|}\right)$*. This lower bound on the periods is not usually stated, but follows from the time it takes to*

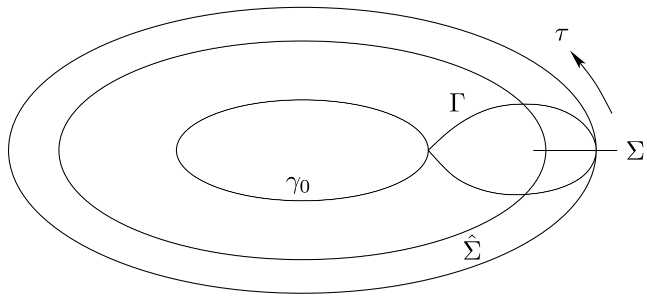

FIGURE 1. Orbit and cross section Σ

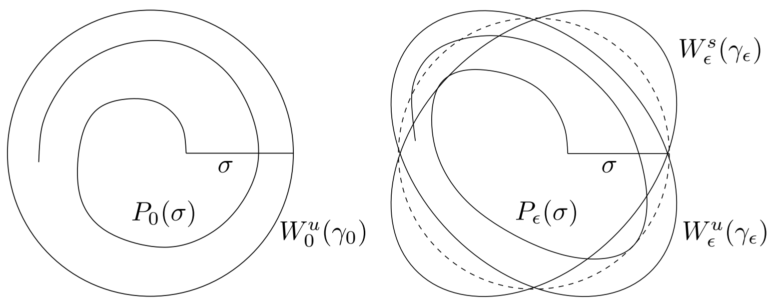

FIGURE 2. Sketch in $\hat{\Sigma}$. **a.** $P_0(\sigma)$ for $\epsilon = 0$, **b.** $P_\epsilon(\sigma)$ for $\epsilon \neq 0$

go past the fixed point. Since the subharmonic orbits for all $m \geq m_0$ exist uniformly for $0 < |\epsilon| \leq \epsilon^*_{sub}$, the ones for $m_0 \leq m < J_\epsilon$ are not in Λ_ϵ and are most likely elliptic.

Remark. *We note that* [2] *and* [3] *both state that periodically forced oscillators like 3 or 4 (even without symmetry) have periodic subharmonic orbits uniformly in the period ($m \geq m_0$) as ϵ goes to zero. They also assume that the unforced system has a hyperbolic fixed point (which is a hyperbolic periodic orbit when τ is added), and that the homoclinic Melnikov function has a nondegenerate zero. However, their argument is very geometric and it is not clear if it applies to the Sitnikov problem since the "periodic orbit at infinity" is not hyperbolic but only saddle-like due to higher order terms.*

Our proof also gives a different argument for the uniformity in periodically forced systems with a symmetry when the original system has a hyperbolic fixed point. We basically highlight the basic ingredients of the proof in Moser's book [4] *and indicated why it implies the uniformity.*

2. **Proof of Theorem 1.2(a).** In this section we consider a forced Hamiltonian system with symmetry (4).

Let $\hat{\Sigma} = \Sigma \times \{0 \leq \tau \leq 1\}$ be the transversal in $\mathbb{R}^2 \times S^1$ out near $\mathbf{q}^0(0) \times S^1$. See Figure 1. Let $\sigma = \{\, (\mathbf{q}^h(0), 0) : h_0 \leq h \leq 0 \,\} \subset \hat{\Sigma} \cap \{\tau = 0\}$ be a line segment transverse to $W^s_0(\gamma_0)$. Let $\mathbf{P}_\epsilon : \hat{\Sigma} \to \hat{\Sigma}$ be the Poincaré map past the closed orbit and back to $\hat{\Sigma}$. Since the return time $T_{h,\epsilon}$ of the orbit with initial conditions

$(\mathbf{q}^h(0), 0)$ goes to infinity in a uniform manner with $\frac{d}{dh}T_{h,\epsilon} > 0$, and using the λ-Lemma, we get that $\mathbf{P}_\epsilon(\sigma)$ spirals around $\hat{\Sigma}$ and converges to $W^u_\epsilon(\gamma_\epsilon)$ in a C^1 manner. See Figure 2. Therefore, $\mathbf{P}_\epsilon(\sigma)$ intersects σ in a countably infinite number of points $\{\mathbf{p}_j(\epsilon)\}_{j\geq j_0}$. Because of the symmetry $\hat{\mathcal{R}}$, these points also return to σ under $\mathbf{P}_\epsilon^{-1}$. Therefore $\mathbf{P}_\epsilon$ takes $\{\mathbf{p}_j(\epsilon)\}_{j\geq j_0}$ to itself (except possibly the furtherest point out from $W^u_\epsilon(\gamma_\epsilon)$.) For $\epsilon = 0$, the system also preserves the energy and so $\mathbf{P}_0(\mathbf{p}_j(0)) = \mathbf{p}_j(0)$ for each j. Because the set of points is discrete and the points vary continuously, $\mathbf{P}_\epsilon(\mathbf{p}_j(\epsilon)) = \mathbf{p}_j(\epsilon)$ for $|\epsilon|$ less than some $\epsilon^*_{\text{sub}} > 0$ and these points are all periodic. Between each pair of points $\mathbf{p}_j(\epsilon)$ and $\mathbf{p}_{j+1}(\epsilon)$, $\mathbf{P}_\epsilon(\sigma)$ wraps one more time around S^1 in the τ-variable, so these points lie on subharmonic orbits of all periods above a single m_0. By reindexing the points by the period m, $\mathbf{p}_m(\epsilon)$ can be made a subharmonic of period m for all $m \geq m_0$ uniformly for $|\epsilon| \leq \epsilon^*_{\text{sub}}$.

Note that the C^1 convergence of $\mathbf{P}_\epsilon(\sigma)$ to $W^u_\epsilon(\gamma_\epsilon)$ is used only to show that it is monotone in τ and crosses Σ transversally. The length of the line segment σ can be taken for $\epsilon = 0$, and then this uniform length will continue to spiral in a correct fashion for $|\epsilon| \leq \epsilon^*_{\text{sub}}$.

3. **Proof of Theorem 1.2(b) for the Sitnikov problem.** The argument given above for the forced Hamiltonian with symmetry is essentially the one given by Moser in [4] for the Sitnikov problem. The details of the argument use McGehee coordinates and the non-hyperbolic saddle periodic orbit at infinity (when the massless body had $z = \pm\infty$.) We leave to the reader to check the details involving coordinates and other details in [4].

The cross section Σ corresponds to $z = 0$. The Poincaré map $\mathbf{P}_\epsilon$ corresponds to the return map from $z = 0$ back to $z = 0$. Moser's proof of Lemma 3 on pages 163 – 165 of [4] shows that the homoclinic Melnikov function has a nondegenerate zero at $\tau = 0$. Then, Moser's proof of Lemma 4 on pages 167 – 171 of [4] shows that $\mathbf{P}_\epsilon(\sigma)$ spirals around $\hat{\Sigma}$ (in our notation) with return time that goes to infinity at in Figure 2. Lemma 5 gives C^1 control of the curve $\mathbf{P}_\epsilon(\sigma)$. Moser proves the existence of the points of intersection of $\mathbf{P}_\epsilon(\sigma)$ and σ with the argument on page 98 of [4]. In terms of our presentation, for $\epsilon = 0$, the points of intersection have to come back to themselves, so they are subharmonic periodic orbits. The length of the line segment σ for which $\mathbf{P}_\epsilon(\sigma)$ spirals around approaching $W^u_\epsilon(\gamma_\epsilon)$ and intersection σ is uniform for $|\epsilon| \leq \epsilon^*_{\text{sub}}$.

Thus for the Sitnikov, Moser has provided a replacement for the part of the above proof in which we used the hyperbolicity of the periodic orbit. The periods of the orbits that are obtained are determined by the length of the line segment σ which is uniform in ϵ. Therefore, his proof gives the uniformity of the subharmonics as claimed in Theorem 1.2(b).

4. **Comments on the case without symmetry.** For the general hyperbolic case without symmetry, to make the above argument apply it would be necessary to show that near a nondegenerate zero of the homoclinic Melnikov function there is a curve $\sigma_\epsilon \subset \hat{\Sigma}$ of uniform length of points each of which $\mathbf{P}_\epsilon$ brings back with the same level of energy H. The argument given above then would show that $\mathbf{P}_\epsilon(\sigma_\epsilon) \cap \sigma_\epsilon$ would be a countable set of subharmonic periodic points. In our proof, the symmetry produced the set σ and points on the intersection $\mathbf{P}_\epsilon(\sigma) \cap \sigma$ automatically preserve the energy.

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