# THE SUBHARMONIC MELNIKOV METHOD

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We consider a Hamiltonian system with two degrees of freedom which completely decouples for $\epsilon = 0$. Therefore we assume there is a Hamiltonian

$$H^\epsilon(\mathbf{z}_1, \mathbf{z}_2) = F_1(\mathbf{z}_1) + F_2(\mathbf{z}_2) + \epsilon H_1(\mathbf{z}_1, \mathbf{z}_2) + O(\epsilon^2).$$

We also write $H_0(\mathbf{z}_1, \mathbf{z}_2) = F_1(\mathbf{z}_1) + F_2(\mathbf{z}_2)$. This induces a system of Hamiltonian differential equations which we write as

$$(*) \qquad \begin{pmatrix} \dot{\mathbf{z}}_1 \\ \dot{\mathbf{z}}_2 \end{pmatrix} = X^\epsilon(\mathbf{z}_1, \mathbf{z}_2) = X(\mathbf{z}_1, \mathbf{z}_2) + \epsilon Y(\mathbf{z}_1, \mathbf{z}_2) + O(\epsilon^2).$$

For $\epsilon = 0$, we write $X(\mathbf{z}_1, \mathbf{z}_2) = \begin{pmatrix} X_1(\mathbf{z}_1) \\ X_2(\mathbf{z}_2) \end{pmatrix}$ so the differential equations divide into two subequations, $\dot{\mathbf{z}}_1 = X_1(\mathbf{z}_1)$ and $\dot{\mathbf{z}}_2 = X_2(\mathbf{z}_2)$. We write $\phi(t, \mathbf{z}, \epsilon)$ for the flow for $X^\epsilon$ with $\phi(0, \mathbf{z}, \epsilon) = \mathbf{z}$ and $\frac{d}{dt}\phi(t, \mathbf{z}, \epsilon) = X^\epsilon \circ \phi(t, \mathbf{z}, \epsilon)$.

We assume further that the unperturbed system for $\epsilon = 0$ has the following properties:

A1 The subsystem $\dot{\mathbf{z}}_1 = X_1(\mathbf{z}_1)$ has a fixed point at $\mathbf{z}_{1,0}$ and a homoclinic orbit to $\mathbf{z}_{1,0}$ enclosing a region filled with periodic orbits of the unperturbed system. Let $\Sigma_1$ be a single transversal for the subsystem $\dot{\mathbf{z}}_1 = X_1(\mathbf{z}_1)$. For the subsystem $\dot{\mathbf{z}}_1 = X_1(\mathbf{z}_1)$, let $\hat{\mathbf{z}}_1^\mu(t)$ be the family of periodic orbits with $\hat{\mathbf{z}}_1^\mu(0) \in \Sigma_1$.

A2 The subsystem $\dot{\mathbf{z}}_2 = X_2(\mathbf{z}_2)$ has a region filled with periodic orbits. Let $\hat{\mathbf{z}}_2^\lambda(t, t_0)$ be this family of periodic orbits $F_2$ with two parameters: the parameter $\lambda$ varies the orbit and $\hat{\mathbf{z}}_2^\lambda(t, t_0) = \hat{\mathbf{z}}_2^\lambda(t + t_0, 0)$ so $t_0$ indicates the phase of the periodic orbit.

Let

$$\hat{\mathbf{z}}(t; \mu, \lambda, t_0) = (\hat{\mathbf{z}}_1^\mu(t), \hat{\mathbf{z}}_2^\lambda(t, t_0)).$$

Because the system decouples for $\epsilon = 0$, the function $\hat{\mathbf{z}}(0; \mu, \lambda, t_0)$ makes $(\mu, \lambda, t_0)$ into coordinates on $\Sigma_1$, i.e., $\hat{\mathbf{z}}(0; \mu, \lambda, t_0)$ uniquely determines a point on $\Sigma_1$.

In the situation for the usual Melnikov function for a homoclinic orbit, the transverse homoclinic orbit of the purturbation must lie on the perturbed stable and unstable manifolds; this determines one dimension in the phase space. In the current situation, we must determine the points which persist in a resonance between the two oscillation. To define a function which give this resonance, we first define the return times of the Poincaré maps.

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Using the transversal $\Sigma_1$, let $\tau_1(\mathbf{z}_1)$ be the return times of the Poincaré map for the periodic orbits for the subsystem $\dot{\mathbf{z}}_1 = X_1(\mathbf{z}_1)$. For $\epsilon \neq 0$, $\Sigma_1$ can be considered a transversal for the full system by letting the $\mathbf{z}_2$ be arbitrary. Using this extended transversal $\Sigma_1$, let $\tau_1(\mathbf{z}_1, \mathbf{z}_2, \epsilon)$ be the return times of the Poincaré map for $\Sigma_1$ for the full system $\dot{\mathbf{z}} = X^\epsilon(\mathbf{z})$. Notice that for $\epsilon = 0$, $\tau_1$ can be defined for all points, but that $\tau_1$ is only defined on $\Sigma_1$ for nonzero $\epsilon$.

For the subsystem $\dot{\mathbf{z}}_2 = X_2(\mathbf{z}_2)$, take a family of transversals, $\Sigma_{2,\mathbf{z}_2}$: if the system is written in action angle coordinates $(I, \theta)$ then the transversals can be taken to be $\theta = \theta_0$. Again $\Sigma_{2,\mathbf{z}_2}$ can be considered a family of transversals in the full system. Let $\tau_2(\mathbf{z}_1, \mathbf{z}_2, \epsilon)$ be the return times of the Poincaré map for $\Sigma_{2,\mathbf{z}_2}$ with initial conditions $(\mathbf{z}_1, \mathbf{z}_2)$.

We are interested in finding periodic orbits which are close to unperturbed orbits which are in resonance between the two subsystems. Let $(m, n)$ be a fixed set of positive integers. For points on $\Sigma_1$, we consider the difference of the multiples of the periods,

$$\Psi^{m,n}(\mathbf{z}_1, \mathbf{z}_2, \epsilon) = n\tau_1(\mathbf{z}_1, \mathbf{z}_2, \epsilon) - m\tau_2(\mathbf{z}_1, \mathbf{z}_2, \epsilon).$$

Zeroes of $\Psi^{m,n}$ correspond to points where the return times are in a $m$ to $n$ resonance. The orbits which are in resonace are isolated; to this end we need to assume that the periods vary within the energy surface, i.e., that the gradent of $H_0$ and $\Psi^{m,n}$ are independent.

A3 Let $(\mu_0, \lambda_0) = (\mu(m, n, h_0), \lambda(m, n, h_0))$ be parameters of the two families of orbits for $\epsilon = 0$ for which $\Psi^{m,n}(\hat{\mathbf{z}}(0; \mu_0, \lambda_0, t_0)) = 0$ and $H_0(\hat{\mathbf{z}}(0; \mu_0, \lambda_0, t_0)) = h_0$. (Note that both $\Psi^{m,n}$ and $H_0$ are independent of $t_0$.) Assume that the gradients of $H_0$ and $\Psi^{m,n}$, as functions on $\Sigma_1$ are independent at $\hat{\mathbf{z}}(0; \mu_0, \lambda_0, t_0)$. In terms of the parameterization $\hat{\mathbf{z}}(0; \mu, \lambda, t_0)$, this assumption can be expressed as follows. Since the orbits $\hat{\mathbf{z}}_2^\lambda(t, t_0)$ are periodic, the gradient of $F_2$ is nonzero and we can solve for $\lambda = \lambda(\mu, h_0)$ such that

$$H_0 \circ \hat{\mathbf{z}}(0; \mu, \lambda(\mu, h_0), t_0) = F_1 \circ \hat{\mathbf{z}}_1^\mu(0)) + F_2 \circ \hat{\mathbf{z}}_2^\lambda(0, t_0) = h_0.$$

Using this notation, the independence of the gradients can be expressed by

$$\frac{\partial}{\partial\mu}\Psi^{m,n}(\hat{\mathbf{z}}(0; \mu, \lambda(\mu, h_0), t_0), 0)\Big|_{\mu_0} \neq 0.$$

Notice that $\Sigma_1 \cap \{t_0 = 0\}$ is two dimensional, and $H_0^{-1}(h_0) \cap \Sigma_1 \cap \{t_0 = 0\}$ is one dimensional with a point determined by the parameter $\mu$. Since

$$\frac{\partial}{\partial\mu}\Psi^{m,n}(\hat{\mathbf{z}}(0; \mu, \lambda(\mu, h_0), 0), 0)\Big|_{\mu_0} \neq 0,$$

the set of solutions $\mu(m, n, h_0)$ and $\lambda(m, n, h_0) = \lambda(\mu(m, n, h_0), h_0)$ is a set of locally isolated parameters.

Next we want to define the subharmonic Melnikov function which can be used to determine a value of $t_0$ for which the resonance persists for $\epsilon \neq 0$. For $\mathbf{z} \in \Sigma_1$, let

$$\hat{G}^{m,n}(\mathbf{z}, \epsilon) = F_1 \circ \phi(n\tau_1(\mathbf{z}, \epsilon), \mathbf{z}, \epsilon) - F_1(\mathbf{z}).$$

(It works just as well to use time $m\tau_2(\mathbf{z},\epsilon)$ instead of $n\tau_1(\mathbf{z},\epsilon)$.) For $\epsilon = 0$, $\hat{G}^{m,n}(\mathbf{z}, 0) \equiv 0$, so we can write

$$\hat{G}^{m,n}(\mathbf{z},\epsilon) = \epsilon G^{m,n}(\mathbf{z},\epsilon).$$

Using these choice of parameters, define the *subharmonic Melnikov function* as follows:

$$M^{m,n}(t_0,h_0) = \frac{\partial}{\partial\epsilon}\hat{G}^{m,n}(\hat{\mathbf{z}}(0;\mu_0,\lambda_0,t_0),\epsilon)\Big|_{\epsilon=0} = G^{m,n}(\hat{\mathbf{z}}(0;\mu_0,\lambda_0,t_0),0).$$

where $\mu_0 = \mu(m,n,h_0)$ and $\lambda_0 = \lambda(m,n,h_0)$ are the parameters of the two families of orbits for $\epsilon = 0$ given above.

There are two main theorem. The first theorem relates nondegenerate zeroes of $M^{m,n}(t_0,h_0)$ with periodic orbits. The second theorem relates $M^{m,n}(t_0,h_0)$ to an integral.

**Theorem 1.** *Make assumptions (A1-A3) and define $M^{m,n}$ as above. Assume there is a $(t_0^*,h_0^*)$ such that $M^{m,n}(t_0^*,h_0^*) = 0$ and $\dfrac{\partial M^{m,n}}{\partial t_0}(t_0^*,h_0^*) \neq 0$. Then there exists a positive number $\epsilon_0(m,n,h_0^*) > 0$ such that for $|\epsilon| \leq \epsilon_0(m,n,h_0^*)$, the perturbed system $(*)$ has a periodic orbit with period approximately*

$$n\tau_1(\hat{\mathbf{z}}_1^{\mu(m,n,h_0^*)}(0),0) = m\tau_2(\hat{\mathbf{z}}_2^{\lambda(m,n,h_0^*)}(0,t_0^*),0)$$

*with initial conditions close to $\hat{\mathbf{z}}(0;\mu(m,n,h_0^*),\lambda(m,n,h_0^*),t_0^*)$ and which stays within $O(\epsilon)$ of*

$$\hat{\mathbf{z}}(t;\mu(m,n,h_0^*),\lambda(m,n,h_0^*),t_0^*)$$

*for all time.*

**Theorem 2.** *Using these choice of parameters, the subharmonic Melnikov function is given by the following integral:*

$$M^{m,n}(t_0,h_0) = \int_0^{n\tau_1} (DF_1\cdot Y)_{\hat{\mathbf{z}}(t;\mu_0,\lambda_0,t_0^*)}\,dt$$

*where $\mu_0 = \mu(m,n,h_0)$, $\lambda_0 = \lambda(m,n,h_0)$, and $n\tau_1 = n\tau_1(\hat{\mathbf{z}}_1^{\mu_0}(0))$.*

*Proof of Theorem 1.* We consider the function $\Theta^{m,n} : \Sigma_1 \times [-1,1] \to \mathbb{R}^3$ given by

$$\Theta^{m,n}(\mathbf{z},\epsilon) = \begin{pmatrix} H^\epsilon(\mathbf{z}) \\ \Psi^{m,n}(\mathbf{z},\epsilon) \\ G^{m,n}(\mathbf{z},\epsilon)\end{pmatrix}.$$

The function $\hat{\mathbf{z}}(0;\mu,\lambda,t_0) = (\hat{\mathbf{z}}_1^\mu(0),\hat{\mathbf{z}}_2^\lambda(0,t_0))$, induces the coordinates $(\mu,\lambda,t_0)$ on $\Sigma_1$. Let

$$(\mu_0^*,\lambda_0^*) = (\mu(m,n,h_0^*),\lambda(m,n,h_0^*)).$$

Since $G^{m,n}(\hat{\mathbf{z}}(0;\mu_0^*,\lambda_0^*,t_0),0) = M^{m,n}(t_0,h_0^*)$, the derivative of $\Theta^{m,n}(\hat{\mathbf{z}}(0;\mu,\lambda,t_0),\epsilon)$ with respect to $(\mu,\lambda,t_0)$ for $t=0$, $\epsilon=0$, $\mu=\mu_0^*$, $\lambda=\lambda_0^*$, and $t_0=t_0^*$ is

$$\begin{aligned} &D_{(\mu,\lambda,t_0)}(\Theta^{m,n}\circ\hat{\mathbf{z}})_{((0;\mu_0^*,\lambda_0^*,t_0^*,0)} \\ &\quad = \begin{pmatrix} \dfrac{\partial(H_0\circ\hat{\mathbf{z}})}{\partial\mu} & \dfrac{\partial(H_0\circ\hat{\mathbf{z}})}{\partial\lambda} & 0 \\ \dfrac{\partial(\Psi^{m,n}\circ\hat{\mathbf{z}})}{\partial\mu} & \dfrac{\partial(\Psi^{m,n}\circ\hat{\mathbf{z}})}{\partial\lambda} & 0 \\ * & * & \dfrac{\partial M^{m,n}}{\partial t_0} \end{pmatrix}_{(0;\mu^*,\lambda^*,t_0^*,0),h_0^*}. \end{aligned}$$

Because (i) the gradients of $H_0$ and $\Psi^{m,n}$ are independent and (ii)

$$\frac{\partial M^{m,n}}{\partial t_0}(t_0^*,h_0^*)\neq 0,$$

this matrix is nonsingular. By the Implicit Function Theorem, the variables $\mu$, $\lambda$, and $t_0$ can be solved for in terms of $\epsilon$, $(\mu(\epsilon),\lambda(\epsilon),t_0(\epsilon))$, such that

$$\Theta^{m,n}((\mu(\epsilon),\lambda(\epsilon),t_0(\epsilon),\epsilon)\equiv\begin{pmatrix} h_0\\ 0\\ 0\end{pmatrix}.$$

Let $\mathbf{z}^*(\epsilon)=\hat{\mathbf{z}}(0;\mu(\epsilon),\lambda(\epsilon),t_0(\epsilon))$. Because $\Psi^{m,n}(\mathbf{z}^*(\epsilon),\epsilon)=0$, $n\tau_1(\mathbf{z}^*(\epsilon),\epsilon)=m\tau_2(\mathbf{z}^*(\epsilon),\epsilon)$, and $\mathbf{z}^*(\epsilon)$ and $\phi(n\tau_1(\mathbf{z}^*(\epsilon),\epsilon),\mathbf{z}^*(\epsilon),\epsilon)$ are both points in the same $\Sigma_{2,\mathbf{z}^*(\epsilon)}$. (This means they have the same $t_0$ coordinate.) Also, they are both points in $\Sigma_1$ by the definition of $\tau_1$. Because $\hat{G}^{m,n}(\mathbf{z}^*(\epsilon),\epsilon)=\epsilon G^{m,n}(\mathbf{z}^*(\epsilon),\epsilon)=0$, $F_1$ has the same value at $\mathbf{z}^*(\epsilon)$ and $\phi(n\tau_1(\mathbf{z}^*(\epsilon),\epsilon),\mathbf{z}^*(\epsilon),\epsilon)$. Because $H^\epsilon$ is preserved, its value is the same at these two points. Because $H_0$ and $F_1$ are independent, for small $\epsilon$, $H^\epsilon$ and $F_1$ are independent on $\Sigma_1\cap\Sigma_{2,\mathbf{z}^*(\epsilon)}$ and it follows that

$$\phi(n\tau_1(\mathbf{z}^*(\epsilon),\epsilon),\mathbf{z}^*(\epsilon),\epsilon)=\mathbf{z}^*(\epsilon)$$

and $\mathbf{z}^*(\epsilon)$ is a periodic orbit for $X^\epsilon$ with period $n\tau_1(\mathbf{z}^*(\epsilon),\epsilon)$. (Notice that $F_1$ is independent from $H^\epsilon$ on a fixed $\Sigma_1\cap\Sigma_{2,\mathbf{z}^*(\epsilon)}$ which is two dimensional.) Because all the variables are determined by the Implicit Function Theorem, the period, the initial conditions, and the orbits are within $O(\epsilon)$ for all time. □

*Proof of Theorem 2.* We consider

$$\begin{aligned} \frac{d}{dt}\frac{\partial}{\partial\epsilon}F_1\circ\phi(t,\mathbf{z},\epsilon)\Big|_{\epsilon=0} &= \frac{\partial}{\partial\epsilon}\frac{d}{dt}F_1\circ\phi(t,\mathbf{z},\epsilon)\Big|_{\epsilon=0} \\ &= \frac{\partial}{\partial\epsilon}[DF_1\cdot(X+\epsilon Y)]_{\phi(t,\mathbf{z},\epsilon)}\Big|_{\epsilon=0} \\ &= [DF_1\cdot Y]_{\phi(t,\mathbf{z},0)}+\frac{\partial}{\partial\epsilon}[DF_1\cdot X]_{\phi(t,\mathbf{z},\epsilon)}\Big|_{\epsilon=0} \\ &= [DF_1\cdot Y]_{\phi(t,\mathbf{z},0)}. \end{aligned}$$

The last equality follows because $[DF_1 \cdot X]_{\mathbf{z}} = 0$ at all points $\mathbf{z}$. Letting

$$\mathbf{z}_{m,n} = \hat{\mathbf{z}}(0; \mu(m,n,h_0), \lambda(\mu(m,n,h_0)), t_0)$$

and integrating from $t = 0$ to $n\tau_1 = n\tau_1(\mathbf{z}_{m,n}, 0)$ we get

$$\begin{aligned} M^{m,n}(t_0,h_0) &= \frac{\partial}{\partial\epsilon} F_1 \circ \phi(n\tau_1, \mathbf{z}_{m,n}, \epsilon)\big|_{\epsilon=0} - \frac{\partial}{\partial\epsilon} F_1(\mathbf{z}_{m,n})|_{\epsilon=0} \\ &= \int_0^{n\tau_1} \frac{d}{dt}\frac{\partial}{\partial\epsilon} F_1 \circ \phi(t, \mathbf{z}_{m,n}, \epsilon)\big|_{\epsilon=0}\, dt \\ &= \int_0^{n\tau_1} [DF_1 \cdot Y]_{\phi(t,\mathbf{z}_{m,n},0)}\, dt \\ &= \int_0^{n\tau_1} [DF_1 \cdot Y]_{\hat{\mathbf{z}}(t;\mu(m,n,h_0),\lambda(m,n,h_0),t_0)}\, dt. \end{aligned}$$

This gives the desired result. □

## References

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