THE MODULI SPACE OF QUADRATIC RATIONAL MAPS

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Abstract. Let M_2 be the space of quadratic rational maps $f : \mathbf{P}^1 \to \mathbf{P}^1$, modulo the action by conjugation of the group of Möbius transformations. In this paper a compactification X of M_2 is defined, as a modification of Milnor's $\overline{M}_2 \simeq \mathbf{CP}^2$, by choosing representatives of a conjugacy class $[f] \in M_2$ such that the measure of maximal entropy of f has conformal barycenter at the origin in $\mathbf{R}^3$, and taking the closure in the space of probability measures. It is shown that X is the smallest compactification of M_2 such that all iterate maps $[f] \mapsto [f^n] \in M_{2^n}$ extend continuously to $X \to \overline{M}_{2^n}$, where $\overline{M}_d$ is the natural compactification of M_d coming from geometric invariant theory.

1. Introduction

For each $d \geq 2$, let $M_d = \mathrm{Rat}_d/\mathrm{PSL}_2\mathbf{C}$ denote the space of degree d rational maps $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$, modulo the action by conjugation of the group of Möbius transformations. The moduli space is a complex orbifold of dimension $2d-2$. Iteration defines a sequence of regular maps

$$\Phi_n : M_d \to M_{d^n},$$

given by $[f] \mapsto [f^n]$, where $[f]$ denotes the conjugacy class of $f \in \mathrm{Rat}_d$.

The aim of this paper is to define a compactification of the moduli space which is natural from the point of view of dynamics, and in particular, one on which iteration is well-defined. Two approaches to this end are presented here, one using results in geometric invariant theory and one in terms of measures of maximal entropy. In degree $d = 2$, the two approaches are shown to be equivalent.

A formal solution. In [Si], Silverman studied a compactification $\overline{M}_d$ of M_d, for each $d \geq 2$, by computing the stability criteria for the conjugation action of $\mathrm{SL}_2\mathbf{C}$ on $\mathrm{Rat}_d \hookrightarrow \mathbf{P}^{2d+1}$, according to Mumford's geometric invariant theory (GIT). The boundary points can be identified with rational maps of degree $< d$. The iterate maps Φ_n, however, do not define regular maps from $\overline{M}_d$ to $\overline{M}_{d^n}$ for any $d \geq 2$ and $n \geq 2$ (see §10).

It is possible to define a compactification of the moduli space M_d on which iteration is well-defined, by resolving the indeterminacy of each rational iterate map $\overline{M}_d \dashrightarrow \overline{M}_{d^n}$ and passing to an inverse limit. Namely, we can define Γ_n to be the closure of M_d as it sits inside the finite product $\overline{M}_d \times \overline{M}_{d^2} \times \cdots \times \overline{M}_{d^n}$ via the first n iterate maps $(\mathrm{Id}, \Phi_2, \ldots, \Phi_n)$. There is a natural projection from Γ_{n+1} to Γ_n for

Date: January 5, 2006.

Research partially supported by an NSF Postdoctoral Fellowship.

2000 Mathematics Subject Classification: Primary 37F45, Secondary 14L24, 57M50.

every n, so we may take the inverse limit over n,

$$\hat{M}_d = \varprojlim \Gamma_n.$$

The moduli space M_d is a dense open subset of $\hat{M}_d$, where a conjugacy class $[f] \in M_d$ is identified with the sequence $([f],[f^2],[f^3],\ldots)$ in $\hat{M}_d$. The iterate map $\Phi_n : M_d \to M_{d^n}$ extends continuously to $\hat{M}_d \to \hat{M}_{d^n}$, by sending the sequence $([f],[f^2],[f^3],\ldots)$ to the sequence $([f^n],[f^{2n}],[f^{3n}],\ldots)$. It remains to understand the structure of this space and if there exists a concrete model for $\hat{M}_d$.

Maximal measures and the barycenter. Given a rational map $f \in \mathrm{Rat}_d$, let μ_f denote the unique probability measure on $\hat{\mathbf{C}}$ of maximal entropy [Ly],[FLM],[Ma1]. The support of μ_f is equal to the Julia set of f, and the measure is invariant under iteration, $\mu_{f^n} = \mu_f$ for all $n \geq 1$. The conformal barycenter of μ_f is its hyperbolic center of mass, where the the unit ball in $\mathbf{R}^3$ is chosen as a model for $\mathbf{H}^3$, and the unit sphere S^2 is identified with the Riemann sphere $\hat{\mathbf{C}}$ via stereographic projection [DE] (see §8).

For each conjugacy class $[f] \in M_d$, we can choose a barycentered representative $f \in \mathrm{Rat}_d$, one such that the conformal barycenter of μ_f is at the origin in $\mathbf{R}^3$. The representative is unique up to conjugation by the compact group of rotations $\mathrm{SO}(3) \subset \mathrm{PSL}_2\mathbf{C}$. If $M^1_{bc}(\hat{\mathbf{C}})$ denotes the space of barycentered probability measures on $\hat{\mathbf{C}}$ (with the weak-$*$ topology), then we obtain a continuous map

$$B : M_d \to M^1_{bc}(\hat{\mathbf{C}})/\mathrm{SO}(3),$$

defined by $[f] \mapsto [\mu_f]$ where SO(3) acts by push-forward on the space of measures (see §9). Let $\overline{M^1_{bc}(\hat{\mathbf{C}})}$ denote the closure of $M^1_{bc}(\hat{\mathbf{C}})$ in the space of all probability measures, and consider the closure of the graph of B in the product of this space of measures with the GIT compactification of M_d,

$$\overline{\overline{M}}_d = \overline{\mathrm{Graph}(B)} \subset \overline{M}_d \times \overline{M^1_{bc}(\hat{\mathbf{C}})}/\mathrm{SO}(3).$$

Then $\overline{\overline{M}}_d$ is a compact topological space with M_d as a dense open subset.

Quadratic rational maps. In degree $d = 2$, Milnor showed that the moduli space, $M_2 = \mathrm{Rat}_2/\mathrm{PSL}_2\mathbf{C}$, is an orbifold with underlying complex manifold isomorphic to $\mathbf{C}^2$ [Mi, Lemma 3.1]. In the compactification $\overline{M}_2 \simeq \mathbf{P}^2$, Milnor naturally identified the boundary of M_2 with the family of conjugacy classes of degree 1 rational maps and one degree 0 map. In fact, Milnor's $\overline{M}_2$ is isomorphic to the geometric invariant theory compactification in degree 2 [Si, Thm 1.5].

The main theorem of this paper shows that the compactification $\overline{\overline{M}}_2$ of M_2 by barycentered measures is an explicit model for the formal construction of $\hat{M}_2$ which resolves the iterate maps.

Theorem 1.1. *The compactifications $\hat{M}_2$ and $\overline{\overline{M}}_2 = \overline{\mathrm{Graph}(B)}$ of M_2 are canonically homeomorphic.*

In other words, there is a homeomorphism $\hat{M}_2 \to \overline{\overline{M}}_2$ which restricts to the identity on the moduli space M_2. It is not true in general that the compactifications $\hat{M}_d$ and

$\overline{\overline{M}}_d$ are homeomorphic for every $d \geq 2$. Examples are given in Section 10. However, for $d = 2$ we have the following corollary.

Corollary 1.2. *The iterate maps $\Phi_n : M_2 \to M_{2^n}$ extend continuously to $\overline{\overline{M}}_2 \to \overline{\overline{M}}_{2^n}$ for every $n \geq 1$.*

Proof. Let $\{[f_k]\}_{k=0}^{\infty}$ be a sequence in M_2 such that $[f_k] \to p \in \partial M_2 \subset \overline{\overline{M}}_2$ as $k \to \infty$. By the definition of $\overline{\overline{M}}_2$, there exist representatives $f_k \in \text{Rat}_2$ with barycentered measures of maximal entropy μ_{f_k} which converge weakly to a probability measure ν as $k \to \infty$. By Theorem 1.1, the point p is identified with a unique point in $\partial M_2 \subset \hat{M}_2$, and therefore it has well defined iterates $p^n \in \overline{M}_{2^n}$ for all $n \geq 1$. By the continuity of $\Phi_n : \hat{M}_2 \to \overline{M}_{2^n}$ and the iterate-invariance of the measures, $\mu_{f_k^n} = \mu_{f_k}$, the sequence of iterates $\Phi_n([f_k])$ must converge in $\overline{\overline{M}}_{2^n}$ to the point (p^n, ν). □

The structure of $\overline{\overline{M}}_2 = \hat{M}_2$. As an inverse limit construction, the space $\hat{M}_2$ could have very complicated structure. In fact, the boundary of M_2 in this space can be understood concretely. The first result says $\hat{M}_2$ can not be embedded into any finite dimensional projective space.

Theorem 1.3. *No finite sequence of blow-ups of $\overline{M}_2 \simeq \mathbf{P}^2$ is enough to resolve all of the rational iterate maps $\Phi_n : \overline{M}_2 \dashrightarrow \overline{M}_{2^n}$ simultaneously.*

On the other hand, the space Γ_n, which is the closure of the graph of $(\Phi_2, \Phi_3, \ldots, \Phi_n)$ in $\overline{M}_2 \times \overline{M}_4 \times \cdots \times \overline{M}_{2^n}$ has a fairly simple structure, described in Section 7. In particular, there are no ideal points in the inverse limit space $\hat{M}_2 = \lim \Gamma_n$:

Theorem 1.4. *Every element of $\hat{M}_2 \subset \prod_1^\infty \overline{M}_{2^n}$ is uniquely determined by finitely many entries.*

Topologically, the boundary of M_2 in $\hat{M}_2$ is obtained from the boundary of M_2 in $\overline{M}_2 \simeq \mathbf{P}^2$ by successively attaching 2-spheres at a countable collection of points in $\overline{M}_2$, as in Figure 1.

Outline of the paper. The notation is fixed in Section 2, and a summary of results from [De] is provided. The geometric invariant theory compactification $\overline{M}_d$ is defined in Section 3. In Section 4, we observe that the iterate map $M_d \to M_{d^n}$ is proper. Sections 5 and 6 are devoted to the study of iteration in degree 2, and we give the proof of Theorem 1.3. Section 7 contains a study of the structure of $\hat{M}_2$ and the proof of Theorem 1.4. The space of barycentered measures $M^1_{bc}(\hat{\mathbf{C}})/\text{SO}(3)$ is studied in Section 8, and the proof of Theorem 1.1 is contained in Section 9. Section 10 is devoted to a study of the iterate map $M_d \to M_{d^n}$ in general degrees $d \geq 2$. Some concluding remarks about the definitions of $\hat{M}_d$ and $\overline{\overline{M}}_d$ are given in Section 11.

Acknowledgements. The analysis of quadratic rational maps used here appeared first in [Mi] and, in greater detail, in [Ep] where Epstein studied the structure of hyperbolic components in M_2 and gave the first examples of discontinuity of the iterate map at the boundary. In fact, as shown in Sections 5 and 6, Epstein's examples are the only examples which demonstrate discontinuity in degree 2. The

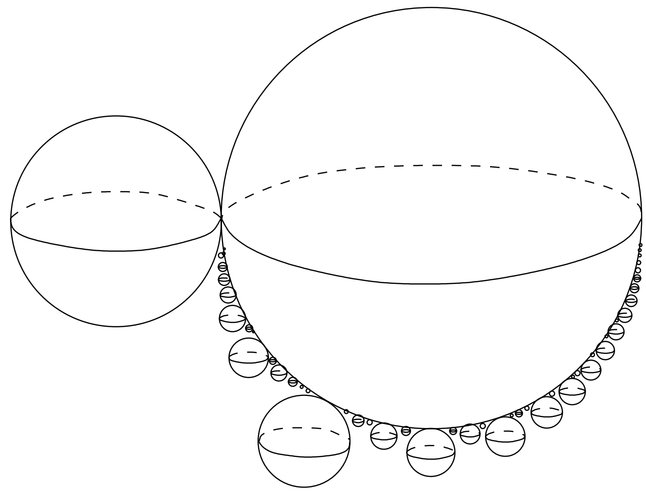

FIGURE 1. Boundary of M_2 in $\overline{\overline{M}}_2 = \hat{M}_2$.

geometric invariant theory approach relies on the results in [Si]. The indeterminacy of the iterate maps at the boundary of M_d in $\overline{M}_d$ is analogous to the indeterminacy at the boundary of $\mathbf{C}^2$ in $\mathbf{P}^2$ when iterating a complex Hénon mapping [HPV]. I am grateful to A. Epstein, J. Harris, J. Hubbard, C. McMullen, J. Milnor, and R. Pandharipande for helping me formulate the results in this paper.

2. $\overline{\mathrm{Rat}}_d$ AND THE PROBABILITY MEASURES AT THE BOUNDARY

In this section, we fix notation and terminology. We state some facts about the iterate map $\mathrm{Rat}_d \to \mathrm{Rat}_{d^n}$ and the measures of maximal entropy from [De].

The compactification $\overline{\mathrm{Rat}}_d$. Let Rat_d denote the space of holomorphic maps $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ of degree d with the topology of uniform convergence. For each d, Rat_d is naturally identified with the complement of a hypersurface in

$$\overline{\mathrm{Rat}}_d = \mathbf{P}\,\mathrm{H}^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(d,1)),$$

by sending $f \in \mathrm{Rat}_d$ to the section which vanishes along the graph of f. The space of rational maps Rat_d is therefore a smooth, affine variety, and we obtain an isomorphism $\overline{\mathrm{Rat}}_d \simeq \mathbf{P}^{2d+1}$. Alternatively, each point $f \in \mathrm{Rat}_d$ defines a map $f : \mathbf{P}^1 \to \mathbf{P}^1$, and, in homogeneous coordinates $(z : w)$ on $\mathbf{P}^1$, it determines a pair of degree d homogeneous polynomials,

$$f(z : w) = (P(z,w) : Q(z,w)).$$

The pair (P,Q) is unique up to scale, and the space of all such pairs is $\mathbf{P}^{2d+1}$, parametrized by the coefficients of (P,Q). In particular,

$$\mathrm{Rat}_d \simeq \mathbf{P}^{2d+1} \setminus V(\mathrm{Res}),$$

where $V(\mathrm{Res}) = \{(P,Q) : \mathrm{Res}(P,Q) = 0\}$ is the resultant hypersurface. In $\overline{\mathrm{Rat}}_d$, the hypersurface $V(\mathrm{Res})$ corresponds to the collection of all sections with reducible zero locus.

Given a pair (P, Q), the zeroes of the homogeneous polynomial $H = \gcd(P, Q)$, as points in $\mathbf{P}^1$, will be called the **holes** of the associated $f \in \overline{\mathrm{Rat}}_d$ and the multiplicity of a zero the **depth** of the hole. Each $f \in \overline{\mathrm{Rat}}_d$ determines a holomorphic map

$$\hat{f} = (P/H : Q/H) : \mathbf{P}^1 \to \mathbf{P}^1$$

of degree $\leq d$. We will often write

$$f = (P : Q) = H_f \hat{f}$$

where $H_f = \gcd(P, Q)$. The holes are removable singularities for $\hat{f}$, and they correspond to vertical components in the graph of f (see Figure 2).

Coordinates on $\mathbf{P}^1$. A point $(z : w) \in \mathbf{P}^1$ will regularly be identified with $z/w \in \hat{\mathbf{C}}$. Any distances on $\hat{\mathbf{C}}$ will be measured in the spherical metric. A ball of radius r about a point $p \in \hat{\mathbf{C}}$ will be denoted $B(p, r)$.

The measure of maximal entropy. Fix $d \geq 2$. Given a rational map $f \in \mathrm{Rat}_d$ and any smooth probability measure ω on $\hat{\mathbf{C}}$, the weak limit

$$\mu_f = \lim_{n \to \infty} \frac{1}{d^n} (f^n)^* \omega$$

exists and is independent of ω. The probability measure μ_f is f-invariant ($f_* \mu_f = \mu_f$) and of maximal entropy ($\log d$) with support equal to the Julia set of f. Furthermore, for any $A \in \mathrm{Aut}(\hat{\mathbf{C}})$, we have $\mu_{AfA^{-1}} = A_* \mu_f$ [Ly], [FLM], [Ma1]. Mañé showed that the function $f \mapsto \mu_f$ is continuous from Rat_d to the space of probability measures with the weak-$*$ topology [Ma2].

Iteration on $\overline{\mathrm{Rat}}_d$. We provide here a summary of relevant definitions and statements from [De]. The main object of study in [De] is the relation between the iterate maps, $f \mapsto f^n$, extended to $\overline{\mathrm{Rat}}_d$ and the extension of the map of maximal measures, $f \mapsto \mu_f$, to $\overline{\mathrm{Rat}}_d$.

The **indeterminacy locus** $I(d) \subset \overline{\mathrm{Rat}}_d$ is the collection of $f = H_f \hat{f}$ with $\deg \hat{f} = 0$ and such that the constant value of $\hat{f}$ is a hole of f. It has codimension $d+1$ in $\overline{\mathrm{Rat}}_d$. See Figure 2.

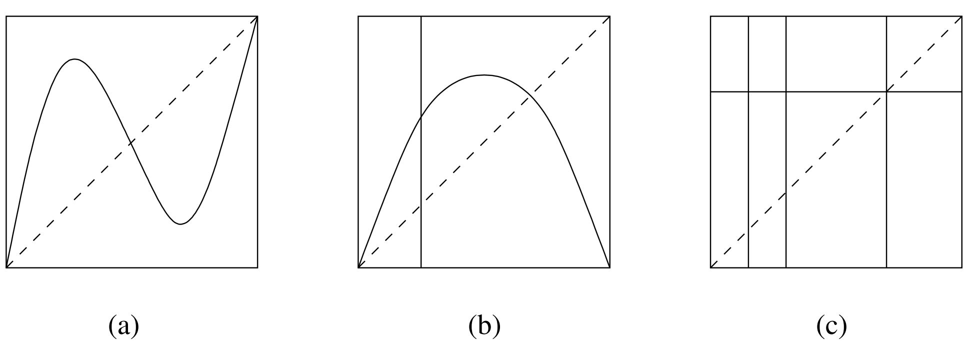

FIGURE 2. Graphs in $\mathbf{P}^1 \times \mathbf{P}^1$ of $f \in \overline{\mathrm{Rat}}_3$: (a) $f \in \mathrm{Rat}_3$, (b) $f = H_f \hat{f} \in \partial \mathrm{Rat}_3$ with $\deg \hat{f} = 2$, (c) $f \in I(3)$.

Theorem 2.1. [De, Thm 0.2] *The indeterminacy locus of the iterate map* $\overline{\mathrm{Rat}}_d \dashrightarrow \overline{\mathrm{Rat}}_{d^n}$ *is* $I(d)$ *for all* $n \geq 2$.

Consequently, any element $f = H_f \hat{f} \in \overline{\mathrm{Rat}}_d \setminus I(d)$ has well-defined forward iterates f^n for all $n \geq 2$. A direct computation yields the formula ([De, Lemma 2.2]),

$$f^n = \left(\prod_{k=0}^{n-1} (H_f \circ \hat{f}^k)^{d^{n-k-1}} \right) \hat{f}^n.$$

Atomic probablity measures. For each $f \in \partial\mathrm{Rat}_d$, we define a purely atomic probability measure μ_f which plays the role of the measure of maximal entropy. For $f = H_f \hat{f} \in \partial\mathrm{Rat}_d$ such that $\deg \hat{f} > 0$, the measure μ_f is given by the following triple sum,

$$\mu_f = \sum_{n=0}^{\infty} \frac{1}{d^{n+1}} \sum_{\{H_f(h)=0\}} \sum_{\{\hat{f}^n(z)=h\}} \delta_z,$$

where the middle sum is over all holes of f, the inner sum is over all preimages of those holes, and the outer sum is over all iterates of $\hat{f}$, all counted with mulitiplicity. Because the number of holes is $d - \deg \hat{f}$, it is easy to check that μ_f has total mass one. For $\deg \hat{f} = 0$, we define the probability measure by

$$\mu_f = \frac{1}{d} \sum_{\{H_f(h)=0\}} \delta_h.$$

One can define pull-back of measures by any $f \notin I(d)$, and the measure μ_f is the unique probability measure satisfying $f^*\mu_f = d \cdot \mu_f$ [De, Prop 3.2].

Theorem 2.2. [De, Thm 0.1(a)] *Given any sequence* $\{f_k\}$ *in* Rat_d *converging to* $f \in \partial\mathrm{Rat}_d \setminus I(d)$ *in* $\overline{\mathrm{Rat}}_d$*, the measures of maximal entropy* μ_{f_k} *converge weakly to* μ_f.

We will use the following three lemmas throughout this text. The first two follow directly from a comparison of the formula for an iterate of $f \in \partial\mathrm{Rat}_d$ with the definition of μ_f. For $f = H_f \hat{f} \in \overline{\mathrm{Rat}}_d$, let $d_h(f)$ denote the depth of $h \in \mathbf{P}^1$ as a hole of f and let $m_h(\hat{f})$ be the multiplicity of $z = h$ as a solution to $\hat{f}(z) = \hat{f}(h)$. Note that $m_h(\hat{f}) = 1$ if and only if h is not a critical point of $\hat{f}$. By convention, $m_h(\varphi) = 0$ for all h if φ is constant, and the 0-th iterate φ^0 is the identity map.

Lemma 2.3. [De, Lemma 1.3] *For each* $f = H_f \hat{f} \in \overline{\mathrm{Rat}}_d$ *and* $z \in \hat{\mathbf{C}}$*, we have*

$$\mu_f(\{z\}) = \frac{1}{d} \sum_{n=0}^{\infty} \frac{m_z(\hat{f}^n) d_{\hat{f}^n(z)}(f)}{d^n}.$$

Lemma 2.4. [De, Cor 2.3] *For each* $f \in \partial\mathrm{Rat}_d - \setminus I(d)$*, the depths of the holes of the iterates of* f *are given by*

$$d_z(f^n) = d^{n-1} \cdot d_z(f) + \sum_{k=1}^{n-1} d^{n-1-k} m_z(\hat{f}^k) d_{\hat{f}^k(z)}(f).$$

Therefore, the sequence $\{d_z(f^n)/d^n : n \geq 1\}$ *is non-decreasing, and*

$$\mu_f(\{z\}) = \lim_{n\to\infty} \frac{d_z(f^n)}{d^n}.$$

Lemma 2.5. [De, Lemmas 4.1, 4.2] *Suppose* $\{f_k\}$ *is a sequence in* Rat_d *converging to* $f = H_f \hat{f} = (P : Q)$ *in* $\overline{\mathrm{Rat}}_d$*.*

(i) *The sequence of rational maps* f_k *converges to* $\hat{f}$ *locally uniformly on the complement of the holes of* f *in* $\hat{\mathbf{C}}$*, and*
(ii) *if* f *has a hole at* h *of depth* d_h *and neither* P *nor* Q *is* $\equiv 0$*, then any neighborhood of* h *contains at least* d_h *zeroes and poles of* f_k *(counted with multiplicity) for all sufficiently large* k*.*

We also need some more general results on the structure of the composition map. Recall the notation from Lemma 2.3.

Lemma 2.6. *The composition map*

$$\mathcal{C}_{d,e} : \overline{\mathrm{Rat}}_d \times \overline{\mathrm{Rat}}_e \dashrightarrow \overline{\mathrm{Rat}}_{de},$$

which sends a pair (f, g) *to the composition* $f \circ g$*, is continuous away from*

$$I(d, e) = \{(f, g) = (H_f \hat{f}, H_g \hat{g}) : \hat{g} \equiv c \text{ and } H_f(c) = 0\}.$$

Furthermore, for each $(f, g) \in \overline{\mathrm{Rat}}_d \times \overline{\mathrm{Rat}}_e$ *such that* $\deg \hat{g} > 0$*,*

$$d_z(f \circ g) = d \cdot d_z(g) + m_z(\hat{g}) \cdot d_{\hat{g}(z)}(f).$$

Proof. In the coordinates on $\overline{\mathrm{Rat}}_d$ and $\overline{\mathrm{Rat}}_e$ given by the coefficients of f and g, the composition map is defined by polynomial functions, so it suffices to show that $\mathcal{C}_{d,e}(f, g)$ is well-defined for each pair $(f, g) \notin I(d, e)$. Write $f = (H_f P_f : H_f Q_f) = H_f \hat{f} \in \overline{\mathrm{Rat}}_d$ and $g = (H_g P_g : H_g P_g) = H_g \hat{g} \in \overline{\mathrm{Rat}}_e$. Let $d' = \deg \hat{f}$. The composition $f \circ g$ can be computed directly by

$$\begin{aligned}
\mathcal{C}_{d,e}(f, g) &= (H_f(H_g P_g, H_g Q_g) P_f(H_g P_g, H_g Q_g) : \\
&\qquad H_f(H_g P_g, H_g Q_g) Q_f(Q_g P_g, H_g Q_g)) \\
&= (H_g)^{d-d'} H_f(P_g, Q_g)(H_g)^{d'} (P_f(P_g, Q_g) : Q_f(P_g, Q_g)) \\
&= (H_g)^d H_f(P_g, Q_g)\, \hat{f} \circ \hat{g}.
\end{aligned}$$

By hypothesis, neither $H_f(P_g, Q_g)$ nor $(H_g)^d$ vanishes identically and $\hat{f} \circ \hat{g}$ is a well-defined rational map.

The formula for the depth of the holes of the composition $f \circ g$ follows directly from the formula for $\mathcal{C}_{d,e}(f, g)$ above. □

3. The GIT stability conditions

The action of $\mathrm{SL}_2\mathbf{C}$ by conjugation on Rat_d extends to $\overline{\mathrm{Rat}}_d = \mathbf{P}\,\mathrm{H}^0(\mathbf{P}^1 \times \mathbf{P}^1, O(d, 1))$ by the diagonal action on $\mathbf{P}^1 \times \mathbf{P}^1$. In this section we describe the stability conditions for this action according to geometric invariant theory (GIT), computed in [Si]. (See also [MFK].) We relate this notion of stability to the atomic probability measures μ_f for $f \in \partial\mathrm{Rat}_d$, defined in §2.

Silverman showed that the moduli space $M_d = \mathrm{Rat}_d/\mathrm{PSL}_2\mathbf{C}$ exists as a geometric quotient scheme which is affine, integral, connected, and of finite type over $\mathbf{Z}$. Furthermore, $\overline{M}_d$ is a geometric quotient for d even and a categorical quotient for d odd, and it is proper over $\mathbf{Z}$ [Si, Thm 2.1]. His computations led to the following proposition:

Proposition 3.1. [Si, Prop 2.2] *A point $f \in \overline{\mathrm{Rat}}_d$ is stable (respectively, semistable) for the conjugation action of* $\mathrm{SL}_2\mathbf{C}$ *if and only if there are no elements in the conjugacy class of f of the form*

$$(a_0 z^d + a_1 z^{d-1} w + \cdots + a_d w^d : b_0 z^d + b_1 z^{d-1} w + \cdots + b_d w^d)$$

with $a_i = 0$ for all $i < (d-1)/2$ (respectively, $i \leq (d-1)/2$) and $b_j = 0$ for all $j < (d+1)/2$ (respectively, $j \leq (d+1)/2$).

Denote the set of stable points by $\mathrm{Rat}_d^s \subset \overline{\mathrm{Rat}}_d$ and the semistable points by Rat_d^{ss}, and note that $\mathrm{Rat}_d^s = \mathrm{Rat}_d^{ss}$ if and only if d is even. Therefore, the compact GIT quotients are defined by $\overline{M}_d = \mathrm{Rat}_d^s/\mathrm{PSL}_2\mathbf{C}$ for d even and $\overline{M}_d = \mathrm{Rat}_d^{ss}//\mathrm{PSL}_2\mathbf{C}$ for d odd. Roughly speaking, Rat_d^s is the largest open $\mathrm{PSL}_2\mathbf{C}$-invariant subset of $\overline{\mathrm{Rat}}_d$ in which all $\mathrm{PSL}_2\mathbf{C}$-orbits are closed, and so the quotient space $\mathrm{Rat}_d^s/\mathrm{PSL}_2\mathbf{C}$ is Hausdorff. When d is odd, elements of Rat_d^{ss} represent the same point in $\overline{M}_d$ if the closures of their orbits intersect in Rat_d^{ss}.

The stable and semistable points. The following is a reformulation of Proposition 3.1 in the language of this paper. Let $f = H_f \hat{f}$ and $g = H_g \hat{g}$ be two elements in $\overline{\mathrm{Rat}}_d$. Then f and g are in the same $\mathrm{PSL}_2\mathbf{C}$-orbit if and only if there exists $A \in \mathrm{PSL}_2\mathbf{C}$ such that $\hat{g} = A\hat{f}A^{-1}$ and the holes of g are the image under A of the holes of f (and of corresponding depths). For even $d \geq 2$, a point $f = H_f \hat{f} \in \overline{\mathrm{Rat}}_d$ is stable (or semistable) if

(i) the depth of each hole is $\leq d/2$, and
(ii) if the depth of $h \in \mathbf{P}^1$ is $d/2$ then $\hat{f}(h) \neq h$.

For odd $d \geq 3$, a point $f \in \overline{\mathrm{Rat}}_d$ is stable if

(i) the depth of each hole is $\leq (d-1)/2$, and
(ii) if the depth of $h \in \mathbf{P}^1$ is $(d-1)/2$ then $\hat{f}(h) \neq h$,

and $f \in \overline{\mathrm{Rat}}_d$ is semistable if

(i) the depth of each hole is $\leq (d+1)/2$, and
(ii) if the depth of $h \in \mathbf{P}^1$ is $(d+1)/2$ then $\hat{f}(h) \neq h$.

Instability of stability. The property of stability is not generally preserved by iteration. For example, consider the point $h = (zw : z^2) \in \overline{\mathrm{Rat}}_2$ which is stable (h has one hole at $z = 0$ and $\hat{h}(z) = 1/z$). This point h does not lie in the indeterminacy locus $I(2)$ and therefore has a well-defined second iterate, namely,

$$h^2 = (z^3 w : z^2 w^2) \in \overline{\mathrm{Rat}}_4.$$

The second iterate h^2 coincides with the identity map away from a hole of depth 2 at $z = 0$ and a hole of depth 1 at $z = \infty$. Consequently, h^2 is *not* stable.

As another example, note that any degenerate polynomial is either unstable itself or will eventually be unstable after iteration. That is, for any $0 < k \le d$, consider

$$p = (w^k Q(z,w) : w^d)$$

where Q is a homogeneous polynomial of degree $d-k$ such that $Q(1,0) \neq 0$, so that $\hat{p}(z)$ is the polynomial $Q(z,1)$ and p has a hole of depth k at $z = \infty$. The iterates of p are of the form,

$$p^n = (w^{kd^{n-1}+k(d-k)d^{n-2}+\cdots+k(d-k)^{n-1}} Q^n(z,w) : w^{d^n}),$$

and p^n is unstable for all n such that $\deg Q^n = (d-k)^n$ is less than or equal to $d^n/2$.

For the examples just mentioned, we can compute the associated probability measures using Lemma 2.3,

$$\mu_h = \frac{2}{3}\delta_0 + \frac{1}{3}\delta_\infty$$

and $\mu_p = \delta_\infty$. The following propositions show that we can read from the measures that some iterate of h and p will be unstable.

Proposition 3.2. *Suppose d is even and $f \notin I(d)$. Then $f^n \in \overline{\mathrm{Rat}}_{d^n}$ is stable for all $n \ge 1$ if and only if $\mu_f(\{z\}) \le 1/2$ for all $z \in \mathbf{P}^1$.*

Proof. Let $d_z(f^n)$ denote the depth of z as a hole of f^n. Write $f = H_f \hat{f}$ and note that $\widehat{f^n} = \hat{f}^n$ (see §2). From Lemma 2.4, the hypothesis on μ_f implies that $d_z(f^n) \le d^n/2$ for all z and all $n \ge 1$. Suppose for some n and z we have $d_z(f^n) = d^n/2$ and $\hat{f}^n(z) = z$. The depth of z as a hole of the composition $f^{2n} = f^n \circ f^n$ must satisfy $d_z(f^{2n}) \ge d^n(d^n/2) + 1 > d^{2n}/2$ by Lemma 2.6, since z is also one of the preimages of the hole at z for f^n, providing a contradiction.

Conversely, suppose that f^n is stable for all n. Then, in particular, $d_z(f^n) \le d^n/2$ for all z and so again by Lemma 2.4, $\mu_f(\{z\}) = \lim_{n\to\infty} d_z(f^n)/d^n$ cannot exceed $1/2$. □

Proposition 3.3. *Suppose d is odd and $f \notin I(d)$. Then $f^n \in \overline{\mathrm{Rat}}_{d^n}$ is semi-stable for all $n \ge 1$ if and only if $\mu_f(\{z\}) \le 1/2$ for all $z \in \mathbf{P}^1$. Furthermore, if $\mu_f(\{z\}) < 1/2$ for all $z \in \mathbf{P}^1$, then f^n is stable for all $n \ge 1$.*

Proof. Let $d_z(f^n)$ denote the depth of z as a hole of f^n, and write $f = H_f \hat{f}$. By Lemma 2.4, $\mu_f(\{z\}) \le 1/2$ implies that $d_z(f^n) \le (d^n - 1)/2$ for all n and z. This gives the second statement immediately: f^n is semi-stable for all n. Conversely, if f^n is semi-stable for all n, then $d_z(f^n) \le (d^n+1)/2$ for all n. But in the limit, this implies that $\mu_f(\{z\}) \le 1/2$.

To prove the final statement, note again that $\mu_f(\{z\}) < 1/2$ implies that $d_z(f^n) \le (d^n - 1)/2$ for all n. Suppose that for some n, $d_z(f^n) = (d^n-1)/2$ and $\hat{f}^n(z) = z$. Then, by Lemma 2.3 applied to f^n,

$$\mu_{f^n}(\{z\}) \ge \frac{1}{d^n}\sum_{l=0}^{\infty} \frac{(d^n-1)/2}{d^{nl}} = \frac{1}{2},$$

which is a contradiction since $\mu_{f^n} = \mu_f$. □

Note that the converse to the second statement of Proposition 3.3 is false. There exist odd degrees d and $f \in \overline{\mathrm{Rat}}_d$ for which f^n is stable for all $n \ge 1$ but with

$\mu_f(\{z\}) = 1/2$ for some point $z \in \mathbf{P}^1$. Consider, for example, $f = (z^4w : zw^4) \in \overline{\mathrm{Rat}}_5$ which has holes at 0 and ∞, each of depth 1, and $\hat{f}(z) = z^3$. Using Lemmas 2.3 and 2.4, it is straightforward to compute that $\mu_f(0) = \mu_f(\infty) = 1/2$ while the depth of each of the two holes for f^n is $(5^n - 3^n)/2 < (5^n - 1)/2$.

4. Properness of the iterate map on M_d

The iterate map $\mathrm{Rat}_d \to \mathrm{Rat}_{d^n}$, given by $f \mapsto f^n$, is a regular map between smooth, affine varieties. It is $\mathrm{PSL}_2\mathbf{C}$-equivariant since $(AfA^{-1})^n = Af^nA^{-1}$ for all Möbius transformations A and all $n \geq 1$. Therefore it descends to a regular map on the affine moduli spaces,

$$\Phi_n : M_d \to M_{d^n}.$$

Iteration on Rat_d is proper (the preimage of any compact set in Rat_{d^n} is compact) if and only if the degree d is at least 2 [De, Cor 0.3]. This implies the following.

Proposition 4.1. *The iterate map $\Phi_n : M_d \to M_{d^n}$ is proper for every $d \geq 2$ and $n \geq 1$.*

Proof. If $\{[f_k]\}_{k=0}^{\infty}$ is an unbounded sequence in M_d, then every sequence of representatives $f_k \in \mathrm{Rat}_d$ is unbounded. By properness of iteration on Rat_d, every sequence of iterates $\{(f_k)^n\}_{k=0}^{\infty}$ is unbounded in Rat_{d^n} [De, Cor 3]. Consequently, $\Phi_n([f_k]) = [(f_k)^n]$ is unbounded in M_{d^n}. □

Since M_d is an open dense subset of the GIT compactification $\overline{M}_d$, the map Φ_n defines a rational map on the closures,

$$\Phi_n : \overline{M}_d \dashrightarrow \overline{M}_{d^n}.$$

In Section 10, we prove that Φ_n is not regular on $\overline{M}_d$ for any $d \geq 2$ and $n \geq 2$ (Theorem 10.1). We aim to study the indeterminacy of this rational map in the boundary of M_d. The following lemma begins to address the relationship between the indeterminacy of Φ_n on $\overline{M}_d$, the stability conditions in $\overline{\mathrm{Rat}}_d$, and the indeterminacy locus $I(d) \subset \partial\mathrm{Rat}_d$ for the general degree $d \geq 2$.

Lemma 4.2. *Suppose $f \in \overline{\mathrm{Rat}}_d$ satisfies $f \notin I(d)$ and f^n is stable for some $n > 1$. Then*

(i) *f is stable, and*
(ii) *the iterate map Φ_n is continuous at $[f] \in \overline{M}_d$.*

Proof. Write $f = H_f\hat{f}$. For $z \in \mathbf{P}^1$, let $d_z(f^n)$ denote the depth of z as a hole of f^n. The sequence $\{d_z(f^n)/d^n : n \geq 1\}$ is non-decreasing by Lemma 2.4.

Suppose first that the degree d is even. Stability of f^n implies that $d_z(f^n) \leq d^n/2$ for all z, and therefore $d_z(f^k) \leq d^k/2$ for all $k \leq n$ and all z. If $d_z(f^n) = d^n/2$, then $\hat{f^n}(z) \neq z$, and therefore $\hat{f}(z) \neq z$ so we see that f is stable. It therefore determines a well-defined point $[f]$ in $\overline{M}_d$.

If d is odd, stability of f^n implies that $d_z(f^n) \leq (d^n - 1)/2$ for all z. Therefore $d_z(f)/d \leq (d^n - 1)/2d^n < 1/2$ and so $d_z(f) \leq (d - 1)/2$ for all z. Furthermore, if we have $d_z(f) = (d - 1)/2$ with $\hat{f}(z) = z$, then

$$d_z(f^n) \geq d^{n-1} \sum_0^{n-1} \frac{d-1}{2d^k} = \frac{d^n - 1}{2}$$

and $\hat{f}^n(z) = z$, which contradicts stability.

Recalling that stability in $\overline{\mathrm{Rat}}_d$ is an open condition, we see that if $[f_t]$ is a family in $\overline{M}_d$ converging to $[f]$, then there is a family of stable representatives f_t converging to f in $\overline{\mathrm{Rat}}_d$. Since $f \notin I(d)$, the iterates $(f_t)^n$ converge to f^n in $\overline{\mathrm{Rat}}_{d^n}$. Also, $(f_t)^n$ must be stable for all sufficiently small t, and therefore the iterates of $[f_t]$ converge in $\overline{M}_{d^n}$ to $[f^n]$. Therefore, Φ_n is continuous at $[f] \in \overline{M}_d$. □

Question. Is the converse true, in the sense that if $f \in I(d)$ is stable, then Φ_n is indeterminate at $[f]$? And if $f \notin I(d)$ is stable but f^n is not stable, then is Φ_n indeterminate at $[f]$?

The answer is yes in degree $d = 2$, as we shall see in the following sections. What makes degree 2 particularly easy for computation is the following observation.

Lemma 4.3. *The intersection of the semistable points with the indeterminacy locus,* $\mathrm{Rat}_d^{ss} \cap\ I(d) \subset \overline{\mathrm{Rat}}_d$*, is empty if and only if* $d = 2$.

Proof. The statement is immediate from the definition of $I(d)$ and Proposition 3.1. □

5. The moduli space $M_2 \simeq \mathbf{C}^2$

In this section, we collect some fundamental facts about the moduli space of quadratic rational maps, and we describe completely the indeterminacy locus of the iterate map

$$\Phi_n : \overline{M}_2 \dashrightarrow \overline{M}_{2^n}$$

which sends $[f]$ to $[f^n]$. We give the proof of Theorem 1.3.

The analysis in this section is based on the work of Milnor and Epstein [Mi], [Ep] and the isomorphism between Milnor's compactification $\overline{M}_2 \simeq \mathbf{P}^2$ and the geometric invariant theory compactification of M_2 [Si, Thm 1.5].

The compactification $\overline{M}_2 \simeq \mathbf{P}^2$. Every rational map $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ of degree 2 has three fixed points (solutions to $f(z) = z$), counted with multiplicity. The derivative of f evaluated at a fixed point is called the **multiplier** of the fixed point. The multipliers of the fixed points are the solutions to a unique monic polynomial,

$$x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3 = 0,$$

and the residue formula (applied to the form $dz/(z - f(z))$) implies the relation $\sigma_3 = \sigma_1 - 2$. As the multipliers are conjugacy invariant, the σ_i define functions on the moduli space $M_2 = \mathrm{Rat}_d/\mathrm{PSL}_2\mathbf{C}$. Milnor showed that the pair (σ_1, σ_2) naturally parametrizes the moduli space, defining an isomorphism $M_2 \simeq \mathbf{C}^2$ [Mi, Lemma 3.1]. Consequently, a sequence $\{[f_k]\}$ is unbounded in M_2 if and only if some fixed point multiplier of f_k tends to infinity.

The boundary of M_2 in the compactification $\overline{M}_2 \simeq \mathbf{P}^2$, arising naturally from Milnor's isomorphism, corresponds to unordered triples of fixed point multipliers of the form $\{a, 1/a, \infty\}$ for $a \in \hat{\mathbf{C}}$. These triples can be identified with the conjugacy classes of degree 1 and constant maps of the form $z \mapsto az + 1$. Indeed, under the identification of Milnor's $\overline{M}_2$ with the GIT compactification ([Si, Thm 1.5]), the line at infinity is parametrized in a two-to-one fashion by $a \mapsto [\Lambda_a] = [\Lambda_{1/a}]$ where $[\Lambda_a]$ is the class of the following points in $\overline{\mathrm{Rat}}_2$:

$$\Lambda_a(z:w) = \begin{cases} (az(z-w) : w(z-w)), \text{ for } a \in \hat{\mathbf{C}} - \{0,1,\infty\}, \\ ((z+w)(z-w) : w(z-w)), \text{ for } a = 1, \\ ((z+w)(z-w) : 0), \text{ for } a = \infty, \text{ and} \\ (0 : (z+w)(z-w)), \text{ for } a = 0. \end{cases} \tag{5.1}$$

That is, Λ_1 is the parabolic Möbius transformation $z \mapsto z+1$ with a hole at $z = 1$, Λ_∞ is the constant infinity map with holes at 1 and -1, Λ_0 is the constant 0 with holes at 1 and -1, and for each $a \neq 0, 1, \infty$, Λ_a is given by $z \mapsto az$ with hole at $z = 1$. Recall by Lemma 2.5 that any sequence in Rat_2 converging to Λ_a in $\overline{\mathrm{Rat}}_2$ will converge to the corresponding degree 0 or 1 map, locally uniformly on the complement of the holes of Λ_a.

In this section we prove,

Theorem 5.1. *For each $n \geq 2$, iteration defines a rational map*

$$\Phi_n : \overline{M}_2 \dashrightarrow \overline{M}_{2^n}$$

with indeterminacy locus given by

$$I(\Phi_n) = \{[\Lambda_a] \in \partial M_2 : a \neq 1 \text{ and } a^q = 1 \text{ for some } 1 < q \leq n\}.$$

In particular, we have $I(\Phi_2) \subset I(\Phi_3) \subset I(\Phi_4) \cdots$.

Proof of Theorem 1.3. From Theorem 5.1, we know that the indeterminacy set of $\Phi_n : \overline{M}_2 \dashrightarrow \overline{M}_{2^n}$ is strictly increasing with n. Therefore, no finite sequence of blow-ups over points in $I(\Phi_n)$ will suffice to resolve the indeterminacy of all iterate maps simultaneously. ☐

For $a \in \hat{\mathbf{C}}$, let $\Lambda_a \in \overline{\mathrm{Rat}}_2$ be defined by (5.1). Checking the stability conditions of §3, we see that each of the points Λ_a is stable, and these elements represent all of the stable conjugacy classes in $\overline{\mathrm{Rat}}_2 - \mathrm{Rat}_2$. Note also that $\Lambda_a \notin I(2)$ for every $a \in \hat{\mathbf{C}}$ and therefore all forward iterates $\Lambda_a^n \in \overline{\mathrm{Rat}}_{2^n}$ are well-defined by Theorem 2.1.

Lemma 5.2. *The iterates $\Lambda_a^n \in \overline{\mathrm{Rat}}_{2^n}$ are stable for all $n \geq 1$ if and only if $a \in \hat{\mathbf{C}}$ is not a primitive q-th root of unity for any $q \geq 2$. For any primitive q-th root of unity ζ, the iterate Λ_ζ^n is stable if and only if $n < q$.*

Proof. By the definition of the atomic probability measures μ_{Λ_a} given in Section 2, we have

$$\mu_{\Lambda_0} = \mu_{\Lambda_\infty} = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1},$$

$$\mu_{\Lambda_1} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k}\delta_{1-k},$$

and

$$\mu_{\Lambda_a} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k}\delta_{1/a^k}$$

for all $a \neq 0, 1, \infty$. We see immediately that $\mu_{\Lambda_a}(\{z\}) \leq 1/2$ for all $z \in \mathbf{P}^1$ if only if a is not a primitive q-th root of unity for any $q > 1$. By Proposition 3.2, all iterates of Λ_a must then be stable.

Now let ζ be a primitive q-th root of unity for some $q > 1$. Writing $\Lambda_\zeta = H_\zeta \hat{\Lambda}_\zeta$, note that $\hat{\Lambda}_\zeta^k(z) = \zeta^k z$, so that $\hat{\Lambda}_\zeta^q$ is the identity map. For each $n < q$ and $l = 0, 1, \ldots, q-1$, it is easy to compute from Lemma 2.4 that the depths of the holes of Λ_ζ^n are given by

$$d_{1/\zeta^l}(\Lambda_\zeta^n) = 2^{n-1-l} \leq 2^n/2,$$

and $1/\zeta^l$ is not fixed by $\hat{\Lambda}_\zeta^n$, so that $\Lambda_\zeta^n \in \overline{\mathrm{Rat}}_{2^n}$ is stable. On the other hand, we find that $d_1(\Lambda_\zeta^q) = 2^{q-1}$ with $\hat{\Lambda}_\zeta^q(1) = 1$ so that Λ_ζ^q is not stable. For each $n > q$, $d_1(\Lambda_\zeta^n) > 2^{q-1}$ so that Λ_ζ^n is not stable. □

Epstein's normal forms. We now follow [Ep]. Suppose $f \in \mathrm{Rat}_2$ has distinct fixed points at 0, ∞, and 1, with multipliers α, β, and $\gamma = (2-\alpha-\beta)/(1-\alpha\beta)$, respectively. Then f can be written

$$f_{\alpha,\beta}(z) = z\frac{(1-\alpha)z + \alpha(1-\beta)}{\beta(1-\alpha)z + (1-\beta)}. \tag{5.2}$$

If the two critical points of f are distinct from the fixed point at 1, then f is conjugate to

$$F_{\gamma,\delta}(z) = \frac{\gamma z}{z^2 + \delta z + 1} \tag{5.3}$$

for some $\delta \in \mathbf{C}$, where F has critical points at 1 and -1 and a fixed point of multiplier γ at 0. Interchanging the labelling of the critical points replaces δ with $-\delta$.

Fix $a \in \hat{\mathbf{C}} - \{0, 1, \infty\}$ and any continuous path $p : (0,1] \to M_2$ such that $p(t) \to [\Lambda_a]$ in $\overline{M}_2$ as $t \to 0$. For any representative of $p(t)$ in Rat_2, the fixed points are eventually distinct, so we can label the multipliers continuously so that $\alpha(t) \to a$, $\beta(t) \to 1/a$, and $\gamma(t) \to \infty$ as $t \to 0$. Choose a continuous path $\tilde{p} : (0,1] \to \mathrm{Rat}_2$ so that $[\tilde{p}] = p$ and $\tilde{p}(t)$ is normalized as in (5.2). Then $\tilde{p}(t) \to \Lambda_a$ in $\overline{\mathrm{Rat}}_2$ as $t \to 0$. If we also label the critical points of $\tilde{p}(t)$, then there is a unique Möbius transformation A_t which transforms $\tilde{p}(t)$ into normalization (5.3). Then A_t satisfies

$$A_t(z) = 1 + z\sqrt{\varepsilon(t)} + o(\sqrt{\varepsilon(t)}), \tag{5.4}$$

for an appropriate choice of the square root of $\varepsilon(t) := 1 - \alpha(t)\beta(t)$, locally uniformly for $z \in \mathbf{C}$ [Ep, §3]. See Figure 3.

Iteration on M_2. Fix $a \in \hat{\mathbf{C}} - \{0, 1, \infty\}$, and let $\{f_t \in \mathrm{Rat}_2 : t \in (0,1]\}$ be a continuous family such that $f_t \to \Lambda_a$ in $\overline{\mathrm{Rat}}_2$ as $t \to 0$. Recall the definition of the indeterminacy locus $I(2)$ in $\overline{\mathrm{Rat}}_2$ given in §2. By Theorem 2.1, $\Lambda_a \notin I(2)$ implies that the iterates f_t^n converge in $\overline{\mathrm{Rat}}_{2^n}$ to

$$\Lambda_a^n = \left(a^n z \prod_{i=0}^{n-1} (z - w/a^i)^{2^{n-1-i}} : w \prod_{i=0}^{n-1} (z - w/a^i)^{2^{n-1-i}} \right), \tag{5.5}$$

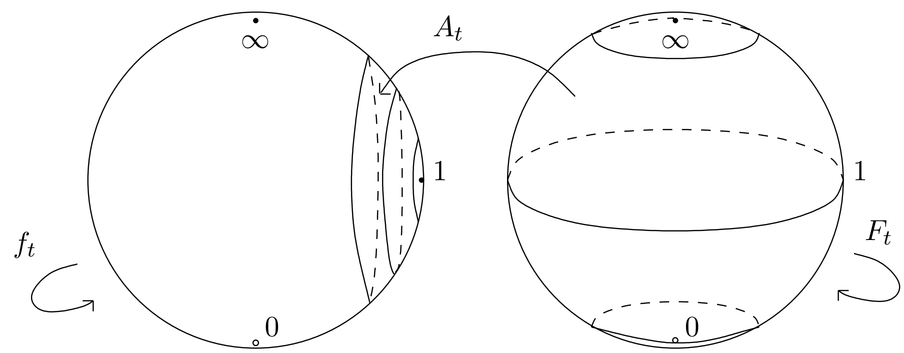

FIGURE 3. Normal forms (5.2) and (5.3) and transformation A_t for small t.

by [De, Lemma 2.2], and therefore

$$f_t^n(z) \to a^n z$$

as $t \to 0$, locally uniformly on $\hat{\mathbf{C}} - \{1, 1/a, 1/a^2, \ldots, 1/a^{n-1}\}$ by Lemma 2.5.

Suppose further that $a = \zeta$ is a primitive q-th root of unity for some $q > 1$. Then the family of q-th iterates $\{f_t^q\}$ converges to the identity function as $t \to 0$, locally uniformly on $\hat{\mathbf{C}} - \{1, \zeta, \ldots, \zeta^{q-1}\}$. However, Epstein showed that if we conjugate this family by a Möbius transformation satisfying (5.4), then the limit of the q-th iterate is a degree 2 map with a parabolic fixed point:

Proposition 5.3. [Ep, Prop 2] *Let ζ be a primitive q-th root of unity for $q > 1$, and suppose that $\{f_t : t \in (0,1]\} \subset \mathrm{Rat}_2$ is a continuous family normalized as in (5.2) such that $\alpha(t) \to \zeta$ and $\beta(t) \to 1/\zeta$ as $t \to 0$. Suppose also that as $t \to 0$,*

$$\frac{\alpha(t)^q - 1}{\sqrt{\varepsilon(t)}} \to \tau \in \hat{\mathbf{C}}$$

for some choice of the square root of $\varepsilon(t) = 1 - \alpha(t)\beta(t)$. Let $F_t = A_t^{-1} f_t A_t$ where A_t satisfies (5.4) for this same choice of $\sqrt{\varepsilon(t)}$. Then

$$F_t^q(z) \to \begin{cases} G_\tau(z) = z + \tau + \frac{1}{z} & \text{for } \tau \in \mathbf{C} \\ \infty & \text{for } \tau = \infty \end{cases}$$

as $t \to 0$, locally uniformly on $\mathbf{C}^*$.

The τ^2-value for holomorphic disks. In order to fully understand the iterate map,

$$\Phi_n : \overline{M}_2 \dashrightarrow \overline{M}_{2^n},$$

defined by $[f] \mapsto [f^n]$ on M_2, we will need to analyse in more detail the behavior of the iterates near the boundary point $\Lambda_\zeta \in \overline{\mathrm{Rat}}_2$, $\zeta^q = 1$.

Fix $q \geq 2$ and ζ a primitive q-th root of unity. Let $\Delta : \mathbf{D} \hookrightarrow \overline{M}_2$ be a holomorphic disk such that $\Delta(0) = [\Lambda_\zeta]$. For $q \geq 3$ (so that $\zeta \neq 1/\zeta$), we can holomorphically parameterize two of the fixed point multipliers $\alpha(t) \to \zeta$ and $\beta(t) \to 1/\zeta$ as $t \to 0$ and set $\varepsilon(t) = 1 - \alpha(t)\beta(t)$. We define the value $\tau^2 \in \hat{\mathbf{C}}$ for the disk Δ by

$$\tau^2(\Delta) = \lim_{t\to 0} \frac{(\alpha(t)^q - 1)^2}{\varepsilon(t)}$$

Set $\tau^2(\Delta) = \infty$ for any $\Delta \subset \partial M_2$, since $\varepsilon(t) \equiv 0$. If we interchange the labeling of α and β, the τ^2 value is unchanged: for any $q \geq 2$, the definition of $\varepsilon = 1 - \alpha\beta$ implies that

$$\tau^2(\Delta) = \lim_{t\to 0} \frac{(\alpha(t)^q - 1)^2}{\varepsilon(t)} = \lim_{t\to 0} \frac{(\beta(t)^q - 1)^2}{\varepsilon(t)}. \tag{5.6}$$

In the case of $q = 2$, we are not necessarily able to holomorphically label the fixed multipliers $\alpha(t)$ and $\beta(t)$, since both tend to $-1 = \zeta$ as $t \to 0$. Nevertheless, we can compute $\tau^2(\Delta)$ independent of any choice because of the equality in (5.6).

Theorem 5.4. *Let ζ be a primitive q-th root of unity for some $q \geq 2$, and let Δ and Δ' be two holomorphic disks in $\overline{M}_2$ such that $\Delta(0) = \Delta'(0) = [\Lambda_\zeta]$. Then for each $n \geq q$, we have*

$$\lim_{t\to 0} \Phi_n(\Delta(t)) = \lim_{t\to 0} \Phi_n(\Delta'(t))$$

in $\overline{M}_{2^n}$ if and only if $\tau^2(\Delta) = \tau^2(\Delta')$.

We will give the proof of Theorem 5.4 in the next section. For now, we complete the proof of Theorem 5.1, which describes the indeterminacy locus of the iterate map Φ_n.

Proof of Theorem 5.1. Let $I(\Phi_n)$ denote the indeterminacy locus of the iterate map $\Phi_n : \overline{M}_2 \dashrightarrow \overline{M}_{2^n}$, and consider the family $\Lambda_a \in \overline{\mathrm{Rat}}_2$ for $a \in \hat{\mathbf{C}}$ defined by (5.1). Since $\overline{M}_2 \simeq \mathbf{P}^2$ is smooth, it suffices to show that Φ_n is discontinuous at $[f] \in \overline{M}_2$ if and only if $[f] = [\Lambda_\zeta]$ for a primitive q-th root of unity ζ, with $1 < q \leq n$.

Suppose $a \in \hat{\mathbf{C}}$ is not a primitive q-th root of unity. By Lemma 5.2, $\Lambda_a^n \in \overline{\mathrm{Rat}}_{2^n}$ is stable for all $n \geq 1$, so that by Lemma 4.2, $[\Lambda_a] \notin I(\Phi_n)$ for all $n \geq 1$.

Fix $q \geq 2$ and let ζ be a primitive q-th root of unity. By Lemma 5.2, the iterate $\Lambda_\zeta^n \in \overline{\mathrm{Rat}}_{2^n}$ is stable if and only if $1 \leq n < q$, and so by Lemma 4.2, $[\Lambda_\zeta] \notin I(\Phi_n)$ for all $n < q$.

Now suppose $n \geq q$. By Theorem 5.4, it suffices to show there exist holomorphic disks Δ and Δ' in $\overline{M}_2$ such that $\Delta(0) = \Delta'(0) = [\Lambda_\zeta]$ and $\tau^2(\Delta) \neq \tau^2(\Delta')$. Define $f_t \in \mathrm{Rat}_2$ by normal form (5.2) with $\alpha(t) = \zeta + t$ and $\beta(t) = 1/\zeta + 2t$. Then it is easy to compute that $\tau^2(\Delta) = 0$ for $\Delta(t) = [f_t]$. On the other hand, $\tau^2(\Delta') = \infty$ for any disk $\Delta' \subset \partial M_2$. □

6. The iterate map in degree 2

Fix $q \geq 2$ and let ζ be a primitive q-th root of unity. Let $\Lambda_\zeta \in \overline{\mathrm{Rat}}_2$ be defined by equation (5.1). In this section, we prove Theorem 5.4, that the limiting values of the iterate map $\Phi_n : \overline{M}_2 \dashrightarrow \overline{M}_{2^n}$ on a holomorphic disk passing through $[\Lambda_\zeta]$ depend only on the τ^2-value of the disk. We treat the cases of $\tau^2 \in \mathbf{C}$ and $\tau^2 = \infty$ separately (Propositions 6.1 and 6.4).

Proposition 6.1. *Fix $q > 1$ and ζ a primitive q-th root of unity. Suppose that $\{f_t : t \in (0,1]\} \subset \mathrm{Rat}_2$ is a continuous family, normalized as in (5.2), such that $\alpha(t) \to \zeta$, $\beta(t) \to 1/\zeta$, and $(\alpha(t)^q - 1)/\sqrt{\varepsilon(t)} \to \tau \in \mathbf{C}$, for some choice of $\sqrt{\varepsilon(t)}$ as $t \to 0$. Let $A_t \in \mathrm{Aut}\,\hat{\mathbf{C}}$ satisfy (5.4) for this choice of $\sqrt{\varepsilon(t)}$. Then the the n-th iterate of $F_t = A_t^{-1} f_t A_t$ converges in $\overline{\mathrm{Rat}}_{2^n}$ as $t \to 0$ to the following:*

$$F_{q,\tau,n} = \begin{cases} (z^{2^{n-1}} w^{2^{n-1}} : 0) & \text{for } 1 \leq n < q, \\ (z^{2^{q-1}-1} w^{2^{q-1}-1}(z^2 + \tau z w + w^2) : z^{2^{q-1}} w^{2^{q-1}}) & \text{for } n = q, \\ F_{q,\tau,(n \bmod q)} \circ (F_{q,\tau,q})^{\lfloor n/q \rfloor} & \text{for } n > q, \end{cases}$$

where $F_{q,\tau,0}$ is the identity map.

Recall that by Proposition 5.3, the q-th iterate of F_t converges to $G_\tau(z) = z + \tau + 1/z$, locally uniformly on $\mathbf{C}^*$. By Proposition 6.1 (together with Lemma 2.5), $F_t^n(z) \to \infty$ as $t \to 0$ locally uniformly on the complement of a finite set in $\hat{\mathbf{C}}$ for all n which are not multiples of q, and for every $n = mq$, $F_t^{mq}(z) \to G_\tau^m(z)$ as $t \to 0$ locally uniformly on the complement of a finite set in $\hat{\mathbf{C}}$.

Lemma 6.2. [Ep, §4, (17)] *Fix $a \in \hat{\mathbf{C}} - \{0, 1, \infty\}$ and suppose that $\{f_t : t \in (0,1]\} \subset \mathrm{Rat}_2$ is a continuous family normalized as in (5.2) such that $f_t \to \Lambda_a$ in $\overline{\mathrm{Rat}}_2$ as $t \to 0$. Let $\varepsilon(t) = 1 - \alpha(t)\beta(t)$ and let $z : (0,1] \to \hat{\mathbf{C}}$ be a continuous path. Then*

$$\frac{f_t(z(t))}{z(t)} = \begin{cases} \alpha(t) + o(1) & \text{if } \varepsilon(t) = o(z(t) - 1) \\ \alpha(t) + o(\sqrt{\varepsilon(t)}) & \text{if } \sqrt{\varepsilon(t)} = o(z(t) - 1) \end{cases}$$

as $t \to 0$.

Proof of Proposition 6.1. For each fixed $z \in \mathbf{C}^*$, we have $\varepsilon(t) = o(A_t(z) - 1)$ and therefore, by Lemma 6.2,

$$f_t(A_t(z))/A_t(z) = \alpha(t) + o(1).$$

In particular, $f_t(A_t(z)) \to \zeta$, locally uniformly on $\mathbf{C}^*$. Since $\zeta \neq 1$, we obtain

$$F_t(z) = A_t^{-1} f_t A_t(z) \to \infty,$$

locally uniformly in $\mathbf{C}^*$. By induction we find that $\varepsilon(t) = o(f_t^{n-1}(A_t(z)) - 1)$ for each $1 \leq n \leq q$, so that

$$f_t^n(A_t(z)) = \alpha(t) f_t^{n-1}(A_t(z)) + o(1) \to \zeta^n \tag{6.7}$$

as $t \to 0$. Consequently, $F_t^n(z) \to \infty$ locally uniformly in $\mathbf{C}^*$ for each $n < q$.

It follows that for each $n < q$, the limit of F_t^n exists in $\overline{\mathrm{Rat}}_{2^n}$ as $t \to 0$: by Lemma 2.5(ii) it must be of the form $F_{q,\tau,n} = (z^k w^l : 0)$ for non-negative integers k and l such that $k + l = 2^n$ because the convergence of $F_t^n(z) \to \infty$ is uniform away from 0 and ∞. To determine k and l, it suffices (again by Lemma 2.5) to count the preimages of 0 by F_t^n near both 0 and ∞.

Fix $n < q$. The iterate $f_t^n \to \Lambda_\zeta^n$ in $\overline{\mathrm{Rat}}_{2^n}$ as $t \to 0$, where Λ_ζ^n is given by equation (5.5), and Lemma 2.5 implies that $f_t^n(z) \to \zeta^n z$ as $t \to 0$, locally uniformly on $\hat{\mathbf{C}} - \{1, 1/\zeta, \ldots, 1/\zeta^{n-1}\}$. Therefore, for all sufficiently small t, there is exactly one preimage by f_t^n of $z = 1$ very close to $z = 1/\zeta^n$. Fix small disks $\overline{D}_1 \subset D_2$ around $z = 1$. Counting the depths of the holes outside the disk D_1, Lemma 2.5 implies that there are exactly 2^{n-1} preimages of $z = 1$ by f_t^n in $\hat{\mathbf{C}} - D_1$ for sufficiently small

t. Thus, for $F_t^n = A_t^{-1} f_t^n A_t$, there are exactly 2^{n-1} preimages of 0 in $A_t^{-1}(\hat{\mathbf{C}} - D_1)$ for sufficiently small t. Therefore the depth of $z = \infty$ for $F_{q,\tau,n}$ is at least 2^{n-1}.

On the other hand, let D_0 be any disk around $z = 0$. As in the argument to show (6.7), $f_t^n(\overline{D}_1 - A_t(D_0)) \subset \zeta^n D_2$ for all sufficiently small t (and $n < q$). Therefore, $F_t^n \to \infty$ uniformly on $A_t^{-1}(D_1) - D_0$, so that F_t^n has at most 2^{n-1} preimages of 0 outside D_0 for small t. Therefore, the depth of $F_{q,\tau,n}$ at ∞ is exactly 2^{n-1}, and we can conclude that

$$F_{q,\tau,n} = (z^{2^{n-1}} w^{2^{n-1}} : 0).$$

Now suppose $n = q$. Since $F_t^q(z) \to G_\tau(z) = z + \tau + 1/z$ locally uniformly in $\mathbf{C}^*$ by Proposition 5.3, the limit of F_t^q must exist in $\overline{\mathrm{Rat}}_{2^q}$ and be of the form $F_{q,\tau,q} = (z^k w^l (z^2 + \tau z w + w^2) : z^{k+1} w^{l+1})$ for integers k and l with $k + l = 2^q - 2$. To compute k and l, we will count preimages of $z = \infty$ near ∞.

Without loss of generality, we may assume that A_t fixes ∞ for all t. As before, let D_0 be a disk centered at $z = 0$ and let D_1 be a small disk around $z = 1$. Because of the depths of the holes of Λ_ζ^q, we find that there are exactly 2^{q-1} preimages of ∞ by f_t^q in $\hat{\mathbf{C}} - D_1$ for all sufficiently small t ($2^{q-1} - 1$ of them accumulate on the q-th roots of unity and one preimage is at ∞). Also, from (6.7) when $n = q$, there are no preimages of ∞ by f_t^q in $D_1 - A_t(D_0)$. Therefore, F_t^q has exactly 2^{q-1} preimages of ∞ in $\hat{\mathbf{C}} - D_0$ for all sufficiently small t. Consequently, $F_{q,\tau,q}$ has a hole of depth exactly $2^{q-1} - 1$ at ∞, and therefore,

$$F_{q,\tau,q} = (z^{2^{q-1}-1} w^{2^{q-1}-1} (z^2 + \tau z w + w^2) : z^{2^{q-1}} w^{2^{q-1}}).$$

Finally we need to compute the limits of F_t^n in $\overline{\mathrm{Rat}}_{2^n}$ for every $n > q$. Write $n = k + mq$ for integers $0 \le k < q$ and $m > 0$. For $k = 0$ the desired form follows immediately from Theorem 2.1 since $F_{q,\tau,q} \notin I(2^q)$. For $k > 0$, the result follows from Lemma 2.6, since $F_t^n = F_t^k \circ F_t^{mq}$. □

Lemma 6.3. *For $q \ge 2$ and $\tau \in \mathbf{C}$, let $\{F_t\}$ be a family of rational maps as in Proposition 6.1. Then the measures μ_{F_t} converge weakly to $\mu_{F_{q,\tau,q}}$ as $t \to 0$. Furthermore, $\mu_{F_{q,\tau,q}}(\{z\}) < 1/2$ for all $z \in \hat{\mathbf{C}}$.*

Proof. The convergence of the measures follows immediately from Theorem 2.2 because $F_{q,\tau,q} \notin I(2^q)$. Let $\mu_\tau = \mu_{F_{q,\tau,q}}$ as defined in Section 2. By Lemma 2.3, we can compute the values of μ_τ,

$$\mu_\tau(\{\infty\}) = \mu_\tau(\{0\}) = \frac{2^{q-1} - 1}{2^q} \sum_{l=0}^{\infty} \frac{1}{2^{ql}} = \frac{2^{q-1} - 1}{2^q - 1} < \frac{1}{2}.$$

Then, for any point $p \in \mathbf{C}^*$, we have

$$\mu_\tau(\{p\}) \le 1 - \mu_\tau(\{0\}) - \mu_\tau(\{\infty\}) = \frac{1}{2^q - 1} < \frac{1}{2}.$$

□

Proposition 6.4. *Fix $q > 1$ and ζ a primitive q-th root of unity. Suppose that $\{f_t : t \in (0, 1]\} \subset \mathrm{Rat}_2$ is a continuous family of rational maps normalized as in (5.2), such that $\alpha(t) \to \zeta$, $\beta(t) \to 1/\zeta$, and $(\alpha(t)^q - 1)/\sqrt{\varepsilon(t)} \to \infty$ as $t \to 0$.*

Conjugating by $A_t(z) = 1 + z(\alpha(t)^q - 1) \in \operatorname{Aut}\hat{\mathbf{C}}$*, the iterates of* $P_t = A_t^{-1} f_t A_t$ *converge in* $\overline{\mathrm{Rat}}_{2^n}$ *as* $t \to 0$ *to the following:*

$$P_{q,n} = \begin{cases} (z^{2^{n-1}} w^{2^{n-1}} : 0) & \textit{for } 1 \le n < q, \\ (z^{2^{q-1}} w^{2^{q-1}-1}(z+w) : z^{2^{q-1}} w^{2^{q-1}}) & \textit{for } n = q, \\ P_{q,(n \bmod q)} \circ (P_{q,q})^{\lfloor n/q \rfloor}, & \textit{for } n > q, \end{cases}$$

where $P_{q,0}$ *is the identity map.*

In particular, Lemma 2.5 implies that $P_t^n(z) = A_t^{-1} f_t^n A_t(z) \to \infty$ as $t \to 0$, locally uniformly on the complement of a finite set in $\hat{\mathbf{C}}$, for all n which are not multiples of q, and $P_t^{mq}(z) \to z+m$ as $t \to 0$, locally uniformly on $\hat{\mathbf{C}} - \{0, -1, -2, \ldots, -m+1, \infty\}$ for every $m \ge 1$.

Proof. The proof is similar to the proof of Proposition 6.1. For fixed $z \in \mathbf{C}^*$, we have $A_t(z) - 1 = z(\alpha(t)^q - 1)$, and so $\sqrt{\varepsilon(t)} = o(A_t(z) - 1)$. By Lemma 6.2,

$$\frac{f_t(A_t(z))}{A_t(z)} = \alpha(t) + o(\sqrt{\varepsilon(t)}).$$

In particular, $f_t(A_t(z)) \to \zeta$, locally uniformly in $\mathbf{C}^*$. Thus $\sqrt{\varepsilon(t)} = o(f_t(A_t(z)) - 1)$ also and so

$$\frac{f_t^2(A_t(z))}{f_t(A_t(z))} = \alpha(t) + o(\sqrt{\varepsilon(t)}).$$

By induction, we have

$$\frac{f_t^q(A_t(z))}{A_t(z)} = \prod_1^q \frac{f_t^n(A_t(z))}{f_t^{n-1}(A_t(z))} = \alpha(t)^q + o(\sqrt{\varepsilon(t)}), \tag{6.8}$$

and we conclude that

$$P_t^q(z) = A_t^{-1} f_t^q A_t(z) = \frac{\alpha(t)^q - 1}{\alpha(t)^q - 1} + \frac{\alpha(t)^q(\alpha(t)^q - 1)z}{\alpha(t)^q - 1} + \frac{o(\sqrt{\varepsilon(t)})}{\alpha(t)^q - 1} \to z + 1$$

as $t \to 0$, and that $P_t^n(z) \to \infty$ for each $n < q$ as $t \to 0$, locally uniformly on $\mathbf{C}^*$.

It remains to determine the limit of the iterates of P_t in $\overline{\mathrm{Rat}}_{2^n}$. Fix $n < q$. The proof that $P_t^n \to P_{q,n}$ in $\overline{\mathrm{Rat}}_{2^n}$ is identical to the proof of Proposition 6.1 in the case $n < q$, and we omit it.

Let $n = q$. Since $P_t^q(z) \to z + 1$ locally uniformly on $\mathbf{C}^*$, Lemma 2.5 implies that the limit of P_t^q must exist in $\overline{\mathrm{Rat}}_{2^q}$ and be of the form $P_{q,q} = (z^k w^l (z+w) : z^k w^{l+1})$ for integers k and l such that $k + l = 2^q - 1$. As in the proof of Proposition 6.1, the convergence of $f_t^q \to \Lambda_\zeta^q \in \overline{\mathrm{Rat}}_{2^q}$ and the estimate on $f_t^q(A_t(z))$ in (6.8) imply that P_t^q will have exactly 2^{q-1} preimages of ∞ near ∞ for all sufficiently small t. Therefore, $l + 1 = 2^{q-1}$ and $P_{q,q}$ will have a hole of depth $2^{q-1} - 1$ at ∞.

For each $n > q$, the formula for $P_{q,n}$ follows from Lemma 2.6. □

Stability of the iterates. Recall the criteria for GIT stability of points in $\overline{\mathrm{Rat}}_{2^n}$ given in Section 3. For each $n \ge 2$, define $F_{q,\tau,n}$ and $P_{q,n}$ in $\overline{\mathrm{Rat}}_{2^n}$ as in Propositions 6.1 and 6.4. The following lemma shows that $F_{q,\tau,n}$ and $P_{q,n}$ define points in $\overline{M}_{2^n}$ if and only if $n \ge q$.

Lemma 6.5. *Let* ζ *be a primitive* q*-th root of unity for* $q \ge 2$ *and fix* $\tau \in \mathbf{C}$*. Then each of* $F_{q,\tau,n}$ *and* $P_{q,n} \in \overline{\mathrm{Rat}}_{2^n}$ *is stable if and only if* $n \ge q$*.*

Proof. For $n < q$, $F_{q,\tau,n} = P_{q,n} = (z^{2^{n-1}} w^{2^{n-1}} : 0)$ has a hole of depth $2^{n-1} = 2^n/2$ at ∞, and the associated rational map of lower degree is the constant ∞. Thus, $F_{q,\tau,n} = P_{q,n}$ is not stable.

Fix $\tau \in \mathbf{C}$. Note that $F_{q,\tau,q} \notin I(2^q)$, so it has well-defined forward iterates. By Lemma 6.3, the measure $\mu_\tau = \mu_{F_{q,\tau,q}}$ has no atoms of mass $\geq 1/2$, and therefore $F_{q,\tau,q}$ and all its forward iterates $F_{q,\tau,qm}$ are stable by Proposition 3.2.

Fix integers $0 < k < q$ and $m \geq 1$. Lemma 2.4 allows us to compute the depths of 0 and ∞ as holes of $F_{q,\tau,qm}$:

$$d_0(F_{q,\tau,qm}) = d_\infty(F_{q,\tau,qm}) = 2^{qm}\left(\frac{2^{q-1}-1}{2^q}\sum_{l=0}^{m-1}\frac{1}{2^{ql}}\right) = \frac{2^{q-1}-1}{2^q-1}\left(2^{qm}-1\right).$$

With Lemma 2.6 we can compute the depths of the holes of $F_{q,\tau,k+mq}$:

$$\begin{aligned} d_0(F_{q,\tau,k+mq}) = d_\infty(F_{q,\tau,k+mq}) &= 2^k\frac{2^{q-1}-1}{2^q-1}(2^{qm}-1) + 2^{k-1} \\ &= 2^{k+qm}\left(\frac{2^{q-1}-1}{2^q-1} + \frac{1}{2^{qm}}\left(\frac{1}{2} - \frac{2^{q-1}-1}{2^q-1}\right)\right) < \frac{2^{k+qm}}{2}, \end{aligned}$$

and therefore for any $z \neq 0, \infty$,

$$d_z(F_{q,\tau,k+mq}) \leq 2^{k+mq} - 2d_0(F_{q,\tau,k+mq}) \leq 2^{k+mq}/(2^q-1).$$

This implies that $F_{q,\tau,k+mq}$ is stable.

Now consider $P_{q,q} = (z^{2^{q-1}} w^{2^{q-1}-1}(z+w) : z^{2^{q-1}} w^{2^{q-1}})$. By the definition of the measure $\mu_{P_{q,q}}$,

$$\mu_{P_{q,q}} = \frac{1}{2^q}\sum_{k=0}^{\infty}\frac{2^{q-1}}{2^{qk}}\delta_{-k} + \frac{1}{2^q}\sum\frac{2^{q-1}-1}{2^{qk}}\delta_\infty = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^{qk}}\delta_{-k} + \frac{2^{q-1}-1}{2^q-1}\delta_\infty.$$

By Proposition 3.2, P_{mq} is stable for all $m \geq 1$.

Fix integers $0 < k < q$ and $m \geq 1$. Lemma 2.4 allows us to compute the depths of the holes of $P_{q,qm}$: $d_0(P_{q,qm}) = 2^{qm-1}$, $d_z(P_{q,qm}) \leq 2^{qm-2}$ for all $z \neq 0, \infty$, and

$$d_\infty(P_{q,qm}) = \frac{2^{qm}}{2^q}\sum_{l=0}^{m-1}\frac{2^{q-1}-1}{2^{ql}} = \frac{2^{q-1}-1}{2^q-1}(2^{qm}-1).$$

Therefore, by Lemma 2.6,

$$d_\infty(P_{q,k+qm}) = 2^k\frac{2^{q-1}-1}{2^q-1}(2^{qm}-1) + 2^{k-1} < 2^{k+qm}/2,$$

$d_0(P_{q,k+qm}) = 2^{k+qm}/2$, and $d_z(P_{q,k+qm}) \leq 2^{q+km}(1/4 + 1/2^{qm+1}) < 2^{k+qm}/2$ for all $z \neq 0, \infty$. Observing that $\hat{P}_{q,k+qm}(0) \neq 0$, we see that $P_{q,k+qm}$ is stable. □

A non-constant map from $\hat{\mathbf{C}}$ to the boundary of M_{2^n}. Recall the definitions of $F_{q,\tau,n}$ and $P_{q,n}$ from Propositions 6.1 and 6.4. The next two lemmas show that there are regular maps of degree 2 from $\mathbf{P}^1$ to the boundary of M_{2^n} in $\overline{M}_{2^n}$ parameterized by the families $F_{q,\tau,n}$ for each $q \leq n$.

Lemma 6.6. *Fix integers $n \geq q \geq 2$ and $\sigma, \tau \in \mathbf{C}$. The following are equivalent:*

(i) $[F_{q,\tau,n}] = [F_{q,\sigma,n}]$ *in* $\overline{M}_{2^n}$,

(ii) $\mu_{F_{q,\tau,q}} = A_*\mu_{F_{q,\sigma,q}}$ *for some* $A \in \operatorname{Aut}\hat{\mathbf{C}}$, *and*

(iii) $\sigma = \pm\tau$.

Furthermore, $[P_{q,n}] \neq [F_{q,\tau,n}]$ *for all* $\tau \in \mathbf{C}$.

Proof. Write $n = k + qm$ for integers $0 \leq k < q$ and $m > 0$, and let $\mu_\tau = \mu_{F_{q,\tau,q}}$. Define $A \in \operatorname{Aut}\hat{\mathbf{C}}$ by $A(z : w) = (-z : w)$, and note that $AF_{q,\tau,q}A^{-1} = F_{q,-\tau,q}$. Therefore,

$$AF_{q,\tau,qm}A^{-1} = A(F_{q,\tau,q})^m A^{-1} = F_{q,-\tau,qm},$$

so that $[F_{q,\tau,qm}] = [F_{q,-\tau,qm}]$ and $\mu_\tau = A_*\mu_{-\tau}$. For $k \geq 1$, note that $AF_{q,\tau,k}A^{-1} = (\pm z^{2^{k-1}} w^{2^{k-1}} : 0) = F_{q,\tau,k}$ and is independent of τ, so that

$$AF_{q,\tau,k+qm}A^{-1} = AF_{q,\tau,k}F_{q,\tau,mq}A^{-1} = F_{q,\tau,k}F_{q,-\tau,qm} = F_{q,-\tau,k+qm},$$

and therefore, $[F_{q,\tau,n}] = [F_{q,-\tau,n}]$. This proves that (iii) implies both (i) and (ii).

To see that (i) implies (iii), note that for each $\tau \in \mathbf{C}$ and $n \geq q$, the points 0 and ∞ in $\hat{\mathbf{C}}$ are distinguished by their depths as holes of $F_{q,\tau,n}$: as computed in the previous lemma, the depth at 0 is the same as the depth at ∞ and greater than at any other point. The two preimages of 0 by the degree two map $z + \tau + 1/z$ are also distinguished: if $n = q$, then they are sent to 0 by the dynamics, and if $n > q$, they are distinguished by their depths which can be computed with Lemma 2.6. Therefore, if $F_{q,\tau,n}$ and $F_{q,\sigma,n}$ are equivalent, the cross-ratio of these four points must coincide.

The preimages of 0 by $z + \tau + 1/z$ lie at the points

$$p_\pm := \frac{1}{2}(-\tau \pm \sqrt{\tau^2 - 4}).$$

We compute the cross ratio χ, normalized so that

$$\chi(0, \infty, 1, z) = z.$$

Then

$$\chi(\tau) := \chi(0, \infty, p_+, p_-) = p_-/p_+ = \frac{1}{4}\left(\tau + \sqrt{\tau^2 - 4}\right)^2.$$

Notice that reversing the labeling of p_+ and p_- or of 0 and ∞ gives $1/\chi(\tau) = \chi(-\tau)$, and so the natural invariant to consider is

$$\chi(\tau) + 1/\chi(\tau),$$

and this is what we will compute. We find,

$$\chi + 1/\chi = \tau^2 - 2.$$

Therefore, if $F_{q,\tau,n}$ and $F_{q,\sigma,n}$ are equivalent, we must have $\tau^2 - 2 = \sigma^2 - 2$, and therefore, $\sigma = \pm\tau$.

The proof that (ii) implies (iii) is similar: the (unordered) pairs of points $\{0, \infty\}$ and $\{p_+, p_-\}$ are distinguished by their masses, as can be computed with Lemma 2.3. If (ii) holds, the cross ratio of these four points must coincide for τ and for σ. As seen above, this implies that $\sigma = \pm\tau$.

Finally, in the proof Lemma 6.5 we showed that $d_0(P_{q,n}) = 2^{n-1}$ which is strictly greater than the depth of any hole for $F_{q,\tau,n}$. Therefore, $[F_{q,\tau,n}] \neq [P_{q,n}]$. □

Lemma 6.7. *For each* $n \geq q \geq 2$, $[F_{q,\tau,n}] \to [P_{q,n}]$ *in* $\overline{M}_{2^n}$ *as* $\tau \to \infty$.

Proof. Write $n = k + mq$ for integers $0 \le k < q$ and $m \ge 1$. Recall the definitions,

$$F_{q,\tau,q} = (z^{2^{q-1}-1} w^{2^{q-1}-1}(z^2 + \tau zw + w^2) : z^{2^{q-1}} w^{2^{q-1}}) \in \overline{\mathrm{Rat}}_{2^q},$$

and

$$P_{q,q} = (z^{2^{q-1}} w^{2^{q-1}-1}(z + w) : z^{2^{q-1}} w^{2^{q-1}}) \in \overline{\mathrm{Rat}}_{2^q}.$$

For each $\tau \in \mathbf{C}$, define $A_\tau \in \mathrm{Aut}\,\hat{\mathbf{C}}$ by

$$A_\tau(z : w) = (\tau^{-1}(z + w) : w).$$

Then

$$A_\tau F_{q,\tau,q} A_\tau^{-1} = (\tau^{2^{q-1}} z^{2^{q-1}-1} w^{2^{q-1}-1}(z^2 + zw) + O(\tau^{2^{q-1}-1}) : \tau^{2^{q-1}} z^{2^{q-1}} w^{2^{q-1}}),$$

and therefore $A_\tau F_{q,\tau,q} A_\tau^{-1} \to P_{q,q}$ in $\overline{\mathrm{Rat}}_{2^q}$ as $\tau \to \infty$. By the regularity of the iterate maps near $P_{q,q}$, we have also that $A_\tau F^m_{q,\tau,q} A_\tau^{-1} \to P^m_{q,q} = P_{q,qm}$ in $\overline{\mathrm{Rat}}_{2^{qm}}$ as $\tau \to \infty$. Note also that for $1 \le k < q$,

$$A_\tau F_{q,\tau,k} A_\tau^{-1} = (\tau^{-1}(\tau z - w)^{2^{k-1}} w^{2^{k-1}} : 0) \to P_{q,k} = (z^{2^{k-1}} w^{2^{k-1}} : 0)$$

in $\overline{\mathrm{Rat}}_{2^k}$ as $\tau \to \infty$. Therefore, by the continuity of the composition map (Lemma 2.6),

$$A_\tau F_{q,\tau,n} A_\tau^{-1} = A_\tau F_{q,\tau,k} F_{q,\tau,qm} A_\tau^{-1} \to P_{q,k} P_{q,qm} = P_{q,n},$$

in $\overline{\mathrm{Rat}}_{2^n}$ as $\tau \to \infty$. Since each of these elements in $\overline{\mathrm{Rat}}_{2^n}$ is stable by Lemma 6.5, we can conclude that $[F_{q,\tau,n}] \to [P_{q,n}]$ in $\overline{M}_{2^n}$ as $\tau \to \infty$. □

The iterate map Φ_n. We are now ready to prove Theorem 5.4. However, with all the notation of this section in place, we can state a more specific result about the limiting values of the iterate map $\Phi_n : M_2 \to M_{2^n}$. Recall the definition of the τ^2-value of a holomorphic disk in $\overline{M}_2$, given just before the statement of Theorem 5.4. For integers $n \ge q \ge 2$ and $\tau \in \mathbf{C}$, let $F_{q,\tau,n} \in \overline{\mathrm{Rat}}_{2^n}$ be defined as in Proposition 6.1 and let $P_{q,n} \in \overline{\mathrm{Rat}}_{2^n}$ be defined as in Proposition 6.4.

Proposition 6.8. *Fix integers $n \ge q \ge 2$ and let ζ be a primitive q-th root of unity. Let $\Delta : \mathbf{D} \hookrightarrow \overline{M}_2$ be a holomorphic disk such that $\Delta(0) = [\Lambda_\zeta]$. If $\tau^2(\Delta) = \infty$, then $\Phi_n(\Delta(t)) \to [P_{q,n}]$ in $\overline{M}_{2^n}$ as $t \to 0$. If τ is a square root of $\tau^2(\Delta) \in \mathbf{C}$, then $\Phi_n(\Delta(t)) \to [F_{q,\tau,n}]$ as $t \to 0$.*

Proof. Suppose that $\Delta \subset \partial M_2$. Then by definition, $\tau^2(\Delta) = \infty$. For any real path $p : [0,1] \to \mathbf{D}$ such that $p(0) = 0$, choose a continuous lift of $\Delta(p(t))$ to $\Lambda_{a(t)} \in \overline{\mathrm{Rat}}_2$ so that $\Delta(p(t)) = [\Lambda_{a(t)}]$ and $a(t) \to \zeta$ as $t \to 0$. Define $A_t \in \mathrm{Aut}\,\hat{\mathbf{C}}$ by $A_t(z, w) = ((a(t)^q - 1)z + w : w)$. Using formula (5.5) for the iterates of $\Lambda_{a(t)}$, it can be computed directly that $P^n_t := A_t^{-1} \Lambda^n_{a(t)} A_t \in \overline{\mathrm{Rat}}_{2^n}$ is given by

$$\Big(z^{2^{n-1}}(a(t)^n(a(t)^q - 1)z + (a(t)^n - 1)w)\left(\prod_{i=1}^{n-1}((a(t)^q - 1)z + w(1 - 1/a(t)^i))^{2^{n-1-i}}\right) :$$
$$(a(t)^q - 1)z^{2^{n-1}} w \left(\prod_{i=1}^{n-1}((a(t)^q - 1)z + w(1 - 1/a(t)^i))^{2^{n-1-i}}\right)\Big).$$

For each $n \le q$, we see immediately that P^n_t converges in $\overline{\mathrm{Rat}}_{2^n}$ to $P_{q,n}$ as $t \to 0$, where $P_{q,n}$ is defined in Proposition 6.4. Lemma 2.6 then implies that $P^n_t \to P_{q,n}$

as $t \to 0$ for all $n \geq q$. By Lemma 6.5, the point $P_{q,n} \in \overline{\mathrm{Rat}}_{2^n}$ is stable (in the sense of GIT) and so defines a conjugacy class in $\overline{M}_{2^n}$. Consequently,

$$\lim_{t\to 0} \Phi_n(\Delta(t)) = [P_{q,n}].$$

If $\Delta \not\subset \partial M_2$, but $\tau^2(\Delta) = \infty$, then by Proposition 6.4,

$$\lim_{t\to 0} \Phi_n(\Delta(t)) = [P_{q,n}].$$

For $\tau^2(\Delta) \in \mathbf{C}$, let $\tau \in \mathbf{C}$ be a square root of $\tau^2(\Delta)$. Then by Proposition 6.1,

$$\lim_{t\to 0} \Phi_n(\Delta(t)) = [F_{q,\tau,n}].$$

□

Proof of Theorem 5.4. By Lemma 6.6, the conjugacy classes $[P_{q,n}]$ and $[F_{q,\tau,n}]$ are all distinct in $\overline{M}_{2^n}$ when $n \geq q$. Therefore, the theorem is an immediate corollary of Proposition 6.8. □

7. The blow-ups of $\overline{M}_2$

Let Φ_n denote the n-th iterate map $M_2 \to M_{2^n}$, and let Γ_n denote the closure of the image of M_2 in the product in $\overline{M}_2 \times \overline{M}_4 \times \cdots \overline{M}_{2^n}$ via the embedding $(\mathrm{Id}, \Phi_2, \ldots, \Phi_n)$. Let $\pi_n : \Gamma_n \to \Gamma_{n-1}$ be the projection to the first $n-1$ factors. In this section we study the structure of the pair (Γ_n, π_n) for each $n \geq 2$, and we give the proof of Theorem 1.4.

Note that the composition $\pi_2\pi_3\cdots\pi_n : \Gamma_n \to \overline{M}_2$ is an isomorphism away from the finite indeterminacy set $I(\Phi_n) \subset \overline{M}_2$, described explicitly in Theorem 5.1. The projection of Γ_n to the n-th factor $\overline{M}_{2^n}$ is a regular extension of Φ_n.

The model for Γ_n. First, let $p_2 : B_2 \to \overline{M}_2 \simeq \mathbf{P}^2$ denote the standard blow-up of $\mathbf{P}^2$ at the unique point in $I(\Phi_2)$. That is, B_2 is the closure of the graph of $\mathbf{P}^2 \dashrightarrow \mathbf{P}^1$ given by $(x : y : z) \mapsto (x : y)$, with the coordinates chosen so that $I(\Phi_2) = \{(0 : 0 : 1)\}$, and p_2 is the projection to the first factor. Inductively define $p_n : B_n \to B_{n-1}$ to be a blow-up of B_{n-1} at each point of $I(\Phi_n) - I(\Phi_{n-1})$, which in local coordinates is given by the blow-up of $\mathbf{C}^2$ along the ideal (x^2, y) where the axis $\{y = 0\}$ represents the boundary of M_2. That is, B_n is locally isomorphic to the closure of the graph of the map $\mathbf{C}^2 \dashrightarrow \mathbf{P}^1$ given by $(x, y) \mapsto (x^2 : y)$, over each point in $I(\Phi_n) - I(\Phi_2)$.

We find,

Theorem 7.1. *For each $n \geq 2$, there is a regular homeomorphism $h_n : B_n \to \Gamma_n$ such that $\pi_n \circ h_n = p_n$.*

In particular, the homeomorphism h_n restricts to the identity on the dense open subset M_2. Note that B_n has a regular double point over each point of $I(\Phi_n) - I(\Phi_2)$, and therefore, Γ_n is singular for all $n \geq 3$. It would be interesting to know if Γ_n and B_n are in fact isomorphic. This would follow, for example, if Γ_n were known to be normal.

Recall that $\hat{M}_2$ is defined to be the inverse limit of the system $\pi_n : \Gamma_n \to \Gamma_{n-1}$. The following immediate corollary to Theorem 7.1 implies that the boundary of M_2 in $\hat{M}_2$ looks (topologically) like the drawing of Figure 1.

Corollary 7.2. *The inverse limit space $\hat{M}_2$ is naturally homeomorphic to the inverse limit $\hat{B}$ of the system $p_n : B_n \to B_{n-1}$.*

We are now able to prove Theorem 1.4 which states that any element in $\hat{M}_2 \subset \prod_{n=1}^{\infty} \overline{M}_{2^n}$ is determined by finitely many entries.

Proof of Theorem 1.4. Let $x = (x_1, x_2, x_2, \ldots)$ denote a point in $\hat{M}_2$. Suppose first that $x_1 \notin I(\Phi_n)$ for any $n \geq 2$. Then every iterate map Φ_n is regular at x_1 so that it has well-defined iterates $x_n \in \overline{M}_{2^n}$ for all n. Consequently the sequence x is determined by the single entry x_1.

Now suppose that $x_1 \in \overline{M}_2$ is in $I(\Phi_n)$ for some $n \geq 2$, and let N be the minimal such n. The claim is that x is uniquely determined by $(x_1, \ldots, x_N)$.

Indeed, let y_N be the point in B_N identified with $(x_1, \ldots, x_N) \in \Gamma_N$ via the homeomorphism of Theorem 7.1. By the definition of B_n for each $n \geq N$, the composition of projections $p_{N+1} \circ \cdots \circ p_n : B_n \to B_N$ is an isomorphism near y_N. Consequently, for every $n \geq N$, there is a unique point $y_n \in B_n$ associated to y_N such that $p_{n+1}(y_{n+1}) = y_n$. Using again Theorem 7.1, we find that $(x_1, \ldots, x_n)$ is determined by $(x_1, \ldots, x_N)$ for every $n \geq N$, proving the claim. □

Proof of Theorem 7.1. For each $a \in \hat{\mathbf{C}}$, let $\Lambda_a \in \overline{\mathrm{Rat}_2}$ be defined by (5.1). Recall from Theorem 5.1 that the indeterminacy locus of Φ_n in $\overline{M}_2$ is the finite set

$$I(\Phi_n) = \{[\Lambda_\zeta] : \zeta \neq 1 \text{ and } \zeta^q = 1 \text{ for some } q \leq n\}.$$

For each $q \leq n$ and primitive q-th root of unity ζ, There exists a regular homeomorphism from $\mathbf{P}^1$ to the the fiber of the composition $\pi_2 \circ \pi_3 \circ \cdots \pi_n : \Gamma_n \to \overline{M}_2$ over $[\Lambda_\zeta]$: in local coordinates, the map is given by

$$\tau^2 \mapsto ([\Lambda_\zeta], [\Lambda_\zeta^2], \ldots, [\Lambda_\zeta^{q-1}], [F_{q,\tau,q}], \ldots, [F_{q,\tau,n}]) \in \Gamma_n,$$

where τ is either square root of $\tau^2 \in \mathbf{C}$, and

$$\infty \mapsto ([\Lambda_\zeta], [\Lambda_\zeta^2], \ldots, [\Lambda_\zeta^{q-1}], [P_{q,q}], \ldots, [P_{q,n}]) \in \Gamma_n.$$

That this map is well-defined follows from Proposition 6.8. Injectivity follows from Lemma 6.6, and continuity from Lemma 6.7. The fiber is not necessarily isomorphic to $\mathbf{P}^1$, as it might have a singularity at the image of ∞.

To compare Γ_n with B_n, consider the composition of projections $q_n = p_2 \circ p_3 \circ \cdots \circ p_n : B_n \to \overline{M}_2$. Observe that the exceptional fiber of $q_n : B_n \to \overline{M}_2$ over $[\Lambda_{-1}]$ is a $\mathbf{P}^1$, corresponding to the family of lines in $\overline{M}_2$ passing through the point $[\Lambda_{-1}]$, and the exceptional fibers over all other points in $I(\Phi_n)$ are each a $\mathbf{P}^1$ in correspondence with a family of conics in $\overline{M}_2$ passing through $[\Lambda_\zeta]$ which are tangent to ∂M_2.

Let ζ be a primitive q-th root of unity, $q \geq 2$. Suppose that $\Delta : \mathbf{D} \hookrightarrow \overline{M}_2$ is a holomorphic disk in $\overline{M}_2$ such that $\Delta(0) = [\Lambda_\zeta]$. From Theorem 5.4, we see that the limiting value of the iterates, $\Phi_n(\Delta(t))$ as $t \to 0$, depends precisely on the limiting value,

$$\lim_{t \to 0} \frac{(\alpha(t)^q - 1)^2}{\varepsilon(t)} = \tau^2(\Delta) \in \hat{\mathbf{C}}.$$

It suffices to put the exceptional fiber of B_n, as a family of lines or conics through $[\Lambda_\zeta]$, in correspondence with the parameter τ^2. Indeed, we will consider limits of the iterate map along a given line or conic passing through $[\Lambda_\zeta]$ and compute the corresponding value of τ^2. We will treat the cases $q = 2$ and $q > 2$ separately.

Coordinates on $\overline{M}_2$. Choose coordinates $(x_1 : x_2 : x_3)$ on $\overline{M}_2 \simeq \mathbf{P}^2$ so that when $x_3 = 1$, we have Milnor's coordinates $x_1 = \sigma_1$ and $x_2 = \sigma_2$ on $M_2 \simeq \mathbf{C}^2$. In these coordinates, the boundary of M_2 is parameterized by $[\Lambda_a] = (1 : a + 1/a : 0)$ for $a \in \hat{\mathbf{C}}$.

Case $q = 2$. Let $\zeta = -1$, so that $[\Lambda_\zeta] = (1 : -2 : 0) \in \overline{M}_2$. The exceptional fiber of B_n over $[\Lambda_\zeta]$ is the fiber of the projectivized tangent bundle of $\mathbf{P}^2$ at $[\Lambda_\zeta]$, and can be identified with the family of lines

$$L_{(a:b)} = \{(x_1 : x_2 : x_3) : 2bx_1 + bx_2 - ax_3 = 0\},$$

for $(a : b) \in \mathbf{P}^1$. Each line can be parametrized near $[\Lambda_\zeta]$ by

$$t \mapsto (1 : -2 + at : bt) =: f_t \in \overline{M}_2,$$

for $t \in \mathbf{D}$, so that $f_0 = [\Lambda_\zeta]$.

First suppose that $b = 0$ and set $a = 1$. Then $f_t = (1 : -2 + t : 0)$ is in ∂M_2 for all $t \in \mathbf{D}$. By definition, this direction of approach corresponds to the parameter $\tau^2 = \infty$.

Now assume that $b \neq 0$. For each $t \neq 0$, we can write,

$$f_t = (\sigma_1(t) : \sigma_2(t) : 1) = (1/bt : (-2 + at)/bt : 1),$$

so that the fixed point multipliers, $\alpha = \alpha(a, b, t)$, $\beta = \beta(a, b, t)$, and $\gamma = \gamma(a, b, t)$, are the three roots of the equation,

$$btx^3 - x^2 + (at - 2)x + 2bt - 1 = 0.$$

By construction, there are exactly two solutions which approach $\zeta = -1$ as $t \to 0$ and one solution tending to ∞. Label them so that $\gamma \to \infty$ and $\alpha + \beta \to -2$. While we can't label α and β individually for all $t \in \mathbf{D}$, we aim to find an expression for $(\alpha^2 - 1)^2$ as a function of t, by using the relations,

(i) $\alpha + \beta + \gamma = \frac{1}{bt}$,
(ii) $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{a}{b} - \frac{2}{bt}$, and
(iii) $\alpha\beta\gamma = \frac{1}{bt} - 2$.

It follows from (i) that the meromorphic function $\gamma(t)$ can be expressed as

$$\gamma(t) = \frac{1}{bt} + 2 + c_1 t + O(t^2),$$

for some constant c_1, and

$$(\alpha + \beta)(t) = -2 - c_1 t + O(t^2).$$

Expressions (ii) and (iii) allow us to compute all of the coefficients in the expansion of γ in terms of a and b, but we need only c_1.

Expressing γ^{-1} as a power series in t, we obtain

$$\gamma^{-1} = bt(1 + 2bt + O(t^2))^{-1} = bt(1 - 2bt + O(t^2)),$$

and therefore, (iii) implies that

$$\alpha\beta = -\gamma^{-1}(2 - 1/bt) = 1 - 4bt + O(t^2).$$

On the other hand, it follows from (ii) that

$$\begin{aligned}\alpha\beta &= -\gamma(\alpha+\beta) + \frac{a}{b} - \frac{2}{bt} \\ &= \frac{a}{b} + \frac{c_1}{b} + 4 + O(t),\end{aligned}$$

and we can solve for c_1 to obtain $c_1 = -3b - a$. Therefore, we can write,

$$\alpha + \beta = -2 + (3b+a)t + O(t^2).$$

Let us note that α and β are the roots of the equation

$$x^2 - (\alpha+\beta)x + \alpha\beta = x^2 - (-2 + (3b+a)t + O(t^2))x + (1 - 4bt + O(t^2)) = 0.$$

According to the quadratic formula, α has the form

$$\alpha = -1 + \frac{1}{2}(a+3b)t + O(t^2) \pm \frac{1}{2}\sqrt{4(b-a)t + O(t^2)},$$

and therefore

$$(\alpha^2 - 1)^2 = 4(b-a)t + o(t).$$

Finally, we are able to compute

$$\tau^2 = \lim_{t\to 0} \frac{(\alpha(t)^2 - 1)^2}{1 - \alpha(t)\beta(t)} = \frac{b-a}{b}.$$

Consequently, the parameter $\tau^2 \in \hat{\mathbf{C}}$ is in one-to-one correspondence with the family of lines $L_{(a:b)}$ for $(a:b) \in \mathbf{P}^1$. This completes the case of $q = 2$.

Case $q > 2$. Let ζ be a primitive q-th root of unity. The exceptional fiber of B_n over $[\Lambda_\zeta] = (1 : \zeta + 1/\zeta : 0)$ can be identified with the family of conics,

$$C_{(a^2:b^2)} = \{(x_1 : x_2 : x_3) : a^2 x_1 x_3 - b^2(x_2 - (\zeta + 1/\zeta)x_1)^2 = 0\},$$

parameterized in a two-to-one fashion by $(a:b) \in \mathbf{P}^1$, each tangent at $[\Lambda_\zeta]$ to the boundary ∂M_2. The curve $C_{(a^2:b^2)}$ can be parameterized near $[\Lambda_\zeta]$ by

$$t \mapsto f_t = (1 : (\zeta + 1/\zeta) + at : b^2t^2) \in \overline{M}_2.$$

First suppose that $b = 0$ and set $a = 1$. Then $f_t = (1 : \zeta + 1/\zeta + t : 0)$ is in ∂M_2 for all $t \in \mathbf{D}$. This direction of approach corresponds to the parameter $\tau^2 = \infty$.

Now assume that $b \neq 0$, so that for each $t \neq 0$, we can write,

$$f_t = (\sigma_1(t) : \sigma_2(t) : 1) = (1/b^2t^2 : (at + \zeta + 1/\zeta)/b^2t^2 : 1).$$

The fixed point multipliers, $\alpha = \alpha(a,b,t)$, $\beta = \beta(a,b,t)$, and $\gamma = \gamma(a,b,t)$, are the three roots of the equation,

$$b^2t^2x^3 - x^2 + (at + \zeta + 1/\zeta)x + 2b^2t^2 - 1 = 0.$$

To compute the value of τ^2, we use the same analysis as in the case of $q = 2$. The difference is that all three of the multipliers can be labelled as meromorphic functions of t such that $\alpha \to \zeta$, $\beta \to 1/\zeta$, and $\gamma \to \infty$ as $t \to 0$. Computing the first few terms in the power series for α and β leads to

$$\tau^2 = \lim_{t\to 0} \frac{(\alpha(t)^2 - 1)^2}{1 - \alpha(t)\beta(t)} = \frac{-q^2a^2\zeta^3}{b^2(\zeta^2 - 1)(\zeta - 1)^2}.$$

Therefore, the parameter $\tau^2 \in \hat{\mathbf{C}}$ is in one-to-one correspondence with the family of conics $C_{(a^2:b^2)}$, and this completes the proof of Theorem 7.1. □

8. The space of barycentered measures

In this section, we study the space of barycentered probablity measures on the Riemann sphere and its quotient by the group of rotations SO(3).

Let $M^1(\hat{\mathbf{C}})$ denote the space of probability measures on the Riemann sphere, with the weak-$*$ topology. Identify $\hat{\mathbf{C}}$ with the unit sphere $S^2 \subset \mathbf{R}^3$ by stereographic projection, and let the unit ball in $\mathbf{R}^3$ be taken as a model for hyperbolic space $\mathbf{H}^3$. Recall that the group of Möbius transformations $\operatorname{Aut}\hat{\mathbf{C}} \simeq \mathrm{PSL}_2\mathbf{C}$ is also the group of orientation preserving isometries of $\mathbf{H}^3$.

The Euclidean center of mass of a probability measure μ on S^2 is given by

$$E(\mu) = \int_{S^2} \zeta \, d\mu(\zeta).$$

Given $\mu \in M^1(\hat{\mathbf{C}})$ such that $\mu(\{z\}) < 1/2$ for all $z \in \hat{\mathbf{C}}$, the conformal barycenter $C(\mu) \in \mathbf{H}^3$ is uniquely determined by the following two properties [DE]:

1. $C(\mu) = 0$ in $\mathbf{R}^3$ if and only if $E(\mu) = 0$, and
2. $C(A_*\mu) = A(C(\mu))$ for all $A \in \operatorname{Aut}\hat{\mathbf{C}} \simeq \mathrm{Isom}^+\mathbf{H}^3$.

The barycenter is a continuous function on the space of probability measures such that $\mu(\{z\}) < 1/2$ for all $z \in \hat{\mathbf{C}}$, and it is undefined if μ has an atom of mass $\geq 1/2$. A measure μ is said to be **barycentered** if $C(\mu) = 0$.

Let $M^1_{bc}(\hat{\mathbf{C}}) \subset M^1(\hat{\mathbf{C}})$ denote the subspace of barycentered measures. It is invariant under the action of the compact group of rotations $\mathrm{SO}(3) \subset \mathrm{PSL}_2\mathbf{C}$, the stabilizer of the origin in $\mathbf{R}^3$. Let $\overline{M^1_{bc}(\hat{\mathbf{C}})}$ denote the closure of $M^1_{bc}(\hat{\mathbf{C}})$ in $M^1(\hat{\mathbf{C}})$. We will consider the quotient topological spaces $M^1_{bc}(\hat{\mathbf{C}})/\mathrm{SO}(3) \subset \overline{M^1_{bc}(\hat{\mathbf{C}})}/\mathrm{SO}(3)$.

Theorem 8.1. *The quotient space $M^1_{bc}(\hat{\mathbf{C}})/\mathrm{SO}(3)$ is a locally compact Hausdorff topological space, with the topology induced by the weak topology on $M^1(\hat{\mathbf{C}})$. The quotient space $\overline{M^1_{bc}(\hat{\mathbf{C}})}/\mathrm{SO}(3)$ is the one-point compactification of $M^1_{bc}(\hat{\mathbf{C}})/\mathrm{SO}(3)$.*

This first lemma will imply that the quotient of $M^1_{bc}(\hat{\mathbf{C}})$ is Hausdorff.

Lemma 8.2. *Suppose $\{\mu_k\}$ and $\{\nu_k\}$ are sequences in $M^1_{bc}(\hat{\mathbf{C}})$ such that $\nu_k = g_{k*}\mu_k$ for a sequence of automorphisms $g_k \in \mathrm{SO}(3)$. If $\mu_k \to \mu$ and $\nu_k \to \nu$ weakly, then $\nu = g_*\mu$ for some $g \in \mathrm{SO}(3)$.*

Proof. Let φ be a continuous function on $\hat{\mathbf{C}}$. Passing to a subsequence if necessary, we can assume that $g_k \to g$ in SO(3). Then $\varphi \circ g_k$ converges uniformly to $\varphi \circ g$. Therefore, the quantity

$$\left| \int (\varphi \circ g_k)\mu - \int (\varphi \circ g)\mu \right|$$

can be made as small as desired for sufficiently large k uniformly over all probability measures μ. This estimate together with weak convergence of $\mu_k \to \mu$ shows that

$$\left| \int (\varphi \circ g_k)\mu_k - \int (\varphi \circ g)\mu \right|$$

can be made arbitrarily small as $k \to \infty$. Since this holds for every φ, we conclude that $g_{k*}\mu_k \to g_*\mu$ weakly. On the other hand, $\nu_k = g_{k*}\mu_k \to \nu$ weakly, so $\nu = g_*\mu$. □

With the identification of $\hat{\mathbf{C}}$ and the unit sphere in $\mathbf{R}^3$, note that the antipode of a point $a \in \hat{\mathbf{C}}$ is $-1/\bar{a}$. The following lemma shows what happens to an unbounded sequence in $M^1_{bc}(\hat{\mathbf{C}})$.

Lemma 8.3. *Let μ_k be a sequence of barycentered measures such that $\mu_k \to \mu$ weakly. Then either μ is barycentered, or*

$$\mu = \frac{1}{2}\delta_a + \frac{1}{2}\delta_{-1/\bar{a}}.$$

Proof. Suppose first that $\mu(\{z\}) < 1/2$ for all $z \in \hat{\mathbf{C}}$. Then the barycenter of μ is well-defined. It follows from the continuity of the barycenter that μ is barycentered. Now suppose there is a point $a \in \hat{\mathbf{C}}$ such that $\mu(\{a\}) \geq 1/2$. Then by weak convergence, there exists a sequence $\varepsilon_k \to 0$ and $r_k \to 0$ such that

$$\mu_k(B(a, r_k)) \geq \frac{1}{2} - \varepsilon_k.$$

Without loss of generality, we may assume that $a = (1, 0, 0)$ in $\mathbf{R}^3$. Then $-1/\bar{a} = (-1, 0, 0)$. It suffices to show that for each fixed $r > 0$, there is a sequence $\delta_k \to 0$ so that

$$\mu_k(B(-1/\bar{a}, r)) \geq 1/2 - \delta_k.$$

Then the limiting measure μ must satisfy

$$\mu(\{a\}) = \mu(B(-1/\bar{a}, r)) = 1/2,$$

for all $r > 0$. Letting $r \to 0$, we see that $\mu(\{-1/\bar{a}\}) = 1/2$.

Suppose, upon passing to a subsequence if necessary, that there is a $\delta > 0$ such that

$$\mu_k(B(\hat{a}, r)) \leq 1/2 - \delta$$

for all k. Let ζ_x denote the x-coordinate of a vector $\zeta \in S^2$. Since the measures μ_k are barycentered, the Euclidean center of mass of μ_k is at the origin, and therefore,

$$\begin{aligned} 0 = \int_{\hat{\mathbf{C}}} \zeta_x \mu_k(\zeta) &= \int_{B(a,r_k)} \zeta_x \mu_k(\zeta) + \int_{\hat{\mathbf{C}} - B(a,r_k) \cup B(\hat{a},r)} \zeta_x \mu_k(\zeta) + \int_{B(\hat{a},r)} \zeta_x \mu_k(\zeta) \\ &\geq (1/2 - \varepsilon_k)(\cos \pi r_k) + (\delta + \varepsilon_k)(-\cos \pi r) - (1/2 - \delta) \\ &= \delta(1 - \cos \pi r) + 1/2(\cos \pi r_k - 1) - \varepsilon_k(\cos \pi r + \cos \pi r_k). \end{aligned}$$

For sufficiently large k, the final line is positive, which is a contradiction. □

Proof of Theorem 8.1. That the quotient $M^1_{bc}(\hat{\mathbf{C}})/\mathrm{SO}(3)$ is Hausdorff follows from Lemma 8.2. Local compactness of the metrizable space $M^1_{bc}(\hat{\mathbf{C}})$ and the fact that the SO(3)-orbits are compact implies that $M^1_{bc}(\hat{\mathbf{C}})/\mathrm{SO}(3)$ is also locally compact. Suppose μ_k is an unbounded sequence in $M^1_{bc}(\hat{\mathbf{C}})$ such that $\mu_k \to \nu$ weakly in $M^1(\hat{\mathbf{C}})$. By Lemma 8.3, $\nu = \frac{1}{2}\delta_a + \frac{1}{2}\delta_{-1/\bar{a}}$ for an antipodal pair $(a, -1/\bar{a})$. Under the action of SO(3) on $M^1(\hat{\mathbf{C}})$, all such ν are equivalent. □

The point at infinity. We will refer to the point at infinity of $\overline{M^1_{bc}(\hat{\mathbf{C}})}/\mathrm{SO}(3)$ simply by ∞. One way to detect if a sequence of probability measures is converging to ∞ in $\overline{M^1_{bc}(\hat{\mathbf{C}})/\mathrm{SO}(3)}$ is to find a sequence of "separating annuli", annuli of growing modulus such that half of the measure lies on side of the annulus and half on the other. By classical arguments, at least one complementary component of the annulus must shrink to a point.

Lemma 8.4. *Suppose $\{\mu_k\}$ is a sequence of barycentered probability measures on $\hat{\mathbf{C}}$ such that $\mu_k \to \nu$ weakly, and A_k a sequence of round annuli such that*

(1) $\operatorname{mod} A_k \to \infty$ *as* $k \to \infty$, *and*
(2) *there is a sequence $\varepsilon_k \to 0$ such that $\mu_k(D_k) \geq 1/2 - \varepsilon_k$ for each of the complementary disks D_k of A_k.*

Then $\nu = \infty$ in $\overline{M^1_{bc}(\hat{\mathbf{C}})}/\mathrm{SO}(3)$.

Proof. By condition (1), some subsequence of the closed disks D_k is converging to a point in the Hausdorff topology, say $D_k \to \{a\}$. Then for any $r > 0$, we have

$$\nu(B(a,r)) \geq \lim_{k\to\infty} \mu_k(D_k) \geq 1/2.$$

This holds for all $r > 0$, and therefore $\nu(\{a\}) \geq 1/2$, so by Lemma 8.3, $\mu_k \to \infty$ in $\overline{M^1_{bc}(\hat{\mathbf{C}})}/\mathrm{SO}(3)$. $\square$

9. The homeomorphism $\hat{M}_2 \to \overline{\overline{M}}_2$

In this section we give the proof of Theorem 1.1 which states that the compactification of M_2 by barycentered measures and the inverse limit space which resolves the iterate maps on $\overline{M}_2$ are the same. We begin by studying the properties of

$$B : M_2 \to M^1_{bc}(\hat{\mathbf{C}})/\mathrm{SO}(3),$$

which sends a conjugacy class $[f]$ to the maximal measure μ_f of a barycentered representative.

Lemma 9.1. *The map $B : M_2 \to M^1_{bc}(\hat{\mathbf{C}})/\mathrm{SO}(3)$ is well-defined and continuous.*

Proof. For every $f \in \mathrm{Rat}_2$, the measure of maximal entropy μ_f is non-atomic [Ly], [FLM]. Therefore, it has a well-defined barycenter in $\mathbf{H}^3$. The group of orientation-preserving isometries of $\mathbf{H}^3$ acts transitively, and for every $A \in \mathrm{Aut}(\hat{\mathbf{C}})$, $\mu_{AfA^{-1}} = A_*\mu_f$. Therefore, there exists an automorphism $A \in \mathrm{Aut}(\hat{\mathbf{C}})$ such that $\mu_{AfA^{-1}}$ is barycentered. Consequently, for every conjugacy class $[f]$ in M_2, there exists a representative f with barycentered maximal measure. Now suppose μ_f is barycentered. For $A \notin \mathrm{SO}(3)$, the stabilizer in $\mathrm{Isom}^+\mathbf{H}^3$ of the origin in $\mathbf{R}^3$, the measure $A_*\mu_f$ is not barycentered. Therefore, the map B which sends $[f]$ to the class of a barycentered measure $[\mu_f] \in M^1_{bc}(\hat{\mathbf{C}})/\mathrm{SO}(3)$ is well-defined. Finally, the map $f \mapsto \mu_f$ is continuous from Rat_d with the topology of uniform convergence to $M^1(\hat{\mathbf{C}})$ with the weak-$*$ topology [Ma2]. The barycenter operation is also continuous, and therefore B is continuous. $\square$

Recall that the boundary of M_2 in $\overline{M}_2 \simeq \mathbf{P}^2$ is parameterized by the family of conjugacy classes $[\Lambda_a] = [\Lambda_{1/a}]$ for $a \in \hat{\mathbf{C}}$, where $\Lambda_a \in \overline{\mathrm{Rat}}_2$ is defined in (5.1).

Proposition 9.2. *Suppose $[f_k]$ is a sequence in M_2 such that $[f_k] \to [\Lambda_a]$ in $\overline{M}_2$ as $k \to \infty$, where a is not a root of unity. Then $B([f_k]) \to \infty$ in $\overline{M^1_{bc}(\hat{\mathbf{C}})}/\mathrm{SO}(3)$ as $k \to \infty$.*

Proof. First suppose that a is neither 0 nor ∞. There is a sequence of representatives $f_k \in \mathrm{Rat}_2$ such that $f_k \to \Lambda_a$ in $\overline{\mathrm{Rat}}_2$, so that $\mu_{f_k} \to \mu_{\Lambda_a}$ weakly by Theorem 2.2. Let $\mu_a = \mu_{\Lambda_a}$. By definition,

$$\mu_a = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{2^n}\delta_{1/a^n}.$$

Since a is not a root of unity, we have $\mu_a(\{z\}) \leq 1/2$ for all $z \in \hat{\mathbf{C}}$, and there exists a point $p \in \hat{\mathbf{C}}$ such that $\mu_a(\{p\}) = 1/2$.

Fix $\varepsilon > 0$ and choose $r = r(\varepsilon) > 0$ so that

$$\mu_a(\hat{\mathbf{C}} - \overline{B(p,r)}) \geq 1/2 - 2\varepsilon.$$

By weak convergence of the measures $\mu_k \to \mu_a$, there exists an integer $N(\varepsilon)$ such that

$$\mu_k(\hat{\mathbf{C}} - \overline{B(p,r)}) \geq 1/2 - \varepsilon,$$

and

$$\mu_k(B(p,r^2)) \geq 1/2 - \varepsilon$$

for all $k \geq N(\varepsilon)$. We can assume that $r(\varepsilon) \to 0$ and $N(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Rephrasing, given k, we can let ε_k be the smallest ε such that $k \geq N(\varepsilon)$, and set $r_k = r(\varepsilon_k)$. Then as $k \to \infty$, we have $r_k \to 0$. Consequently, the annulus $B(p,r_k) - \overline{B(p,r_k^2)}$ has μ_k-measure $< 2\varepsilon_k$ and modulus $\to \infty$ as $k \to \infty$.

Now suppose that $g_k \in \mathrm{Aut}\,\hat{\mathbf{C}}$ is chosen so that $g_{k*}\mu_k$ is barycentered. Let $A_k = g_k(B(p,r_k) - \overline{B(p,r_k^2)})$, so that $\mathrm{mod}\,A_k \to \infty$. If ν is any subsequential limit of the measures $g_{k*}\mu_k$, then Lemma 8.4 implies that $\nu = \infty$ in $\overline{M^1_{bc}(\hat{\mathbf{C}})}/\mathrm{SO}(3)$.

Finally, suppose that $a = 0$ or $a = \infty$. Then the probability measure associated to $\Lambda_a \in \overline{\mathrm{Rat}}_2$ is

$$\mu_{\Lambda_a} = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1},$$

and $\mu_k \to \mu_{\Lambda_a}$ weakly. Therefore, for any $r > 0$, any annulus of the form $\hat{\mathbf{C}} - (B(1,r) \cup B(-1,r))$ will have μ_k-measure tending to 0 as $k \to \infty$. By Lemma 8.4, the maximal measures of the barycentered representatives of $[f_k]$ must tend to infinity in $\overline{M^1_{bc}(\hat{\mathbf{C}})}/\mathrm{SO}(3)$. $\square$

Let ζ be a primitive q-th root of unity for some $q \geq 2$. Recall that by Theorem 5.4, the limiting value of the iterate map Φ_n near $[\Lambda_\zeta] \in \partial M_2$ depends on the direction of approach, and the n-th iterate can be computed in terms of a parameter $\tau^2 \in \hat{\mathbf{C}}$. The limiting barycentered measure depends on the limit of the iterates. The key observation is that the measures associated to the GIT stable limits of the q-th iterates (in $\overline{\mathrm{Rat}}_{2^q}$) have well-defined barycenters. Recall the definitions of $F_{q,\tau,q}$ and $P_{q,q} \in \overline{\mathrm{Rat}}_{2^q}$ from Propositions 6.1 and 6.4.

Proposition 9.3. *Fix $q \geq 2$ and ζ a primitive q-th root of unity. Let $[f_k]$ be a sequence in M_2 converging to $[\Lambda_\zeta]$ in $\overline{M}_2$ as $k \to \infty$ such that the q-th iterates $[f_k^q]$ converge in $\overline{M}_{2^q}$ to either (i) $[P_{q,q}]$ or (ii) $[F_{q,\tau,q}]$ for some $\tau \in \mathbf{C}$. Then in case (i), $B([f_k]) \to \infty$ in $\overline{M^1_{bc}(\hat{\mathbf{C}})}/\mathrm{SO}(3)$ as $k \to \infty$. In case (ii), $\lim_{k\to\infty} B([f_k])$ is equivalent (in $\mathrm{PSL}_2\mathbf{C}$) to the measure $\mu_{F_{q,\tau,q}}$.*

Proof. In case (i), there exist representatives $f_k \in \mathrm{Rat}_2$ such that the iterates f_k^q converge in $\overline{\mathrm{Rat}}_{2^q}$ to $P_{q,q}$ as $k \to \infty$. Since $P_{q,q} \notin I(2^q)$, Theorem 2.2 implies that the measures μ_{f_k} converge weakly to

$$\mu_{P_{q,q}} = \frac{1}{2^q} \sum_{j=0}^{\infty} \frac{2^{q-1}}{2^{qj}} \delta_{-j} + \frac{1}{2^q} \sum \frac{2^{q-1}-1}{2^{qj}} \delta_\infty = \frac{1}{2} \sum_{j=0}^{\infty} \frac{1}{2^{qj}} \delta_{-j} + \frac{2^{q-1}-1}{2^q - 1} \delta_\infty$$

as $k \to \infty$. Notice that $\mu_{P_{q,q}}(\{0\}) = 1/2$.

Therefore, there exists a family of annuli A_k separating $z = 0$ from the other points in the support of $\mu_{P_{q,q}}$, such that $\operatorname{mod} A_k \to \infty$ as $k \to \infty$ and the μ_{f_k}-measure of each complementary component of A_k tends to 1/2 as $k \to \infty$. Choose a sequence $g_k \in \operatorname{Aut} \hat{\mathbf{C}}$ so that each $g_{k*}\mu_{f_k}$ is barycentered. Then $g_{k*}\mu_{f_k}$ and the annuli $g_k(A_k)$ satisfy the hypotheses of Lemma 8.4 which shows that $g_{k*}\mu_{f_k} \to \infty$ in $\overline{M^1_{bc}(\hat{\mathbf{C}})/\mathrm{SO}(3)}$.

In case (ii), there exist representatives $f_k \in \mathrm{Rat}_2$ such that the iterates f_k^q converge in $\overline{\mathrm{Rat}}_{2^q}$ to $F_{q,\tau,q}$ as $k \to \infty$. Since $F_{q,\tau,q}$ is not in $I(2^q)$, Theorem 2.2 implies that the measures μ_{f_k} converge weakly to $\mu_{F_{q,\tau,q}}$ as $k \to \infty$. Note that $\mu_{F_{q,\tau,q}}(\{z\}) < 1/2$ for all $z \in \hat{\mathbf{C}}$ by Lemma 6.3. Therefore, $\mu_{F_{q,\tau,q}}$ has a well-defined barycenter. By continuity of the barycenter, we can choose a sequence $g_k \in \mathrm{Aut}(\hat{\mathbf{C}})$ for such that $g_{k*}\mu_{f_k}$ is barycentered for all k and $g_k \to g \in \mathrm{Aut}(\hat{\mathbf{C}})$ with $g_*\mu_{F_{q,\tau,q}}$ barycentered. Therefore, $B([f_k])$ converges in $\overline{M^1_{bc}(\hat{\mathbf{C}})/\mathrm{SO}(3)}$ to a measure which is equivalent (in $\mathrm{Aut}(\hat{\mathbf{C}})$) to $\mu_{F_{q,\tau,q}}$ as $k \to \infty$. □

Proof of Theorem 1.1. Recall the definitions of $\overline{M}_2$, $\hat{M}_2$, and $\overline{\overline{M}}_2$ given in Section 1. We aim to show that the embedding of M_2 into $M_2 \times M^1_{bc}(\hat{\mathbf{C}})/\mathrm{SO}(3)$ via the graph of $B([f]) = [\mu_f]$ (for a barycentered representative) extends to a homeomorphism

$$h : \hat{M}_2 \to \overline{\overline{M}}_2 \subset \overline{M}_2 \times \overline{M^1_{bc}(\hat{\mathbf{C}})/\mathrm{SO}(3)}.$$

Since $\hat{M}_2$ is compact and $\overline{\overline{M}}_2$ is Hausdorff, it suffices to show that h is continuous and bijective. Furthermore, since the image of h contains a dense open subset in $\overline{\overline{M}}_2$, namely M_2 itself, it suffices to show only continuity and injectivity.

Let $x = (x_1, x_2, \dots)$ be a boundary point of M_2 in $\hat{M}_2 \subset \prod_{n=1}^\infty \overline{M}_{2^n}$. Suppose first that $x_1 = [\Lambda_a]$ where a is not a primitive q-th root of unity for any $q \geq 2$. Then by Theorem 5.1, $x_n = [\Lambda_\zeta^n]$ for every $n \geq 2$. That is to say, there is a unique point in $\hat{M}_2$ which projects to x_1 in $\overline{M}_2$. By Proposition 9.2, h extends continuously to x with $h(x) = ([\Lambda_a], \infty) \in \overline{M}_2 \times \overline{M^1_{bc}(\hat{\mathbf{C}})/\mathrm{SO}(3)}$, and we see that $h(x) \in \overline{\overline{M}}_2$ has x as its unique preimage.

Now suppose that $x_1 \in I(\Phi_n)$ for some $n \geq 2$. Then by Theorem 5.1, there is a $q \leq n$ and a primitive q-th root of unity ζ such that $x_1 = [\Lambda_\zeta]$. From Proposition 6.8, the q-th entry x_q in the sequence x must be either (i) $[P_{q,q}]$, or (ii) of the form $[F_{q,\tau,q}]$ for some $\tau \in \mathbf{C}$, and all further entries x_n for $n > q$ are determined by x_q. That is to say, there is a unique point in $\hat{M}_2$ which projects to $(x_1, \dots, x_q)$ under the projection to the first q factors. Let $[f_k]$ be any sequence in M_2 such that $[f_k] \to x$ in $\hat{M}_2$ as $k \to \infty$. By Proposition 9.3, h extends continuously to x with $h(x) = ([\Lambda_\zeta], \infty)$ in case (i) and $h(x) = ([\Lambda_\zeta], g_*\mu_{F_{q,\tau,q}})$ for some $g \in \operatorname{Aut} \hat{\mathbf{C}}$ in case (ii). Furthermore, we see that $h(x) \in \overline{\overline{M}}_2$ has x as its unique preimage by Lemma 6.6. $\square$

10. Higher degrees

Recall the definitions of $\overline{M}_d$, $\hat{M}_d$, and $\overline{\overline{M}}_d$ given in Section 1. In this section, we show that the inverse limit space $\hat{M}_d$ and the compactification by barycentered measures $\overline{\overline{M}}_d$ are not homeomorphic for $d \geq 5$. The examples come from [De, §5]. We also prove the following theorem.

Theorem 10.1. *The iterate map $\Phi_n : M_d \to M_{d^n}$ does not extend continuously to a map from $\overline{M}_d$ to $\overline{M}_{d^n}$ for any $d \geq 2$ and $n \geq 2$.*

The examples used to show discontinuity at the GIT boundary of M_d are also from [De, §5]. In fact, they are a generalization of Epstein's examples, seen in Proposition 5.3 ([Ep, Prop 2]). The idea is to find unbounded families in Rat_d such that the critical points are always at $2d-2$ given points, and such that these critical points are distinct from the holes which develop in the limit.

The second iterate. We give first a complete proof of Theorem 10.1 for the case of the second iterate, $n = 2$.

Fix $d \geq 2$. Let $P(z, w)$ be a homogeneous polynomial of degree $d - 1$ with distinct roots in $\mathbf{P}^1$ which is monic as a polynomial in z and such that $P(0, 1) \neq 0$ and $P(1, 0) \neq 0$. Let

$$g = (wP(z, w) : 0) \in \overline{\mathrm{Rat}}_d.$$

Then g has a hole at ∞ of depth 1 and holes of depth 1 at each of the roots of P. The lower degree map $\hat{g}$ is the constant ∞ map, so we see that $g \in I(d)$. The point g is GIT stable for all $d \geq 4$, semistable for $d = 3$, and unstable for $d = 2$.

Consider the family of rational maps given by

$$g_{a,t} = (atz^d + wP(z, w) : tz^d),$$

for $a \in \mathbf{C}$ and $t \in \mathbf{D}^*$. This family converges to g in $\overline{\mathrm{Rat}}_d$ as $t \to 0$ for every $a \in \mathbf{C}$. In [De, §5], it was computed that the second iterates of the family $g_{a,t}$ converge as $t \to 0$ to

$$f_a = (w^{d-1}P^{d-1}(awP + z^d) : w^d P^d) \in \overline{\mathrm{Rat}}_{d^2}.$$

In the notation $f_a = H_a \hat{f}_a$, we have $H_a = w^{d-1}P^{d-1}$ for all $a \in \mathbf{C}$ and

$$\hat{f}_a = (awP(z, w) + z^d : wP(z, w)) \in \mathrm{Rat}_d.$$

The point f_a has holes at ∞ and the roots of P, each of depth $d - 1$. Note that f_a is *not* in $I(d^2)$.

Lemma 10.2. *For each $d \geq 2$, the conjugacy classes $[g_{a,t}]$ converge in $\overline{M}_d$ as $t \to 0$ to a boundary point independent of $a \in \mathbf{C}$.*

Proof. For each $d > 2$, the point $g \in \overline{\mathrm{Rat}}_d$ is stable or semistable and therefore determines a unique point $[g] \in \overline{M}_d$. Convergence of $g_{a,t}$ to g in $\overline{\mathrm{Rat}}_d$ as $t \to 0$ implies that $[g_{a,t}] \to [g]$ in $\overline{M}_d$ as $t \to 0$ for every $a \in \mathbf{C}$.

For $d = 2$, however, the point g is unstable, so it does not represent a point in $\overline{M}_2$. Conjugating the family $g_{a,t}$ by $A_t(z) = t^{1/2}z$, we obtain new representatives,

$$A_t g_{a,t} A_t^{-1} = (at^{1/2}z^2 + wP(z, t^{1/2}w) : z^2),$$

which converge as $t \to 0$ to

$$h = (zw : z^2).$$

This point h is stable, since it agrees with the lower degree map $\hat{h}(z) = 1/z$ away from a hole of depth 1 at $z = 0$ which is not fixed by $\hat{h}$. Therefore, $[g_{a,t}] \to [h]$ in $\overline{M}_2$ as $t \to 0$ for every $a \in \mathbf{C}$. □

Lemma 10.3. *For each $d \geq 2$,*

(i) *$f_a \in \overline{\mathrm{Rat}}_{d^2}$ is stable for all $a \in \mathbf{C}$, and*
(ii) *the map $\mathbf{C} \to \overline{M}_{d^2}$ given by $a \mapsto [f_a]$ is non-constant.*

Proof. From the definition of f_a we see that each hole has depth $d-1$ and $d - 1 = (d^2 - 1)/(d+1) < (d^2 - 1)/2$ for all $d \geq 2$. This proves (i). Consequently, $[f_a] = [f_b]$ in $\overline{M}_{d^2}$ if and only if f_a and f_b are conjugate by an element of $\mathrm{PSL}_2\mathbf{C}$.

Now suppose that $d > 2$. Any conjugacy between f_a and f_b for $a \neq b$ must preserve the holes at ∞ and the roots of P and conjugate $\hat{f}_a$ to $\hat{f}_b$. There are at least three holes since the degree of P is $d-1$ with distinct roots. On the other hand, the finite fixed points of $\hat{f}_a$ are the $d-1$ solutions to $z^d - zP(z,1) + aP(z,1) = 0$, and so the set of fixed points varies with $a \in \mathbf{C}$. The cross-ratio of three of the holes with a moving fixed point must then vary with $a \in \mathbf{C}$, and this proves that not all f_a are conjugate.

For $d = 2$, note that a conjugacy between f_a and f_b must preserve the holes and also send the critical points of $\hat{f}_a$ to the critical points of $\hat{f}_b$. It can be computed directly that if $P(z, 1) = z - \alpha$, the critical points of $\hat{f}_a$ are at $z = 0$ and $z = 2\alpha$, independent of $a \in \mathbf{C}$. Together with the root α of P and the point at ∞, there are four marked points which must be permuted by any conjugacy. Moreover, the finite fixed point of $\hat{f}_a$ is at $z = a\alpha/(a + \alpha)$, so the cross ratio of 0, α, ∞, and the finite fixed point depends on $a \in \mathbf{C}$. We conclude that not all f_a are conjugate, and the lemma is proved. □

Corollary 10.4. *The second iterate map $\Phi_2 : M_d \to M_{d^2}$ does not extend continuously to $\overline{M}_d$.*

Proof. This is immediate from Lemmas 10.2 and 10.3. □

Higher iterates of $g_{a,t}$. We are now ready to complete the proof of Theorem 10.1.

Lemma 10.5. *For each $d \geq 2$, $n \geq 2$, and $a \in \mathbf{C}$, the limit of the iterates $(g_{a,t})^n \in \overline{\mathrm{Rat}}_{d^n}$ as $t \to 0$ is stable.*

Proof. Let n be an even integer. We have seen that the second iterates of $g_{a,t}$ converge to $f_a \in \overline{\mathrm{Rat}}_{d^2}$ as $t \to 0$. The n-th iterates of the family $g_{a,t}$ will converge to $(f_a)^{n/2}$ as $t \to 0$ by the continuity of the $n/2$-th iterate map at $f_a \notin I(d^2)$. In [De, §5], it was computed that $\mu_{f_a}(\{\infty\}) = 1/(d+1)$ (or it follows from Lemma 2.3). The roots of P are simple and are each mapped to ∞ by $\hat{f}_a$ with multiplicity

1, so Lemma 2.3 implies that $\mu_{f_a}(\{\alpha\}) = 1/(d+1)$ for each root α of P. Since μ_{f_a} is a probability measure, any other point of $\mathbf{P}^1$ must have mass $\leq 1/(d+1)$. By Propositions 3.2 and 3.3, we see that every iterate of f_a must be GIT stable. Therefore, all even iterates of the family $g_{a,t}$ have a stable limit as $t \to 0$.

Now let $n \geq 3$ be an odd integer. Lemma 2.6 implies that the composition map $\mathcal{C}_{d,d^{n-1}}$ is continuous at the pair $(g, (f_a)^{(n-1)/2})$. Consequently, the n-th iterates of $g_{a,t}$ converge to the point $g \circ (f_a)^{(n-1)/2}$ as $t \to 0$. The iterate formula of [De, Lemma 2.2] (see also §2) shows that

$$(f_a)^{(n-1)/2} = \prod_{k=0}^{\frac{n-1}{2}-1} \left(H_a \circ \hat{f}_a^k\right)^{d^{-k-1+(n-1)/2}} (\hat{f}_a)^{(n-1)/2}.$$

It will be useful to write $(\hat{f}_a)^{(n-1)/2}$ in terms of its coordinate functions $(\hat{f}_{az}^{(n-1)/2} : \hat{f}_{aw}^{(n-1)/2})$ so that we can compute the composition $g \circ (f_a)^{(n-1)/2}$. Indeed, substituting the coordinate functions for this iterate of f_a into the formula for g, we obtain

$$g \circ (f_a)^{(n-1)/2} = \left(\prod_{k=0}^{\frac{n-1}{2}-1} \left(H_a \circ \hat{f}_a^k\right)^{d^{-k+(n-1)/2}} \hat{f}_{aw}^{(n-1)/2} P(\hat{f}_{az}^{(n-1)/2}, \hat{f}_{aw}^{(n-1)/2}) : 0 \right),$$

where we have factored out all appearances of H_a. Notice, in particular, that the expression involving H_a appears as the d-th power of the same expression in $(f_a)^{(n-1)/2}$. The estimates on μ_{f_a} given in the previous paragraph together with Lemma 2.4 imply that the depth of any point for $(f_a)^k$ is no greater than $d^{2k}/(d+1)$. Therefore, the depths of the holes of the composition $g \circ (f_a)^{(n-1)/2}$ at ∞ or at the roots of P will not exceed $d(d^{n-1}/(d+1)) + 1$, where the added 1 comes from each of these holes being a simple zero of $\hat{f}_{aw}^{(n-1)/2}$. Notice that this bound is $< d^n/2$ for all even d and $< (d^n - 1)/2$ for all odd d. Therefore the holes at ∞ and the roots of P do not violate the stability criteria.

For any other point in $\mathbf{P}^1$, we know from the above that its depth as a hole of $(f_a)^{(n-1)/2}$ is no more than $d^{n-1}/(d+1)$; its depth as a hole of this composition cannot then exceed $d^n/(d+1) + d^{(n-1)/2}$, where the second term is the degree of $(\hat{f}_a)^{(n-1)/2}$. This upper bound on the depth is less than $d^n/2$ except when $d = 2$ and $n = 3$. In this special case, it is easy to check that the point $g \circ f_a \in \overline{\mathrm{Rat}}_8$ has holes of depth 3 at ∞ and the root of P, and a hole of depth at most 2 at any other point. It follows that the composition $g \circ (f_a)^{(n-1)/2}$ is always stable. □

Proof of Theorem 10.1. Let $f_{a,n} \in \overline{\mathrm{Rat}}_{d^n}$ denote the limit of the iterates $(g_{a,t})^n$ as $t \to 0$. By Lemma 10.5, $f_{a,n}$ is stable for all $n \geq 2$ and all $a \in \mathbf{C}$, and therefore it determines a unique point $[f_{a,n}]$ in $\overline{M}_{d^n}$. Furthermore, stability implies that $[f_{a,n}] = [f_{b,n}]$ if and only if $f_{a,n}$ and $f_{b,n}$ are conjugate.

From Lemma 10.2, the family $[g_{a,t}]$ converges in $\overline{M}_d$ as $t \to 0$ to a point independent of $a \in \mathbf{C}$, for every $d \geq 2$. Lemma 10.3 implies that $a \mapsto [f_{a,2}]$ is non-constant. To conclude the proof, we need to show that $a \mapsto [f_{a,n}] \in \overline{M}_{d^n}$ is non-constant for all $n \geq 2$.

Suppose first that n is even. Any conjugacy between $f_{a,n}$ and $f_{b,n}$ for $a \neq b$ must preserve the holes and conjugate $\hat{f}_a^{n/2}$ to $\hat{f}_b^{n/2}$. In particular, it must preserve the critical points of $\hat{f}_a$, located at the $2d-2$ solutions to $dP(z,1)z^{d-1} - P'(z,1)z^d = 0$, independent of $a \in \mathbf{C}$. Together with the hole at ∞, these give at least three marked points to be permuted by a conjugacy. On the other hand, the finite fixed points of $\hat{f}_a$ are the $d-1$ solutions to $z^d - zP(z,1) + aP(z,1) = 0$, and the set of these will vary with $a \in \mathbf{C}$. Consequently, the cross-ratio of three of the marked points with a moving fixed point of $\hat{f}_a$ also varies with $a \in \mathbf{C}$, so not all $f_{a,n}$ are conjugate.

Now suppose that $n \geq 3$ is odd. Any conjugacy between $f_{a,n}$ and $f_{b,n}$ must preserve holes of the same depth. The formula for $f_{a,n} = g \circ f_a^{(n-1)/2}$, given in the proof of Lemma 10.5, shows that for each $d > 2$, there are at least three holes at ∞ and the roots of P which are of the same depth and do not depend on $a \in \mathbf{C}$. If α is a root of P, then the preimages of α by $\hat{f}_a$ are also holes of $f_{a,n}$ and do depend on $a \in \mathbf{C}$. Therefore, the cross-ratio of ∞ with two roots of P and a moving preimage of α must vary with $a \in \mathbf{C}$, and so not all $f_{a,n}$ are conjugate. For $d = 2$, if $P(z,1) = z - \alpha$, note that the cross-ratio of the holes at ∞ and α with the pair of preimages of α by $\hat{f}_a$ is given by

$$\chi(a) = \frac{a + \alpha + \sqrt{(a-\alpha)^2 + 4\alpha(a-\alpha)}}{a + \alpha - \sqrt{(a-\alpha)^2 + 4\alpha(a-\alpha)}},$$

which depends on $a \in \mathbf{C}$. Therefore, not all $f_{a,n}$ are conjugate. □

The spaces $\hat{M}_d$ and $\overline{\overline{M}}_d$. We conclude by demonstrating that there cannot exist a continuous map $\overline{\overline{M}}_d \to \hat{M}_d$ which restricts to the identity on M_d, for any $d \geq 5$.

The following examples are from [De, Example 5.2]. Fix $d \geq 5$. Let $P = P(z,w)$ be a homogeneous polynomial of degree $d-2$ with distinct roots such that $P(0,1) \neq 0$, $P(1,0) \neq 0$, and P is monic as a polynomial in z. Let $g = (w^2 P(z,w) : 0)$. Then $g \in I(d)$ is stable for all $d \geq 6$ and semistable for $d = 5$ since the depth at ∞ is $2 < d/2$. Therefore g defines a unique point $[g]$ in $\overline{M}_d$.

For each $a \in \mathbf{C}$ and $t \in [0,1]$, consider the family

$$h_{a,t} = (atz^d + w^2 P(z,w) : tz^d) \in \mathrm{Rat}_d,$$

so that $h_{a,t} \to g$ in $\overline{\mathrm{Rat}}_d$ as $t \to 0$, and therefore $[h_{a,t}] \to [g]$ in $\overline{M}_d$ for every $a \in \mathbf{C}$. Computing second iterates and taking a limit as $t \to 0$, we obtain,

$$(h_{a,t})^2 \to h_a := (aw^{2d} P(z,w)^d : w^{2d} P(z,w)^d) \in \overline{\mathrm{Rat}}_{d^2}.$$

Note that h_a is stable for all $a \in \mathbf{C}$ and all $d \geq 5$ since the depth at ∞ is $2d < d^2/2$. Therefore each h_a determines a point $[h_a] \in \overline{M}_{d^2}$ and $[h_a] = [h_b]$ if and only if they lie in the same $\mathrm{PSL}_2\mathbf{C}$-orbit. Write $h_a = H_a \hat{f}_a$. Since P has at least 3 distinct roots and the constant $\hat{f}_a \equiv a$ depends on a, we see that only finitely many of the conjugacy classes $[h_a]$ can coincide.

For each $a \in \mathbf{C}$ such that $P(a,1) \neq 0$, the point h_a is not in $I(d^2)$. Therefore, the measures $\mu_{h_{a,t}}$ converge weakly as $t \to 0$ to

$$\mu_{h_a} = \frac{2}{d}\delta_\infty + \frac{1}{d} \sum_{P(z,1)=0} \delta_z,$$

a measure which is independent of a. The measure μ_{h_a} has no atoms of mass $\geq 1/2$, and so it has a well-defined barycenter. Let $\mu = A_* \mu_{h_a}$ be a barycentered measure for some $A \in \mathrm{PSL}_2\mathbf{C}$. Consequently, for every $a \in \mathbf{C}$ such that $P(a, 1) \neq 0$, the family $[h_{a,t}]$ converges in $\overline{\overline{M}}_d$ as $t \to 0$ to the pair $([g], \mu) \in \overline{M}_d \times \overline{M^1_{bc}(\hat{\mathbf{C}})}/\mathrm{SO}(3)$. On the other hand, the limits of $[h_{a,t}]$ as $t \to 0$ in $\hat{M}_d$ depend on $a \in \mathbf{C}$ since the second iterates have distinct limits in $\overline{M}_{d^2}$.

Question. Does there exist a continuous projection $\hat{M}_d \to \overline{\overline{M}}_d$ restricting to the identity on M_d for all $d \geq 2$? That is, do the GIT stable limits of an unbounded sequence in M_d (and all its iterates) determine the limiting barycentered measure of maximal entropy?

11. Final remarks

The choices made in the definitions of $\hat{M}_d$ and $\overline{\overline{M}}_d$ reflect the following desirable properties in degree 2. A compactification X of the moduli space of quadratic rational maps should satisfy:

- iteration is well-defined on X,
- there exists a projection from X to Milnor's $\overline{M}_2 \simeq \mathbf{P}^2$,
- there exists a projection from X to the space of barycentered measures $\overline{M^2_{bc}(\hat{\mathbf{C}})}/\mathrm{SO}(3)$.

The second condition ensures that there is a projection of the boundary of M_2 to the moduli spaces $M_1 \amalg M_0$, as the conjugacy classes of lower-degree maps arise naturally as limits of degenerating families in M_2. The third condition about the barycentered measures reflects the geometry of the rational maps. Each rational map f (together with its measure of maximal entropy) determines a hermitian metric on the tangent bundle of $\hat{\mathbf{C}}$, up to scale, with curvature equal to the distribution $4\pi\mu_f$. Conjugate rational maps define isometric spheres (when the scale is fixed). See, for example, [De, §6]. Given an unbounded family in M_2 and the associated family of metrics on $\hat{\mathbf{C}}$, the choice of barycentered representatives corresponds to fixing the diameter of the spheres when taking limits in the Gromov-Hausdorff topology. In this language, it would follow from the propositions of §9 that the limiting metrics at the boundary of M_2 are isometric to either (i) compact convex polyhedra (with countably many vertices) or (ii) "needles" (where the curvature is concentrated at two points).

Finally, much can be said about the dynamical properties of unbounded families in M_2, particularly when restricted to a given hyperbolic component. It would be interesting to understand better the explicit examples given in Section 6 which appeared first in [Ep].

Example. Suppose $\{f_t : t \in (0, 1]\}$ is a family of quadratic rational maps with $f_1(z) = z^2 - 1$, and such that

(i) the critical points of f_t are at 0 and ∞ for all $t \in (0, 1]$,
(ii) the critical point at 0 is in a cycle $0 \mapsto -1 \mapsto 0$ for all t, and
(iii) there is an attracting fixed point of multiplier $\alpha(t) \to -1$ as $t \to 0$.

Then this family is contained in the hyperbolic component of f_1. Recall that the Julia set of f_1 is the basilica. The family f_t can be expressed as

$$f_t(z) = \frac{z^2 - 1}{c(t)z^2 + 1},$$

for a function c with $c(1) = 0$. It can be computed directly that the triple of fixed point multipliers of f_t tends to $\{\infty, -1, -1\}$ as $c(t)$ descends from 0 to -1. In the limit, the second iterate of f_t converges to $-2z^2/(z^2+1)$ locally uniformly on $\hat{\mathbf{C}} - \{-1, 1\}$, which is conjugate to $G_{-1}(z) = z - 1 + 1/z$ and to the polynomial $z^2 + 1/4$ with a parabolic fixed point. It follows that the τ^2-value (as defined before the statement of Theorem 5.4) for this family is equal to 1.

The Julia sets of f_t appear to converge (in the Hausdorff topology) to the cauliflower Julia set of $z^2 + 1/4$. The divergence of this family $[f_t]$ in M_2 is an illustration of the obstruction to mating the polynomial f_1 with the root-point of the 1/2-limb of the Mandelbrot (this limb is its own conjugate).

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