

BIFURCATION MEASURES AND QUADRATIC RATIONAL MAPS

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ABSTRACT. We study critical orbits and bifurcations within the moduli space M_2 of quadratic rational maps, $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. We focus on the family of curves, $\text{Per}_1(\lambda) \subset M_2$ for $\lambda \in \mathbb{C}$, defined by the condition that each $f \in \text{Per}_1(\lambda)$ has a fixed point of multiplier λ . We prove that the curve $\text{Per}_1(\lambda)$ contains infinitely many postcritically-finite maps if and only if $\lambda = 0$, addressing a special case of [BD2, Conjecture 1.4]. We also show that the two critical points of f define distinct bifurcation measures along $\text{Per}_1(\lambda)$.

1. INTRODUCTION

In this article, we study the dynamics of holomorphic maps $f : \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$ of degree 2. We concentrate our analysis on the lines $\text{Per}_1(\lambda)$ within the moduli space $M_2 \simeq \mathbb{C}^2$ of quadratic rational maps, introduced by Milnor in [Mi1]. For each $\lambda \in \mathbb{C}$, $\text{Per}_1(\lambda)$ is the set of all (conformal conjugacy classes of) maps f with a fixed point p at which $f'(p) = \lambda$; so $\text{Per}_1(0)$ is the family of maps conjugate to a polynomial.

Our first main result addresses a special case of Conjecture 1.4 of [BD2]. (See also the corrected version in [De3, §5.4] and this case presented in [Si2, §6.5].) The conjecture aims to classify the algebraic subvarieties of M_d containing a Zariski-dense set of postcritically-finite maps, for each degree $d \geq 2$. By definition, a rational map of degree d is postcritically finite if each of its $2d - 2$ critical points has a finite forward orbit. It is known that the postcritically finite maps form a Zariski-dense subset of M_d in every degree $d \geq 2$, but the subvarieties intersecting many of them are expected to be quite special.

Theorem 1.1. *The curve $\text{Per}_1(\lambda)$ in M_2 contains infinitely many postcritically-finite maps if and only if $\lambda = 0$.*

This result is the exact analog of Theorem 1.1 in [BD2] that treated cubic polynomials. As in that setting, one implication is easy: if $\lambda = 0$, the curve $\text{Per}_1(0)$ defines the family of quadratic polynomials, and it contains infinitely many postcritically-finite maps (by a standard application of Montel's theorem on normal families). The converse direction is more delicate; its proof, though similar in spirit to that of [BD2, Theorem 1.1], required different techniques, more in line with our work for the Lattès family of [DWY].

The second theme of this paper is a study of the bifurcation locus in the curves $\text{Per}_1(\lambda)$; refer to §2 for definitions. For each fixed $\lambda \neq 0$, we work with an explicit parametrization

Date: April 23, 2015.

2010 Mathematics Subject Classification. Primary 37F45; Secondary 37F10.

The research was supported by the National Science Foundation.

of $\text{Per}_1(\lambda)^{cm}$, a double cover of $\text{Per}_1(\lambda)$ consisting of maps with marked critical points:

$$f_t(z) = \frac{\lambda z}{z^2 + tz + 1}$$

with $t \in \mathbb{C}$. The map f_t has a fixed point at $z = 0$ with multiplier λ ; the critical points of f_t are $\{\pm 1\}$ for all t ; note that f_t is conjugate to f_{-t} via the conjugacy $z \mapsto -z$ interchanging the two critical points. Each critical point determines a finite bifurcation measure on $\text{Per}_1(\lambda)^{cm}$, which we denote by μ_λ^+ and μ_λ^- . (The symmetry of f_t implies that $\mu_\lambda^- = A_* \mu_\lambda^+$ for $A(t) = -t$.) Our main result in this direction is:

Theorem 1.2. *For every $\lambda \neq 0$, we have $\mu_\lambda^+ \neq \mu_\lambda^-$ in $\text{Per}_1(\lambda)^{cm}$.*

Theorem 1.2 is not unexpected. For any λ , the two critical points should behave independently. In fact, it is not difficult to show that the critical points *cannot* satisfy any dynamical relation of the form $f^n(+1) \equiv f^m(-1)$ along $\text{Per}_1(\lambda)^{cm}$; see Corollary 2.6. However, computational experiments suggested some unexpected alignment of the two bifurcation loci, $\text{Bif}^+ = \text{supp } \mu_\lambda^+$ and $\text{Bif}^- = \text{supp } \mu_\lambda^-$, for certain values of λ . For example, for values of λ near -4 , the two bifurcation loci appear remarkably similar. See Figure 2.3 and Question 2.4.

A key ingredient in our proof of Theorem 1.1 is an equidistribution statement, that parameters t where the critical point ± 1 has finite forward orbit for f_t will be uniformly distributed with respect to the bifurcation measure μ_λ^\pm . Post-critically finite maps have algebraic multipliers, so it suffices to study the case where $\lambda \in \overline{\mathbb{Q}}$; to prove the equidistribution result, we rely on the arithmetic methods introduced in [BR1] and [FRL]. But there are two features of $\text{Per}_1(\lambda)$ that distinguish it from a series of recent articles on this theme (see e.g. [BD1, BD2, GHT1, GHT2, FG]); in particular, we could *not* directly apply the existing arithmetic equidistribution theorems for points of small height on \mathbb{P}^1 .

- (1) The bifurcation locus can be noncompact, and the proof that the potential functions for the bifurcation measures are continuous across $t = \infty$ is more delicate (we show this in Theorem 4.1, with the method we used in [DWY]); and
- (2) the canonical height function defined on $\text{Per}_1(\lambda)$ (associated to each critical point) is only “quasi-adelic,” meaning that it may have nontrivial contributions from infinitely many places of any number field containing λ .

Because of (2), we use a modification of the original equidistribution result (and of its proofs, following [BR2, FRL]) that appears in [Ye]. We deduce the following result. (The full statement of this theorem appears as Theorem 7.1.)

Theorem 1.3. *For every $\lambda \in \overline{\mathbb{Q}} \setminus \{0\}$ with λ not a root of unity, or for $\lambda = 1$, the set*

$$\text{Preper}_\lambda^+ = \{t \in \text{Per}_1(\lambda)^{cm} : +1 \text{ has finite forward orbit for } f_t\}$$

is equidistributed with respect to μ_λ^+ ; similarly for Preper_λ^- and μ_λ^- . More precisely, for any non-repeating sequence of finite sets $S_n \subset \text{Preper}_\lambda^+$, the discrete probability measures

$$\mu_n = \frac{1}{|G \cdot S_n|} \sum_{t \in G \cdot S_n} \delta_t$$

converge weakly to the measure μ_λ^+ , where $G = \text{Gal}(\overline{\mathbb{Q}(\lambda)}/\mathbb{Q}(\lambda))$.

Note that the sets $\text{Preper}_\lambda^\pm$ are invariant under the action of the Galois group G : if $+1$ is preperiodic for t_0 , then $+1$ is preperiodic for all t in its Galois orbit, since these parameters are solutions of an equation of the form $f_t^n(+1) = f_t^m(+1)$, with coefficients in $\mathbb{Q}(\lambda)$. A “classical” setting of Theorem 1.3 would be to take S_n as the full set of solutions to the equation $f_t^n(+1) = f_t^m(+1)$, with any sequence $0 \leq m = m(n) < n$ as $n \rightarrow \infty$.

The equidistribution of Theorem 1.3 for $\lambda = 0$ is well known. It was first shown by Levin (in the classical sense of equidistribution) [Le], and it was shown in the stronger (arithmetic) form by Baker and Hsia [BH, Theorem 8.15]. In fact the equidistribution of Theorem 1.3 holds at each place v of the number field $\mathbb{Q}(\lambda)$, on an appropriately-defined Berkovich space $\mathbb{P}_{\mathbb{C}_v}^{1,an}$, for sets S_n with canonical height tending to 0; see Theorem 7.1.

Outline of the article. In Section 2, we introduce the families $\text{Per}_1(\lambda)^{cm}$, the bifurcation loci within these curves, and the bifurcation measures μ_λ^+ and μ_λ^- . We prove the independence of the critical points (Corollary 2.6) and pose Question 2.4 about the bifurcation loci. In Section 3, we give the proof of Theorem 1.2. In Section 4, we prove that the measures μ_λ^+ and μ_λ^- have continuous potentials on all of \mathbb{P}^1 , assuming that λ is not “too close” to a root of unity (Theorem 4.1). In Section 5, we prove a non-archimedean convergence statement, analogous to Theorem 4.1, for $\lambda \in \overline{\mathbb{Q}}$ that are not equal to roots of unity. In Section 6, we introduce the homogeneous bifurcation sets and compute their homogeneous capacities. In Section 7, we prove the needed equidistribution theorems, including Theorem 1.3. In Section 8, we complete the proof of Theorem 1.1.

Acknowledgements. We would like to thank Ilia Binder, Dragos Ghioca, and Curt McMullen for helpful comments. We also thank Suzanne Boyd for help with her program Dynamics Explorer, used to generate all images in this article.

2. BIFURCATION LOCUS IN THE CURVE $\text{Per}_1(\lambda)$

The moduli space M_2 is the space of conformal conjugacy classes of quadratic rational maps $f : \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$, where two maps are equivalent if they are conjugate by a Möbius transformation; see [Mi1, Si1]. In this section, we provide some basic results about the bifurcation locus within the curves $\text{Per}_1(\lambda)$ in M_2 . By definition, $\text{Per}_1(\lambda)$ is the set of conjugacy classes of quadratic rational maps with a fixed point of multiplier λ . In Milnor’s parameterization of $M_2 \simeq \mathbb{C}^2$, using the symmetric functions in the three fixed point multipliers, each $\text{Per}_1(\lambda)$ is a line [Mi1, Lemma 3.4]. For $\lambda = 0$, $\text{Per}_1(0)$ is the family of quadratic polynomials, usually parametrized by $f_t(z) = z^2 + t$ with $t \in \mathbb{C}$.

Remark. A number of results have appeared since [Mil] that address features of the bifurcations within $\text{Per}_1(\lambda)$. For example, when $|\lambda| < 1$, it is known that the bifurcation locus is homeomorphic to the boundary of the Mandelbrot set; this follows from the straightening theorem of [DH] (see the remark following Corollary 3.4 of [GK]). See [Pe, Uh, BEE] for more in the setting of λ a root of unity. Berteloot and Gauthier have recently studied properties of the bifurcation current on M_2 near infinity [BG].

2.1. Bifurcations. Let X be a complex manifold. A *holomorphic family* of rational maps parametrized by X is a holomorphic map

$$f : X \times \mathbb{P}^1 \rightarrow \mathbb{P}^1.$$

We often write f_t for the restriction $f(t, \cdot) : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ for each $t \in X$. A holomorphic family $\{f_t, t \in X\}$ of rational functions of degree $d \geq 2$ is *stable* at $t_0 \in X$ if the Julia sets $J(f_t)$ are moving holomorphically in a neighborhood of t_0 . In particular, $f_{t_0}|_{J(f_{t_0})}$ is topologically conjugate to all nearby maps when restricted to their Julia sets (and the Julia sets are homeomorphic) [MSS, Mc2]. An equivalent characterization of stability, upon passing to a branched cover of X where the critical points $c_1, c_2, \dots, c_{2d-2}$ can be labelled holomorphically, is that the sequence of holomorphic maps

$$\{t \mapsto f_t^n(c_i(t))\}$$

forms a normal family for each i on some neighborhood of t_0 . The failure of normality can be quantified with the construction of a positive $(1, 1)$ -current on the parameter space X , as follows.

Suppose that we can express f_t in homogeneous coordinates, as a holomorphic family

$$F_t : \mathbb{C}^2 \rightarrow \mathbb{C}^2$$

for $t \in X$. Assume we are given holomorphic functions $\tilde{c}_i : X \rightarrow \mathbb{C}^2 \setminus \{(0, 0)\}$ projecting to the critical points $c_i(t) \in \mathbb{P}^1$ of f_t . We define the *bifurcation current* of c_i on X by

$$T_i := dd^c \left(\lim_{n \rightarrow \infty} \frac{1}{d^n} \log \|F_t^n(\tilde{c}_i(t))\| \right).$$

The current vanishes if and only if the family $\{t \mapsto f_t^n(c_i(t))\}$ is normal. In particular, $T_i = 0$ for all i if and only if the family is stable. In fact, the family F_t and the functions \tilde{c}_i can always be defined locally on X , after passing to a branched cover where the critical points can be labelled, and the current T_i is independent of the choice of \tilde{c}_i and the homogenization F_t . When the parameter space X has dimension 1, note that the current T_i is a measure (where dd^c is simply the Laplacian), and we will refer to it as the *bifurcation measure*. (See [De1, De2, DF].)

The *bifurcation locus* is the set of parameters in X where f_t fails to be stable. It coincides with the union of the supports of the bifurcation currents T_1, \dots, T_{2d-2} . The following lemma is a straightforward application of Montel's theorem; for a proof of Montel's theorem, see [Mi2, §3].

Lemma 2.1. *Let $f : X \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a holomorphic family of rational functions of degree > 1 , with marked critical point $c : X \rightarrow \mathbb{P}^1$. Let T be the bifurcation current of c , and assume that $T \neq 0$. Then there are infinitely many parameters $t \in X$ where $c(t)$ has finite orbit for f_t .*

Proof. Fix $t_0 \in \text{supp } T$. Choose any repelling periodic cycle for f_{t_0} of period ≥ 3 that is not in the forward orbit of $c(t_0)$. By the Implicit Function Theorem, the repelling cycle persists in a neighborhood U of t_0 . If the orbit of the critical point $c(t)$ were disjoint from the cycle for all $t \in U$, then Montel's Theorem would imply that $\{t \mapsto f_t^n(c(t))\}$ forms a normal family on U . This contradicts the fact that t_0 lies in the support of T . Consequently, there is a parameter $t_1 \in U \setminus \{t_0\}$ where $c(t_1)$ is preperiodic for f_{t_1} . Shrinking the neighborhood U , we obtain an infinite sequence of such parameters converging to t_0 . \square

Example 2.1. For the family of quadratic polynomials, $f_t(z) = z^2 + t$, there is only one critical point (at $z = 0$) inducing bifurcations. The associated bifurcation current defines a measure on the parameter space ($t \in \mathbb{C}$). It is equal, up to a normalization factor, to the harmonic measure supported on the boundary of the Mandelbrot set \mathcal{M} [De1, Example 6.1]. In this case, there is no need to pass to homogeneous coordinates; a potential function for the normalized bifurcation measure is given by

$$(2.1) \quad G_{\mathcal{M}}(t) = \lim_{n \rightarrow \infty} \frac{1}{2^n} \log^+ |f_t^n(t)|$$

for $t \in \mathbb{C}$.

2.2. The bifurcation locus in the critically-marked curve. Fix a complex number $\lambda \neq 0$, and set

$$f_{\lambda,t}(z) = \frac{\lambda z}{z^2 + tz + 1}$$

for all $t \in \mathbb{C}$. (We will often write f_t for $f_{\lambda,t}$ when the dependence on λ is clear.) Then f_t has critical points at $c_+(t) = +1$ and $c_-(t) = -1$ for all $t \in \mathbb{C}$. Since f_t is conjugate to f_{-t} by $z \mapsto -z$, the family f_t parametrizes a degree-2 branched cover of the curve $\text{Per}_1(\lambda) \subset \mathcal{M}_2$, which we denote by $\text{Per}_1(\lambda)^{cm}$; the cm in the superscript stands for ‘‘critically marked.’’

We define the bifurcation currents associated to the critical points c_+ and c_- as in §2.1. As the parameter space is 1-dimensional, the currents are in fact measures; we denote these bifurcation measures by μ_λ^+ and μ_λ^- . The supports will be denoted by

$$\text{Bif}^+ = \text{supp } \mu_\lambda^+ \quad \text{and} \quad \text{Bif}^- = \text{supp } \mu_\lambda^-.$$

These bifurcation measures have globally-defined potential functions. We set

$$(2.2) \quad F_t(z_1, z_2) = (\lambda z_1 z_2, z_1^2 + t z_1 z_2 + z_2^2)$$

and

$$(2.3) \quad H_\lambda^\pm(t) = \lim_{n \rightarrow \infty} \frac{1}{2^n} \log \|F_t^n(\pm 1, 1)\|,$$

so that

$$\mu_\lambda^+ = \frac{1}{2\pi} \Delta H_\lambda^+ \quad \text{and} \quad \mu_\lambda^- = \frac{1}{2\pi} \Delta H_\lambda^-.$$

Since $f_{-t}(z) = -f_t(-z)$, we see that

$$H_\lambda^-(t) = H_\lambda^+(-t)$$

and $\text{Bif}^- = -\text{Bif}^+$.

The following proposition follows from the observations of Milnor in [Mi1].

Proposition 2.2. *The bifurcation loci Bif^+ and Bif^- are nonempty for all $\lambda \neq 0$. They are compact in $\text{Per}_1(\lambda)^{cm}$ if and only if $|\lambda| \neq 1$ or $\lambda = 1$.*

Moreover, when the bifurcation locus $\text{Bif} = \text{Bif}^+ \cup \text{Bif}^-$ is compact, the unbounded stable component consists of maps for which both critical points lie in the basin of an attracting (or parabolic, in the case of $\lambda = 1$) fixed point.

Proof. The three fixed points of f_t lie at 0 and

$$Z_\pm(t) = \frac{-t \pm \sqrt{t^2 - 4(1 - \lambda)}}{2}.$$

The set of fixed point multipliers is

$$\{\lambda, (1 - Z_\pm(t)^2)/\lambda\}.$$

For each fixed λ , there are well-defined branches of the square root for $|t| \gg 0$ so that Z_\pm define analytic functions near infinity, with $Z_+(t) \rightarrow 0$ and $Z_-(t) \rightarrow \infty$ as $t \rightarrow \infty$. The set of fixed point multipliers converges to $\{\lambda, 1/\lambda, \infty\}$ as $t \rightarrow \infty$. (Compare [Mi1, Lemma 4.1].)

We first observe that Bif^+ and Bif^- are nonempty. Note that the two fixed points $Z_+(t)$ and $Z_-(t)$ must collide at $t = \pm 2\sqrt{1 - \lambda}$. Consequently, their multipliers are 1 at that point, while they cannot be persistently equal to 1. By the characterizations of stability [Mc2, Theorem 4.2], the parameters $t = \pm 2\sqrt{1 - \lambda}$ will lie in the bifurcation locus.

Suppose that $|\lambda| = 1$ with $\lambda \neq 1$. Then

$$f'_t(Z_+(t)) = \frac{1}{\lambda} \left(1 - \frac{(1 - \lambda)^2}{t^2} + O\left(\frac{1}{t^4}\right) \right)$$

for t large. Fixing any large value of $R > 0$ and letting the argument of $t = Re^{i\theta}$ vary in $[0, 2\pi]$, the absolute value of $(1 - (1 - \lambda)^2/t^2)$ will fluctuate around 1. Consequently, the multiplier $f'_t(Z_+(t))$ will have absolute value 1 for some parameter t with $|t| = R$, for all sufficiently large R . Again by the characterizations of stability, all such parameters will lie in the bifurcation locus. Consequently, Bif^+ and Bif^- are unbounded.

For $|\lambda| \neq 1$, f_t has an attracting fixed point (of multiplier λ or $\approx 1/\lambda$) for all t large. Both critical points will lie in its basin of attraction for all t large, demonstrating stability

of the family f_t . To see this, we may place the three fixed points of f_t at $\{0, 1, \infty\}$ so that f_t is conjugate to the rational function

$$g_t(z) = z \frac{(1 - \lambda)z + \lambda(1 - \beta)}{\beta(1 - \lambda)z + 1 - \beta}.$$

where $\beta = \beta(t) = \beta(-t) \approx 1/\lambda$ is the multiplier of the fixed point at ∞ . In this form, g_t will converge (locally uniformly on $\widehat{\mathbb{C}} \setminus \{1\}$) to the linear map $z \mapsto \lambda z$ as $t \rightarrow \infty$. In particular, there is a neighborhood U containing the attracting fixed point and the point $z = \lambda$ mapped compactly inside itself by g_t for all t large. On the other hand, we can explicitly compute the critical values $v_+(t), v_-(t)$ of g_t and determine that $\lim_{t \rightarrow \infty} v_{\pm}(t) = \lambda$. Consequently, the critical points lie in the basin of attraction for all t large enough, and the bifurcation locus must be bounded.

For $\lambda = 1$, it is convenient to conjugate f_t by $1/(tz)$, to express it in the form

$$g_t(z) = z + 1 + \frac{1}{t^2 z}$$

with a parabolic fixed point at $z = \infty$. In these coordinates, g_t converges (locally uniformly on $\widehat{\mathbb{C}} \setminus \{0\}$) to the translation $z \mapsto z + 1$ as $t \rightarrow \infty$. Again, we compute explicitly the critical values of g_t and their limit as $t \rightarrow \infty$; in this case, they converge to the point $z = 1$. As such, they both lie in the basin of the parabolic fixed point for t large. \square

2.3. Comparing the two bifurcation loci. We begin with a simple observation.

Lemma 2.3. *For $\operatorname{Re} \lambda > 1$, then Bif^+ and Bif^- intersect only at the two points $t = \pm 2\sqrt{1 - \lambda}$. For $|\lambda| < 1$, the sets Bif^+ and Bif^- are disjoint.*

Proof. For $\operatorname{Re} \lambda > 1$ and $t \neq \pm 2\sqrt{1 - \lambda}$, f_t has at least one attracting fixed point. The proof is immediate from the index formula for fixed point multipliers (see [Mi2]). For all

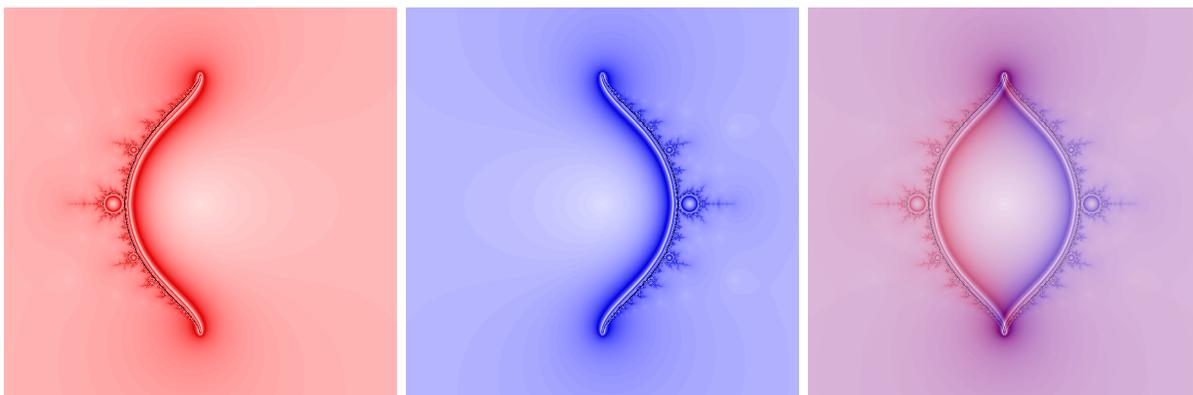


FIGURE 2.1. The bifurcation loci in $\operatorname{Per}_1(2)^{cm}$. At left, an illustration of Bif^+ where $|\operatorname{Re} t|, |\operatorname{Im} t| \leq 3$; the color shading records a rate of convergence of the critical point c_+ to an attracting cycle. In the middle, Bif^- in the same region. At right, the two images superimposed. By Lemma 2.3, $\operatorname{Bif}^+ \cap \operatorname{Bif}^- = \{2i, -2i\}$.

such t then, at least one of the critical points must be stable. This shows that $\text{Bif}^+ \cap \text{Bif}^- \subset \{\pm 2\sqrt{1-\lambda}\}$. A straightforward calculation shows that the fixed point multipliers at $t = 0$ are $\{\lambda, -1 + 2/\lambda, -1 + 2/\lambda\}$, so f_0 must have two distinct attracting fixed points whenever $\text{Re } \lambda > 1$. Consequently, $t = 0$ cannot lie in the unbounded stable component (where there is a unique attracting fixed point by Proposition 2.2). For topological reasons, then, and since $\text{Bif}^+ = -\text{Bif}^-$, the intersection of Bif^+ and Bif^- must consist of at least two points, concluding the proof that $\text{Bif}^+ \cap \text{Bif}^- = \{\pm 2\sqrt{1-\lambda}\}$.

For $|\lambda| < 1$, the point 0 is an attracting fixed point for all t , and its immediate attracting basin must contain at least one critical point. Thus, for all t , at least one critical point remains in an attracting basin under perturbation, and so it is stable; this implies that $\text{Bif}^+ \cap \text{Bif}^- = \emptyset$. \square

For certain values of λ , the bifurcation loci Bif^\pm are remarkably similar. Though Bif^+ does not appear to be equal to Bif^- for any value of λ , the differences can be subtle. We include illustrations of Bif^\pm in $\text{Per}_1(\lambda)^{cm}$ for three values of λ in Figures 2.1-2.3. In Figure 2.4, we illustrate the distribution of the parameters where the critical points are periodic, with $\lambda = -4$; these parameters converge to the bifurcation measures by Theorem 1.3. Theorem 1.2 states that the two measures μ_λ^+ and μ_λ^- are distinct for all λ .

Question 2.4. *Are the bifurcation loci Bif^+ and Bif^- distinct in $\text{Per}_1(\lambda)^{cm}$ for all λ ? What explains their near-coincidence for parameters such as $\lambda = -4$?*

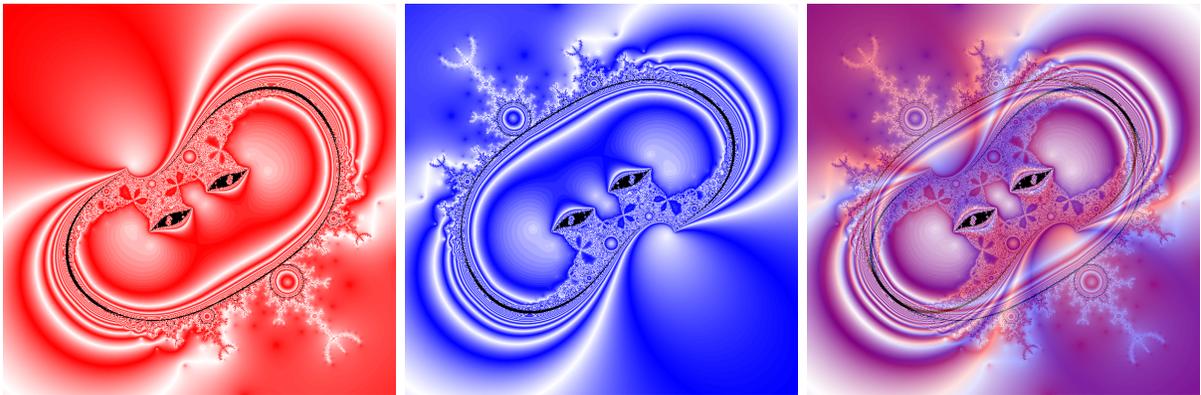


FIGURE 2.2. The bifurcation loci in $\text{Per}_1(1.1i)^{cm}$. At left, an illustration of Bif^+ where $|\text{Re } t|, |\text{Im } t| \leq 6$; the color shading records a rate of convergence of the critical point c_+ to an attracting cycle. In the middle, Bif^- in the same region. At right, the two images superimposed.

2.4. Dynamical independence of the critical points. We conclude this section with the observation that the two critical points $c_+ = +1$ and $c_- = -1$ must be dynamically independent along $\text{Per}_1(\lambda)^{cm}$. We define,

$$\text{Preper}_\lambda^\pm = \{t \in \text{Per}_1(\lambda)^{cm} : \pm 1 \text{ has finite forward orbit for } f_t\}.$$

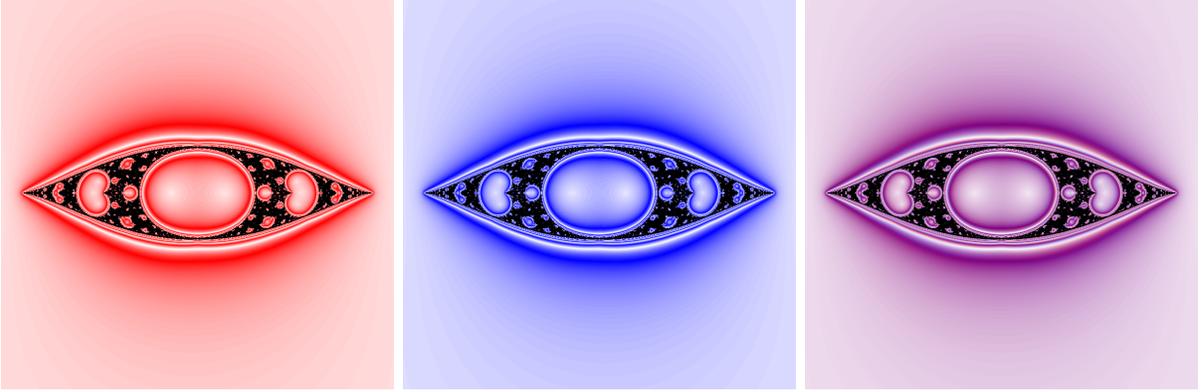


FIGURE 2.3. The bifurcation loci in $\text{Per}_1(-4)^{cm}$. At left, an illustration of Bif^+ where $|\text{Re } t|, |\text{Im } t| \leq 5$; the color shading records a rate of convergence of the critical point c_+ to an attracting cycle. In the middle, Bif^- in the same region. At right, the two images superimposed.

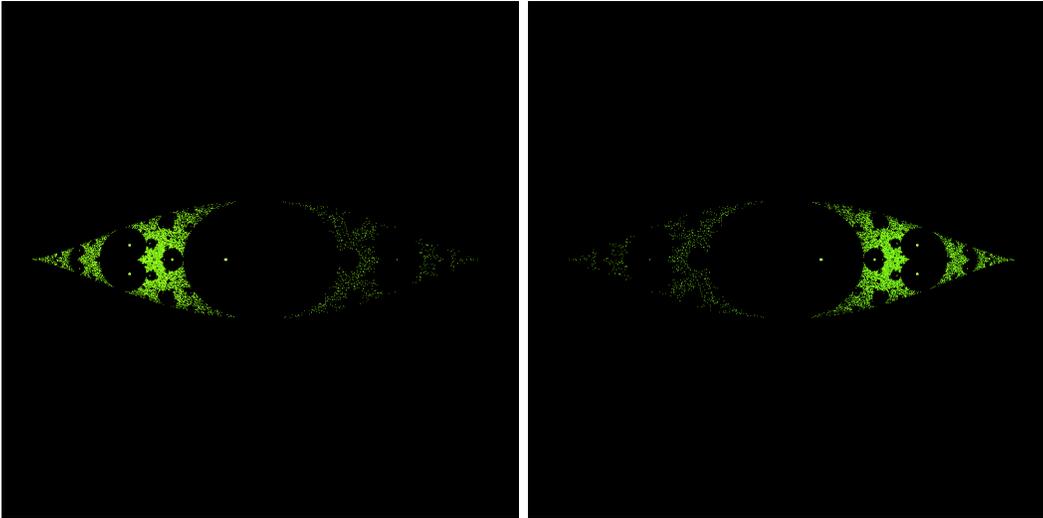


FIGURE 2.4. At left, a plot of parameters t such that $f_t^n(+1) = +1$ in $\text{Per}_1(-4)^{cm}$, with $n \leq 5000$. By Theorem 1.3, these parameters are equidistributed with respect to μ_λ^+ as $n \rightarrow \infty$. At right, the parameters where critical point -1 is periodic, equidistributed with respect to μ_λ^- .

Proposition 2.5. *For all $\lambda \in \mathbb{C}$, we have*

$$\text{Preper}_\lambda^+ \neq \text{Preper}_\lambda^-$$

in $\text{Per}_1(\lambda)^{cm}$.

Proof. The case of $\lambda = 0$ is easy. The curve $\text{Per}_1(0)^{cm}$ has two irreducible components; each may be parameterized by $f_t(z) = z^2 + t$ with one critical point at ∞ and the other

at 0. The critical point at ∞ is fixed for all t , while the orbit of 0 is infinite for all but countably many parameters t .

For $0 < |\lambda| \leq 1$, a stronger statement is true:

$$\text{Preper}_\lambda^+ \cap \text{Preper}_\lambda^- = \emptyset$$

in $\text{Per}_1(\lambda)^{cm}$. Indeed, for every $f \in \text{Per}_1(\lambda)^{cm}$, at least one critical point must have infinite forward orbit, as it is attracted to (or accumulates upon) the fixed point with multiplier λ (or on the boundary of the Siegel disk in case the fixed point is of Siegel type). See, for example, [Mi2, Corollaries 14.4 and 14.5].

For the remainder of this proof, assume that $|\lambda| > 1$ and that $\text{Preper}_\lambda^+ = \text{Preper}_\lambda^-$. Lemma 2.1 shows that Bif^+ is contained in the set of accumulation points of Preper_λ^+ . But, in fact, the characterizations of stability (as in [Mc2, Chapter 4]) imply that elements of Preper_λ^+ cannot accumulate in a stable region. So we have $\text{Bif}^+ = \text{Bif}^-$.

As before, parameterize $\text{Per}_1(\lambda)^{cm}$ as

$$f_t(z) = \frac{\lambda z}{z^2 + tz + 1}.$$

Recall from Proposition 2.2 that f_t has an attracting fixed point for all t near ∞ , with multiplier converging to $1/\lambda$ as $t \rightarrow \infty$. Moreover, both critical points lie in its basin of attraction for all t in the unbounded stable component. In particular, each f_t in the unbounded stable component has a unique attracting fixed point.

The multiplier of the unique attracting fixed point of f_t defines a holomorphic function from the unbounded stable component to the unit disk. Recall that the fixed-point multiplier cannot be equal to $1/\lambda$ for any f_t , and it will converge to $1/\lambda$ if and only if $t \rightarrow \infty$ in $\text{Per}_1(\lambda)^{cm}$ [Mi1]. Moreover, since $\text{Bif}^+ = \text{Bif}^-$, the multiplier must converge to 1 in absolute value as t tends to the bifurcation locus. Consequently, the multiplier of the attracting fixed point determines a proper holomorphic map from the unbounded stable component to the unit disk punctured at $1/\lambda$. It follows that each preimage of the line segment $[0, 1/\lambda)$ defines a path from a parameter t_0 to infinity in this unbounded stable component. In fact, $t_0 = \lambda - 2$ is the unique parameter where the critical point $+1$ is fixed (and similarly, $2 - \lambda$ is the unique parameter at which -1 is fixed), so f_{t_0} is conjugate to a polynomial; we have just shown that t_0 lies in this unbounded stable component.

As the orbit of the critical point -1 for f_{t_0} must converge towards the fixed critical point $+1$, we see that -1 has infinite orbit. This contradicts our assumption that $\text{Preper}_\lambda^+ = \text{Preper}_\lambda^-$ and completes the proof. \square

As an immediate corollary of Proposition 2.5, we see that the two critical points are dynamically independent on $\text{Per}_1^{cm}(\lambda)$ for any $\lambda \neq 0$, in the sense of generalized critical orbit relations introduced by Baker and DeMarco in [BD2, §1.4].

Corollary 2.6. *Fix $\lambda \in \mathbb{C}^*$. The critical points $c_+ = +1$ and $c_- = -1$ cannot satisfy any dynamical relations along $\text{Per}_1^{cm}(\lambda)$. In particular, for each pair of integers $n, m \geq 0$, there exists $f_t \in \text{Per}_1(\lambda)^{cm}$ so that $f_t^n(c_+) \neq f_t^m(c_-)$.*

Proof. Dynamical dependence of points c_+ and c_- , in the sense of [BD2, §1.4], which includes orbit relations such as $f_t^n(c_+) \equiv f_t^m(c_-)$, would imply that c_+ is preperiodic for f_t if and only if c_- is preperiodic for f_t . This contradicts Proposition 2.5. \square

Remark. In contrast with Corollary 2.6, conditions on the multipliers can and do impose relations between the critical points in other settings. For example, if we look at conjugacy classes $f \in M_2$ with two distinct period-3 cycles of the same multiplier, then we obtain the automorphism locus \mathcal{A}_2 [BEP, Theorem 3.1]. The family \mathcal{A}_2 is given by $f_{\lambda,0}(z) = \lambda z/(z^2 + 1)$, for parameter $\lambda \in \mathbb{C}^*$, with the automorphism $A(z) = -z$ for all λ ; the critical points (at ± 1) and their orbits are symmetric by A , and thus they define the same bifurcation locus and equal bifurcation measures.

3. THE BIFURCATION MEASURES ARE DISTINCT

In this section, we provide a proof of Theorem 1.2. Recall the definitions from §2.2.

3.1. Potential functions for the bifurcation measures. Recall the definitions of the measures μ_λ^\pm and their potential functions H_λ^\pm from §2.2. We begin by showing that if the two bifurcation measures were to coincide, their potential functions would have to be equal. The first lemma controls the growth of H_λ^\pm . The lower bound will be used again in the proof of Theorem 1.3. We will work with the norm

$$\|(z_1, z_2)\| = \max\{|z_1|, |z_2|\}$$

on \mathbb{C}^2 .

Lemma 3.1. *For each $\lambda \neq 0$, there are constants $c, C > 0$, such that*

$$c|t|^{-1} \leq \frac{\|F_t(z_1, z_2)\|}{\|(z_1, z_2)\|^2} \leq C|t|$$

for all $|t| \geq 1$ and all $(z_1, z_2) \neq (0, 0)$. Consequently,

$$H_\lambda^\pm(t) = O(\log |t|)$$

as $t \rightarrow \infty$.

Proof. The upper bound is immediate from the expression $F_t(z_1, z_2) = (\lambda z_1 z_2, z_1^2 + t z_1 z_2 + z_2^2)$. We may set

$$C = \max\{|\lambda|, 3\}.$$

For the lower bound, by the symmetry and homogeneity of F_t , we may assume that $z_2 = 1$ and $|z_1| \leq 1$. Then $\|(z_1, z_2)\| = 1$, and we shall estimate the norm of $sF_{1/s}(z_1, 1) = (s\lambda z_1, sz_1^2 + s + z_1)$ with $|s| \leq 1$. Let

$$c = \min\{|\lambda|/2, 1/4\}.$$

For each s with $|s| \leq 1$, either $|s\lambda z_1| \geq c|s|^2$, or $|z_1| < |s|/2$ in which case,

$$|sz_1^2 + s + z_1| \geq |s| - |s|/2 - |s|^3/4 \geq |s|/4 \geq c|s|^2.$$

Consequently, $\|F_{1/s}(z_1, z_2)\| \geq c|s|$ and the lower bound is proved.

By the identity

$$H_\lambda^+(t) = \sum_{i=2}^{+\infty} \frac{1}{2^i} \log \left(\frac{\|F_t^i(1, 1)\|}{\|F_t^{i-1}(1, 1)\|^2} \right) + \frac{1}{2} \log \|F_t(1, 1)\|,$$

we have that $|H_\lambda^+(t)| < 3 \log |t|$ when t is large. The same holds for H_λ^- , since $H_\lambda^-(t) = H_\lambda^+(-t)$. \square

Lemma 3.2. *For any $\lambda \neq 0$, we have*

$$\mu_\lambda^+ = \mu_\lambda^- \implies H_\lambda^+ = H_\lambda^-.$$

Proof. Let $h(t) = H_\lambda^+(t) - H_\lambda^-(t)$. If $\mu_\lambda^+ = \mu_\lambda^-$, then $\Delta h = 0$ in \mathbb{C} . This implies that h is harmonic. By Lemma 3.1, we have $h(t) = O(\log |t|)$ for t near ∞ . Therefore h is constant. Combined with the symmetry $H_\lambda^-(t) = H_\lambda^+(-t)$, we may conclude that $H_\lambda^+ = H_\lambda^-$. \square

3.2. Showing that the potentials differ at a single point. As a consequence of Lemma 3.2, it suffices to show that H^+ and H^- differ at a single point. For $t = \lambda - 2$, we may compute the values. Observe that $f_{\lambda-2}$ is conjugate to a polynomial, as $f_{\lambda-2}(1) = 1$.

Lemma 3.3. *We have*

$$H_\lambda^+(\lambda - 2) = \log |\lambda| \quad \text{and} \quad H_\lambda^-(\lambda - 2) = \frac{1}{2} G_{\mathcal{M}}(c(\lambda)) + \log 2,$$

when $G_{\mathcal{M}}$ is defined in Example 2.1 and $c(\lambda) = \frac{\lambda}{2} - \frac{\lambda^2}{4}$.

Proof. The map F_t defined in (2.2) for $t = \lambda - 2$ satisfies $F_{\lambda-2}(1/\lambda, 1/\lambda) = (1/\lambda, 1/\lambda)$. Therefore,

$$H_\lambda^+(\lambda - 2) = \lim_{n \rightarrow \infty} \frac{1}{2^n} \log \|F_{\lambda-2}^n(1/\lambda, 1/\lambda)\| + \log |\lambda| = \log |\lambda|.$$

For $t = \lambda - 2$, recall that $f_{\lambda-2}$ is conjugate to a polynomial

$$(3.1) \quad q_\lambda(z) = \lambda z(z + 1).$$

Set

$$A(z_1, z_2) = (z_1, z_2 - z_1) \quad \text{and} \quad \tilde{F}_{\lambda,t} = A \circ F_{\lambda,t} \circ A^{-1}.$$

When $t = \lambda - 2$, we have

$$\tilde{F}_{\lambda,\lambda-2}(z_1, z_2) = (\lambda z_1(z_1 + z_2), z_2^2).$$

Note that $A(-1, 1) = 2(-1/2, 1)$, $\tilde{F}_{\lambda, \lambda-2}^n(A(-1, 1)) = 2^{2^n}(q_\lambda^n(-1/2), 1)$, so that

$$\begin{aligned}
(3.2) \quad H_\lambda^-(\lambda - 2) &= \lim_{n \rightarrow \infty} 2^{-n} \log \|F_{\lambda, \lambda-2}^n(-1, 1)\| \\
&= \lim_{n \rightarrow \infty} 2^{-n} \log \|A^{-1} \circ \tilde{F}_{\lambda, \lambda-2}^n(A(-1, 1))\| \\
&= \lim_{n \rightarrow \infty} 2^{-n} \log \|\tilde{F}_{\lambda, \lambda-2}^n(A(-1, 1))\| \\
&= \lim_{n \rightarrow \infty} 2^{-n} \log^+ |q_\lambda^n(-1/2)| + \log 2.
\end{aligned}$$

In this way, we express $H_\lambda^-(\lambda - 2)$ in terms of the escape rate of the critical point $-1/2$ of the polynomial q_λ . We may conjugate q_λ to the quadratic polynomial

$$p_{c(\lambda)}(z) = z^2 + c(\lambda), \quad \text{with } c(\lambda) = \frac{\lambda}{2} - \frac{\lambda^2}{4}.$$

Then we get

$$H_\lambda^-(\lambda - 2) = \frac{1}{2}G_{\mathcal{M}}(c(\lambda)) + \log 2.$$

□

Now we are ready to compare $H_\lambda^+(\lambda - 2)$ and $H_\lambda^-(\lambda - 2)$.

Lemma 3.4. *For any $\lambda \neq 0$ and $\operatorname{Re} \lambda \leq 1$, we have*

$$(3.3) \quad H_\lambda^+(\lambda - 2) \leq H_\lambda^-(\lambda - 2),$$

with equality if and only if $\lambda = -2$.

Proof. By Lemma 3.3, we need to show that

$$(3.4) \quad G_{\mathcal{M}}(c(\lambda)) \geq 2 \log |\lambda/2|,$$

for $\lambda \neq 0$ and $\operatorname{Re} \lambda \leq 1$. The proof relies on the following two claims:

Claim 1: *If $\operatorname{Re} \lambda = 1$, we have*

$$(3.5) \quad G_{\mathcal{M}}(c(\lambda)) > 2 \log |\lambda/2|.$$

Claim 2: *For $|\lambda| = 2$, $\operatorname{Re} \lambda < 1$, the parameter $c(\lambda) = \frac{\lambda}{2} - \frac{\lambda^2}{4}$ lies in the Mandelbrot set \mathcal{M} if and only if $\lambda = -2$.*

Lemma 3.4 follows easily from Claims 1 and 2. Indeed, for $|\lambda| < 2$, we have $G_{\mathcal{M}}(c(\lambda)) \geq 0 > 2 \log |\lambda/2|$. By Claim 2, for $|\lambda| = 2$, $\operatorname{Re} \lambda < 1$ and $\lambda \neq -2$, we have $G_{\mathcal{M}}(c(\lambda)) > 0 = 2 \log |\lambda/2|$, and for $\lambda = -2$, $G_{\mathcal{M}}(c(\lambda)) = G_{\mathcal{M}}(-2) = 0 = 2 \log |\lambda/2|$. By Claim 1, if $\operatorname{Re} \lambda = 1$, we have $G_{\mathcal{M}}(c(\lambda)) > 2 \log |\lambda/2|$. It follows from Claim 2 that $G_{\mathcal{M}}(c(\lambda))$ is harmonic in the region $\Omega = \{\operatorname{Re} \lambda < 1, |\lambda| > 2\}$. Observe that when $\lambda \rightarrow \infty$ in Ω ,

$$G_{\mathcal{M}}(c(\lambda)) - 2 \log |\lambda/2| \rightarrow 0.$$

By the maximum/minimum value theorem, we have $G_{\mathcal{M}}(c(\lambda)) > 2 \log |\lambda/2|$ in Ω . The conclusion then follows.

Proof of Claim 1. For $\operatorname{Re} \lambda = 1$, we have $2 - \lambda = \bar{\lambda}$ and $c(\lambda) = |\lambda|^2/4$. It is equivalent to show that $G_{\mathcal{M}}(c) > \log c$ when $c > 1/4$ and $c \in \mathbb{R}$. For $p_c(z) = z^2 + c$ with $c > 1/4$, we have $p_c(c) = c^2 + c$ and $p_c^n(c) \geq (c^2 + c)^{2^{n-1}}$. Consequently,

$$G_{\mathcal{M}}(c) \geq \lim_{n \rightarrow \infty} 2^{-n} \log(c^2 + c)^{2^{n-1}} = \frac{1}{2} \log(c^2 + c) > \log c.$$

Proof of Claim 2. Let $p_c(z) = z^2 + c$. Recall that the Mandelbrot set \mathcal{M} can be defined by

$$\mathcal{M} = \{c \in \mathbb{C} : |p_c^n(0)| \leq 2 \text{ for any } n \geq 1\}.$$

In order to show $c(\lambda) \notin \mathcal{M}$ when $|\lambda| = 2$, $\operatorname{Re} \lambda < 1$, by the above definition, it suffices to show that

$$(3.6) \quad |p_{c(\lambda)}^2(0)| = |(c(\lambda))(c(\lambda) + 1)| > 2.$$

Let $\lambda = 2(\cos \theta + i \sin \theta)$ with $\theta \in [\pi/3, 5\pi/3]$. Then $\cos \theta \in [-1, 1/2]$. With some computation, one has

$$|(c(\lambda))(c(\lambda) + 1)| = \sqrt{2(5 - 5 \cos \theta - 4 \cos^2 \theta + 4 \cos^3 \theta)}$$

Let $u = \cos \theta \in [-1, 1/2]$, the function $g(u) = 5 - 5u - 4u^2 + 4u^3$ have minimum when $u = -1$ or $1/2$. Consequently, for any λ with $|\lambda| = 2$, $\operatorname{Re} \lambda < 1$ and $\lambda \neq -2$, the inequality (3.6) holds; i.e., $c(\lambda)$ is not in the Mandelbrot set. \square

Finally, we treat the case of $\lambda = -2$.

Proposition 3.5. *For $\lambda = -2$, the bifurcation sets Bif^+ and Bif^- are not equal.*

Proof. For $\lambda = -2$ and $t = 2\sqrt{3}$, the map $f_{\lambda,t} = f_{-2,2\sqrt{3}}$ has a fixed point at $z = -\sqrt{3}$ with multiplier 1. For all $t > 2\sqrt{3}$, one of the fixed points is attracting. Using Proposition 2.2, to show the bifurcation sets are different, it suffices to show the existence of $t_0 > 2\sqrt{3}$ so that f_{-2,t_0} has a second attracting cycle of period > 1 . In that case, both of the critical points cannot lie in the basin of the attracting fixed point.

For $t > 2\sqrt{3}$, consider the first three iterations of 1 under $f_{-2,t}$

$$1 \mapsto \frac{-2}{2+t} \mapsto \frac{8+4t}{8-t^2} \mapsto f_{-2,t}^3(1) = \frac{-8(2+t)(8-t^2)}{(8-t^2)^2 + 4t(t+2)(8-t^2) + 16(2+t)^2}.$$

To see $f_{-2,t}^3(1) = 1$ has a solution for $t > 2\sqrt{3}$, define

$$\begin{aligned} \ell(t) &= (8-t^2)^2 + 4t(t+2)(8-t^2) + 16(2+t)^2 + 8(2+t)(8-t^2) \\ &= 16(2+t)^2 + 4(2+t)^2(8-t^2) + (8-t^2)^2 \end{aligned}$$

Note that

$$\ell(2\sqrt{3}) = 16 > 0; \quad \lim_{t \rightarrow +\infty} \ell(t) = -\infty,$$

this implies that $\ell(t) = 0$ has a solution for some $t_0 > 2\sqrt{3}$. Since $f_{-2,t_0}(1) \neq 1$, the critical point 1 is of periodic 3 for f_{-2,t_0} . \square

3.3. Proof of Theorem 1.2. First, suppose $|\lambda| < 1$ or $\operatorname{Re} \lambda > 1$. By Lemma 2.3, the two bifurcation sets Bif^+ and Bif^- are not equal. As the supports of μ_λ^+ and μ_λ^- are exactly Bif^+ and Bif^- , it follows that $\mu_\lambda^+ \neq \mu_\lambda^-$.

Second, if $\lambda = -2$, by Proposition 3.5, we know $\operatorname{Bif}^+ \neq \operatorname{Bif}^-$ which implies $\mu_\lambda^+ \neq \mu_\lambda^-$.

Finally, assume $\operatorname{Re} \lambda \leq 1$ and $\lambda \neq 0, -2$ and suppose that $\mu_\lambda^+ = \mu_\lambda^-$. Then by Lemma 3.2, we have $H_\lambda^+(\lambda - 2) = H_\lambda^-(\lambda - 2)$. However, this contradicts Lemma 3.4. \square

4. HOMOGENEOUS POTENTIAL FUNCTIONS

In this section, we study the potential functions $H_\lambda^+(t)$ and $H_\lambda^-(t)$ of (2.3) in more detail. From Lemma 3.1 we know that $H_\lambda^\pm(t) = O(\log |t|)$ as $t \rightarrow \infty$. Here, we refine this estimate and prove Theorem 4.1, the first step in our proof of Theorem 1.3.

4.1. The homogeneous potential functions on parameter space. Fix $\lambda \neq 0$. Working in homogeneous coordinates, we write

$$F_t(z_1, z_2) = (\lambda z_1 z_2, z_1^2 + t z_1 z_2 + z_2^2)$$

for a lift of $f_{\lambda,t}$ to \mathbb{C}^2 , with $z = z_1/z_2$. We will also work in homogeneous coordinates over the parameter space. Consider the two sequences of maps,

$$(4.1) \quad F_n^+(t_1, t_2) := t_2^{2^{n-1}} F_{t_1/t_2}^n(+1, 1),$$

and

$$(4.2) \quad F_n^-(t_1, t_2) := t_2^{2^{n-1}} F_{t_1/t_2}^n(-1, 1),$$

for $(t_1, t_2) \in \mathbb{C}^2$.

Theorem 4.1. *The maps F_n^\pm are homogeneous polynomial maps in (t_1, t_2) of degree 2^{n-1} with nonzero resultants. For each $\lambda \neq 0$ such that*

$$\gamma(\lambda) = \frac{1}{2} \sum_{i=1}^{+\infty} \frac{1}{2^i} \log |1 + \lambda + \dots + \lambda^i|$$

converges, the limits

$$\lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} \log \|F_n^\pm(t_1, t_2)\|$$

converge locally uniformly on $\mathbb{C}^2 \setminus \{(0, 0)\}$ to continuous functions G^\pm satisfying

$$G^\pm(t_1, t_2) = \begin{cases} 2H_\lambda^\pm(t_1/t_2) + \log |t_2| & \text{if } t_2 \neq 0, \\ \log |t_1| + \gamma(\lambda) & \text{if } t_2 = 0. \end{cases}$$

Remark. It is easy to see that $\gamma(\lambda)$ is finite for all $\lambda \in \mathbb{C}$ with $|\lambda| \neq 1$ and for $\lambda = 1$. In the next section, we observe that it is finite for all algebraic numbers λ that are not roots of unity.

For the proof of Theorem 4.1, it suffices to consider the maps F_n^+ and the function G^+ ; the results for F_n^- and G^- follow by symmetry. Define polynomials $P_n(t)$ and $Q_n(t)$ by

$$F_t^n(1, 1) = (P_n(t), Q_n(t)).$$

One may verify by induction that the degree of P_n is $2^{n-1} - 1$, and the degree of Q_n is 2^{n-1} . This shows that F_n^+ is polynomial in (t_1, t_2) . Also, since $F_t^{-1}\{(0, 0)\} = \{(0, 0)\}$ for all $t \in \mathbb{C}$, we see that P_n and Q_n have no common roots. Thus, F_n^+ has nonzero resultant in (t_1, t_2) .

For the convergence statement, note that standard arguments from complex dynamics imply that the convergence is uniform away from $t_2 = 0$. In fact, the escape-rate function for F_t will be continuous in both the dynamical variable (z_1, z_2) and the parameter t for any holomorphic family [HP, FS]. It follows immediately from the definitions that

$$G^+(t_1, t_1) = 2H_\lambda^+(t_1/t_2) + \log |t_2|$$

whenever $t_2 \neq 0$.

The remainder of this section is devoted to the proof of uniform convergence near $t_2 = 0$. The proof of Theorem 4.1 will be complete once we have proved Lemmas 4.3 and 4.4 below. We make an effort to include all details, especially the steps that will be repeated in the nonarchimedean setting in the following section.

4.2. Convergence near $t_2 = 0$. Throughout this subsection, we set $t_1 = 1$ and $t_2 = s$. We have

$$F_{n+1}^+(1, s) = F_{1/s}(F_n^+(1, s)).$$

We begin by looking at the coefficients of $F_n^+(1, s)$. Write

$$(4.3) \quad F_n^+(1, s) = (sB_n(\lambda) + s^2A_n(\lambda) + O(s^3), C_n(\lambda) + sD_n(\lambda) + O(s^2)).$$

Note that $B_1(\lambda) = \lambda$, $C_1(\lambda) = 1$, and

$$B_{n+1}(\lambda) = \lambda B_n(\lambda) C_n(\lambda) \quad \text{and} \quad C_{n+1}(\lambda) = C_n(\lambda)(B_n(\lambda) + C_n(\lambda))$$

for all $n \geq 1$. By induction, we obtain explicit expressions

$$(4.4) \quad B_n(\lambda) = \lambda^n (1 + \lambda)^{2^{n-3}} (1 + \lambda + \lambda^2)^{2^{n-4}} \cdots (1 + \lambda + \cdots + \lambda^{n-2})^{2^0}$$

$$(4.5) \quad C_n(\lambda) = B_n(\lambda)(1 + \lambda + \cdots + \lambda^{n-1})/\lambda^n$$

for all $n \geq 3$.

Lemma 4.2. *The coefficients $A_n(\lambda)$, $B_n(\lambda)$, $C_n(\lambda)$, and $D_n(\lambda)$ of (4.3) satisfy*

$$\begin{aligned} e^{\gamma(\lambda)} &= \lim_{n \rightarrow \infty} |B_n(\lambda)|^{\frac{1}{2^{n-1}}} = \lim_{n \rightarrow \infty} |C_n(\lambda)|^{\frac{1}{2^{n-1}}} \\ &\geq \limsup_{n \rightarrow \infty} |A_n(\lambda)|^{\frac{1}{2^{n-1}}}, \limsup_{n \rightarrow \infty} |D_n(\lambda)|^{\frac{1}{2^{n-1}}} \end{aligned}$$

where $\gamma(\lambda)$ is defined in Theorem 4.1.

Proof. With the explicit expressions for B_n and C_n given above, the limiting value is clearly e^γ ; it suffices to show the bound for A_n and D_n . By induction, we find

$$A_{n+1}(\lambda) = \lambda (B_n(\lambda)D_n(\lambda) + C_n(\lambda)A_n(\lambda)),$$

and

$$D_{n+1}(\lambda) = C_n(\lambda)(2D_n(\lambda) + A_n(\lambda)) + D_n(\lambda)B_n(\lambda),$$

with $A_1(\lambda) = 0$ and $D_1(\lambda) = 2$.

The explicit expressions for $B_n(\lambda)$ and $C_n(\lambda)$ and these inductive formulas for $A_n(\lambda)$, $D_n(\lambda)$, show that

$$A_n(\lambda) = (1 + \lambda)^{2^{n-3}-2}(1 + \lambda + \lambda^2)^{2^{n-4}-2} \cdots (1 + \lambda + \cdots + \lambda^{n-3})^0 A_n^*(\lambda),$$

and

$$D_n(\lambda) = (1 + \lambda)^{2^{n-3}-2}(1 + \lambda + \lambda^2)^{2^{n-4}-2} \cdots (1 + \lambda + \cdots + \lambda^{n-3})^0 D_n^*(\lambda),$$

for sequences $A_n^*(\lambda)$ and $D_n^*(\lambda)$ given inductively by

$$A_{n+1}^*(\lambda) = \lambda^{n+1} D_n^*(\lambda) + (\lambda + \lambda^2 + \cdots + \lambda^n) A_n^*(\lambda)$$

and

$$D_{n+1}^*(\lambda) = (2(1 + \lambda + \cdots + \lambda^{n-1}) + \lambda^n) D_n^*(\lambda) + (1 + \lambda + \cdots + \lambda^{n-1}) A_n^*(\lambda)$$

with $A_1^*(\lambda) = 0$ and $D_1^*(\lambda) = 2$. Finally, the induction formulas for $A_n^*(\lambda)$ and $D_n^*(\lambda)$ imply that

$$\begin{aligned} \max(|A_n^*(\lambda)|, |D_n^*(\lambda)|) &\leq (2 + 2|\lambda|)^n \max(|A_{n-1}^*(\lambda)|, |D_{n-1}^*(\lambda)|) \\ &\leq (2 + 2|\lambda|)^{2^{n-1}} \max(|A_{n-2}^*(\lambda)|, |D_{n-2}^*(\lambda)|) \\ &\leq 2(2 + 2|\lambda|)^{1+2+\cdots+n} \end{aligned}$$

Consequently, we have $\lim_{n \rightarrow \infty} \sup |A_n(\lambda)|^{\frac{1}{2^{n-1}}} , \lim_{n \rightarrow \infty} \sup |D_n(\lambda)|^{\frac{1}{2^{n-1}}} \leq e^{\gamma(\lambda)}$. \square

The growth of the coefficients B_n and C_n in Lemma 4.2 provides a uniform upper bound on the size of $2^{1-n} \log \|F_n^+(1, s)\|$ for small s :

Lemma 4.3. *For any given $\varepsilon > 0$, there exists a $\delta > 0$ and an integer $N > 0$ so that*

$$\frac{1}{2^{n-1}} \log \|F_n^+(1, s)\| - \gamma(\lambda) < \varepsilon$$

for all $|s| < \delta$ and all $n \geq N$.

Proof. Define polynomials $p_n(s)$ and $q_n(s)$ by

$$F_n^+(1, s) = (sp_n(s), q_n(s))$$

so that $p_n(0) = B_n(\lambda)$ and $q_n(0) = C_n(\lambda)$. By Lemma 4.2, there is a huge integer N such that

$$|B_N(\lambda)|, |C_N(\lambda)| < (1 + \varepsilon/4)^{2^{N-1}} e^{\gamma(\lambda)2^{N-1}}$$

and

$$\frac{\log(|\lambda| + 3)}{2^{N-1}} \leq \varepsilon/2.$$

Set

$$R := (1 + \varepsilon/4)^{2^{N-1}} e^{\gamma(\lambda)2^{N-1}} (|\lambda| + 3).$$

Since $|B_N(\lambda)|, |C_N(\lambda)| < R/(|\lambda| + 3)$, we can choose a very small $\delta > 0$ such that

$$|p_N(s)|, |q_N(s)| < R/(|\lambda| + 3),$$

for any s with $|s| \leq \delta$.

Recall that $F_{n+1}^+(1, s) = F_{1/s}(F_n^+(1, s))$ for all n . Thus,

$$|p_{N+1}(s)| = |\lambda p_N(s) q_N(s)| < R^2/(|\lambda| + 3)$$

and

$$|q_{N+1}(s)| = |s^2 p_N(s)^2 + q_N(s)^2 + p_N(s) q_N(s)| < R^2/(|\lambda| + 3).$$

Inductively, we find

$$|p_{N+i}(s)| < R^{2^i}/(|\lambda| + 3)$$

$$|q_{N+i}(s)| < R^{2^i}/(|\lambda| + 3)$$

for any s with $|s| \leq \delta$ and any $i \geq 0$. Consequently, as the integer N satisfies $\log(|\lambda| + 3)/2^{N-1} \leq \varepsilon/2$ and $|s| \leq \delta$,

$$\begin{aligned} \frac{\log \|F_{N+i, \lambda}(1, s)\|}{2^{N+i-1}} &< \frac{\log R^{2^i}/(|\lambda| + 3)}{2^{N+i-1}} \\ &< \frac{\log[(1 + \varepsilon/4)^{2^{N-1+i}} e^{\gamma(\lambda)2^{N-1+i}} (|\lambda| + 3)^{2^i}]}{2^{N+i-1}} \\ &< \gamma(\lambda) + \varepsilon/4 + \log(|\lambda| + 3)/2^{N-1} \\ &< \gamma(\lambda) + \varepsilon. \end{aligned}$$

□

The corresponding lower bound on the size of $2^{1-n} \log \|F_n^+(1, s)\|$ for small s is more delicate, and we use Lemma 3.1 together with the estimates of Lemma 4.2.

Lemma 4.4. *For any given $\varepsilon > 0$, there exists a $\delta > 0$ and an integer $N > 0$ so that*

$$\frac{1}{2^{n-1}} \log \|F_n^+(1, s)\| - \gamma(\lambda) > -\varepsilon$$

for all $|s| < \delta$ and all $n \geq N$.

Proof. In contrast with the proof of Lemma 4.3, we define polynomials $p_n(s)$ and $q_n(s)$ by

$$F_n^+(1, s) = (s(B_n(\lambda) + s p_n(s)), C_n(\lambda) + s q_n(s)).$$

By Lemma 4.2, there is a huge integer N , such that

$$|A_N(\lambda)|, |D_N(\lambda)| < (1 + \varepsilon/8)^{2^{N-1}} e^{\gamma(\lambda)2^{N-1}},$$

$$|B_{N+i}(\lambda)| < ((1 + \varepsilon/8)e^{\gamma(\lambda)})^{2^{N-1+i}},$$

and

$$(4.6) \quad ((1 - \varepsilon/8)e^{\gamma(\lambda)})^{2^{N-1+i}} < |C_{N+i}(\lambda)| < ((1 + \varepsilon/8)e^{\gamma(\lambda)})^{2^{N-1+i}},$$

for any $i \geq 0$. By increasing N if necessary, we may also assume that

$$(4.7) \quad \frac{\log(c\varepsilon/8(12|\lambda| + 12)^2)}{2^{N-1}} > -\varepsilon/10,$$

where the constant c is defined in Lemma 3.1. Set

$$R := ((1 + \varepsilon/8)e^{\gamma(\lambda)})^{2^{N-1}}(12|\lambda| + 12).$$

Since $|A_N(\lambda)|, |D_N(\lambda)| < R/(12|\lambda| + 12)$, we can choose a very small $\delta > 0$ such that

$$|p_N(s)|, |q_N(s)| < R/(12|\lambda| + 12),$$

for any s with $|s| \leq \delta$. Recalling that $F_{n+1}^+(1, s) = F_{1/s}(F_n^+(1, s))$ for all n , the estimate (4.6) implies

$$|p_{N+1}(s)| = |\lambda(C_N(\lambda)p_N(s) + (B_N(\lambda) + sp_N(s))q_N(s))| < \frac{R^2}{12|\lambda| + 12}$$

and similarly, $|q_{N+1}(s)| < R^2/(12|\lambda| + 12)$. Inductively, for any $i \geq 0$ and s with $|s| \leq \delta$,

$$(4.8) \quad |p_{N+i}(s)|, |q_{N+i}(s)| < R^{2^i}/(12|\lambda| + 12).$$

Choose an integer $N' > N$, such that

$$(4.9) \quad \delta' := \frac{\varepsilon(1 - \varepsilon/8)^{2^{N'-1}}}{8(1 + \varepsilon/8)^{2^{N'-1}}(12|\lambda| + 12)^{2^{N'-N}}} < \delta.$$

For any $j \geq N'$ and s with

$$|s| \leq \frac{\varepsilon(1 - \varepsilon/8)^{2^{j-1}}}{8(1 + \varepsilon/8)^{2^{j-1}}(12|\lambda| + 12)^{2^{j-N}}} \leq \delta',$$

by (4.6), we have

$$\begin{aligned} \frac{\log \|F_j(1, s)\|}{2^{j-1}} &\geq \frac{\log |C_j(\lambda) + sq_j(s)|}{2^{j-1}} \\ &\geq \frac{\log(|C_j(\lambda)| - |sq_j(s)|)}{2^{j-1}} \\ &\geq \frac{\log\left(\left|(1 - \varepsilon/8)^{2^{j-1}} e^{\gamma(\lambda)2^{j-1}}\right| - \frac{|s|R^{2^{j-N}}}{12|\lambda|+12}\right)}{2^{j-1}}, \text{ by (4.8)} \\ &\geq \gamma(\lambda) + 2\log(1 - \varepsilon/8) \\ &> \gamma(\lambda) - \varepsilon, \text{ as } \varepsilon \text{ is small.} \end{aligned}$$

For any $n \geq N'$ and s with $|s| < \delta'$, if $|s| \leq \frac{\varepsilon(1-\varepsilon/8)^{2^{n-1}}}{8(1+\varepsilon/8)^{2^{n-1}}(12|\lambda|+12)^{2^{n-N}}}$, then the above inequality guarantees

$$\frac{\log \|F_n^+(1, s)\|}{2^{n-1}} > \gamma(\lambda) - \varepsilon.$$

Otherwise, by (4.9), there is a j with $N' \leq j < n$, such that

$$\frac{\varepsilon(1-\varepsilon/8)^{2^j}}{8(1+\varepsilon/8)^{2^j}(12|\lambda|+12)^{2^{j+1-N}}} \leq |s| \leq \frac{\varepsilon(1-\varepsilon/8)^{2^{j-1}}}{8(1+\varepsilon/8)^{2^{j-1}}(12|\lambda|+12)^{2^{j-N}}}.$$

From Lemma 3.1, we have

$$(4.10) \quad \frac{1}{2^{(n+i)-1}} \log \|F_{n+i}^+(1, s)\| - \frac{1}{2^{n-1}} \log \|F_n^+(1, s)\| \geq \frac{1}{2^{n-1}} \log(c|s|),$$

for all s with $|s| \leq 1$ and all $n, i \geq 1$. Indeed, note that $F_{n+1}^+(1, s) = F_{1/s}(F_n^+(1, s))$ for all n . We see that

$$\begin{aligned} \frac{\|F_{n+i}^+(1, s)\|}{\|F_n^+(1, s)\|^{2^i}} &= \frac{\|F_{n+i}^+(1, s)\|}{\|F_{n+i-1}^+(1, s)\|^2} \left(\frac{\|F_{n+i-1}^+(1, s)\|}{\|F_{n+i-2}^+(1, s)\|^2} \right)^2 \cdots \left(\frac{\|F_{n+1}^+(1, s)\|}{\|F_n^+(1, s)\|^2} \right)^{2^{i-1}} \\ &\geq (c|s|)^{2^{i-1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\log \|F_n^+(1, s)\|}{2^{n-1}} &\geq \frac{\log \|F_j^+(1, s)\|}{2^{j-1}} + 2^{1-j} \log(c|s|) \\ &\geq \gamma(\lambda) + 2 \log(1 - \varepsilon/8) + 2^{1-j} \log \frac{c\varepsilon(1-\varepsilon/8)^{2^j}}{8(1+\varepsilon/8)^{2^j}(12|\lambda|+12)^{2^{j+1-N}}} \\ &\geq \gamma(\lambda) + 4 \log(1 - \varepsilon/8) - 2 \log(1 + \varepsilon/8) - \varepsilon/10, \text{ by (4.7)} \\ &> \gamma(\lambda) - \varepsilon, \text{ as } \varepsilon \text{ is small.} \end{aligned}$$

□

5. NON-ARCHIMEDEAN POTENTIAL FUNCTIONS

In this section, we prove a non-archimedean counterpart to Theorem 4.1. If we assume $\lambda \in \overline{\mathbb{Q}}$, many of the computations of the previous section hold (and simplify) for the nonarchimedean absolute values on the number field $k = \mathbb{Q}(\lambda)$. As such, we may conclude that the bifurcation measures μ_λ^\pm are the archimedean components of a pair of *quasi-adelic measures*, equipped with continuous potential functions. We use the term “quasi-adelic” because the measures might be nontrivial at infinitely many places, though the associated height functions (defined as a sum of all local potentials, over all places of k) converge.

5.1. Defining the potential functions at each place. Let k be a number field and let \bar{k} denote a fixed algebraic closure of k . (In this article, we always take $k = \mathbb{Q}(\lambda)$ for $\lambda \in \overline{\mathbb{Q}}$.) Any number field k is equipped with a set \mathcal{M}_k of pairwise inequivalent nontrivial absolute values, together with a positive integer N_v for each $v \in \mathcal{M}_k$, such that

- for each $\alpha \in k^*$, we have $|\alpha|_v = 1$ for all but finitely many $v \in \mathcal{M}_k$; and
- every $\alpha \in k^*$ satisfies the *product formula*

$$(5.1) \quad \prod_{v \in \mathcal{M}_k} |\alpha|_v^{N_v} = 1 .$$

For each $v \in \mathcal{M}_k$, let k_v be the completion of k at v , let \bar{k}_v be an algebraic closure of k_v , and let \mathbb{C}_v denote the completion of \bar{k}_v . We work with the norm

$$\|(z_1, z_2)\|_v = \max\{|z_1|_v, |z_2|_v\}$$

on $(\mathbb{C}_v)^2$. We let $\mathbb{P}_v^{1,an}$ denote the Berkovich projective line over \mathbb{C}_v , which is a canonically defined path-connected compact Hausdorff space containing $\mathbb{P}^1(\mathbb{C}_v)$ as a dense subspace. If v is archimedean, then $\mathbb{C}_v \cong \mathbb{C}$ and $\mathbb{P}_v^{1,an} = \mathbb{P}^1(\mathbb{C})$. See [BR2] for more information.

With the v -adic norms and $t \in \mathbb{C}_v$, we can define $H_{\lambda,v}^\pm$ exactly as in the archimedean case,

$$H_{\lambda,v}^\pm(t) = \lim_{n \rightarrow \infty} \frac{1}{2^n} \log \|F_t^n(\pm 1, 1)\|_v .$$

The definition extends naturally to the Berkovich affine line $\mathbb{A}_v^{1,an}$; see [BR2, Chapter 10]. Recall the definition of F_n from §4.1, now defined on $(\mathbb{C}_v)^2$.

Theorem 5.1. *Fix $\lambda \in \overline{\mathbb{Q}}$ with λ nonzero and not a root of unity, or set $\lambda = 1$. For each place v of $k = \mathbb{Q}(\lambda)$, the limits*

$$\lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} \log \|F_n^\pm(t_1, t_2)\|_v$$

converge locally uniformly on $(\mathbb{C}_v)^2 \setminus \{(0, 0)\}$ to continuous functions G_v^\pm satisfying

$$G_v^\pm(t_1, t_2) = \begin{cases} 2H_{\lambda,v}^\pm(t_1/t_2) + \log |t_2|_v & \text{if } t_2 \neq 0, \\ \log |t_1|_v + \gamma_v(\lambda) & \text{if } t_2 = 0. \end{cases}$$

The function G_v^\pm extends uniquely to define a continuous potential function for a probability measure μ_v^\pm on $\mathbb{P}_v^{1,an}$.

Theorem 5.1 is nearly identical to Theorem 4.1, except there is no longer a condition on the finiteness of $\gamma(\lambda)$, and the convergence holds at all places v . This finiteness is guaranteed by the following lemma.

Lemma 5.2. *For every algebraic number λ which is not a root of unity, or for $\lambda = 1$, the sum*

$$\gamma_v(\lambda) = \frac{1}{2} \sum_{i=1}^{\infty} \frac{1}{2^i} \log |1 + \lambda + \cdots + \lambda^i|_v$$

converges for all places v of $k = \mathbb{Q}(\lambda)$.

Proof. The statement follows from the product formula for the number field k . First assume that $\lambda \neq 1$. Note that $(1+\lambda+\cdots+\lambda^i)(1-\lambda) = 1-\lambda^{i+1}$, so it suffices to prove the convergence of the sum

$$\sum_{i=0}^{\infty} \frac{1}{2^i} \log |1 - \lambda^i|_v$$

at all places v .

Let v be a non-archimedean place of k . If $|\lambda|_v < 1$, then $|1 - \lambda^i|_v = 1$ for all i , and γ_v is 0. If $|\lambda|_v > 1$, then $|1 - \lambda^i|_v = |\lambda|_v^i$ for all i , and again the sum converges. Similarly for archimedean places, as long as $|\lambda|_v \neq 1$, it is easy to see that the sum defining γ_v converges.

For any $\lambda \in \overline{\mathbb{Q}}$, there are only finitely many places of $k = \mathbb{Q}(\lambda)$ for which $|\lambda|_v > 1$. For all such v , we have $|1 - \lambda^i|_v$ growing as $|\lambda|_v^i$ as $i \rightarrow \infty$. For all other v , the absolute value $|1 - \lambda^i|_v$ is uniformly bounded above (by 1 if non-archimedean, and by 2 if archimedean). Let ℓ be the number of archimedean places for which $|\lambda|_v \leq 1$.

Suppose v is a place such that $|\lambda|_v = 1$. It remains to show that $|1 - \lambda^i|_v$ cannot get too small as $i \rightarrow \infty$. As $1 - \lambda^i \neq 0$ for all i , the product formula for k states that

$$\prod_{w \in \mathcal{M}_k} |1 - \lambda^i|_w^{N_w} = 1.$$

Therefore,

$$|1 - \lambda^i|_v^{N_v} = \frac{1}{\prod_{w \neq v \in \mathcal{M}_k} |1 - \lambda^i|_w^{N_w}} \geq \frac{1}{2^\ell \prod_{w: |\lambda|_w > 1} |1 - \lambda^i|_w^{N_w}} \geq c^i$$

for some constant $c > 0$ and all i . It follows that the expression for γ_v converges at this place v .

Finally, assume $\lambda = 1$. Then the sum becomes

$$\gamma_v(1) = \sum_{j=2}^{\infty} \frac{1}{2^j} \log |j|_v.$$

The expression clearly converges at the unique archimedean place $v = \infty$. Setting $v = p$ for any prime p , we have $|j|_p \geq 1/j$ for all $j \in \mathbb{N}$; therefore, the sum is easily seen to converge also in this case. \square

Remark. For a given λ , the value $\gamma_v(\lambda)$ may be nonzero at infinitely many places v . For example, $\gamma_v(1)$ is nonzero at *all* places v of \mathbb{Q} . The conclusion of Lemma 5.2 also appears in [HY, Lemma 4].

5.2. Proof of Theorem 5.1. Fix $\lambda \in \overline{\mathbb{Q}}$, with λ nonzero and not a root of unity; or let $\lambda = 1$. If v is an archimedean place of the number field $k = \mathbb{Q}(\lambda)$, then the Theorem follows immediately from Theorem 4.1 and Lemma 5.2 (for the finiteness of $\gamma_v(\lambda)$).

Now suppose that v is a non-archimedean place of k . The proof of convergence in the archimedean case shows *mutatis mutandis* that the convergence to G_v^\pm is locally uniform for all places v . A line-by-line analysis of the proof of Theorem 4.1 shows that the proof uses

nothing more than the triangle inequality and elementary algebra. As such, the estimates can only be improved when the usual triangle inequality is replaced by the ultrametric inequality in the case of a non-archimedean absolute value.

The extension of G_v^\pm to Berkovich space and the construction of the measure μ_v^\pm as its Laplacian are carried out exactly as in [BR2, §10.1]. \square

There is one lemma (Lemma 3.1) used in the proof of Theorem 4.1 that we will need again in the next section, in the non-archimedean setting. We state it explicitly here.

Lemma 5.3. *For each $\lambda \in \overline{\mathbb{Q}} \setminus \{0\}$, there is constant $c > 0$ so that*

$$\frac{\|F_t(z_1, z_2)\|_v}{\|(z_1, z_2)\|_v^2} \geq c|t|_v^{-1}$$

for all $|t|_v \geq 1$, all $(z_1, z_2) \neq (0, 0)$ in $(\mathbb{C}_v)^2$, and all places v of $k = \mathbb{Q}(\lambda)$.

Proof. Set

$$c = \min\{\min\{|\lambda|_v/2 : v \in \mathcal{M}_k\}, 1/4\}.$$

Note that $c > 0$ since there are only finitely many places v for which $|\lambda|_v < 1$. Now fix $v \in \mathcal{M}_k$. If v is archimedean, then the proof is identical to that of Lemma 3.1. If v is non-archimedean, the estimate can be simplified a bit. Indeed, assume that $z_2 = 1$ and $|z_1|_v \leq 1$. Then $\|(z_1, z_2)\|_v = 1$, and we estimate the norm of $sF_{1/s}(z_1, 1) = (s\lambda z_1, sz_1^2 + s + z_1)$ with $0 < |s|_v \leq 1$. For each such s , either $|s\lambda z_1|_v \geq c|s|_v^2$, or $|z_1|_v < |s|_v/2$ in which case,

$$|sz_1^2 + s + z_1|_v = |s|_v > c|s|_v^2.$$

In either case, $\|F_{1/s}(z_1, z_2)\| \geq c|s|$ and the lower bound is proved. The case of $|z_2|_v < |z_1|_v = 1$ follows by symmetry, and the conclusion of the lemma is obtained from the homogeneity of F_t . \square

6. THE HOMOGENEOUS BIFURCATION SETS

In this section, we study the escape-rate functions G_v^\pm of Theorems 4.1 and 5.1 and compute the homogeneous capacity of the sets

$$K_{\lambda,v}^\pm = \{z \in (\mathbb{C}_v)^2 : G_v^\pm(z) \leq 0\}.$$

We also provide a bound on the diameter of the sets $K_{\lambda,v}^\pm$ that will be used in our proof of Theorem 7.1 (and Theorem 1.3).

6.1. The homogeneous capacity. We will consider compact sets $K \subset \mathbb{C}^2$ that are circled and pseudoconvex: these are sets of the form

$$K = \{(z, w) \in \mathbb{C}^2 : G_K(z, w) \leq 0\}$$

for continuous, plurisubharmonic functions $G_K : \mathbb{C}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$ such that

$$G_K(\alpha z, \alpha w) = G_K(z, w) + \log |\alpha|$$

for all $\alpha \in \mathbb{C}^*$; see [De2, §3]. Such functions are (homogeneous) potential functions for probability measures on \mathbb{P}^1 [FS, Theorem 5.9].

Set $G_K^+ = \max\{G_K, 0\}$. The Levi measure of K is defined by

$$\mu_K = dd^c G_K^+ \wedge dd^c G_K^+.$$

It is known that μ_K is a probability measure supported on $\partial K = \{G = 0\}$. The homogeneous capacity of K is defined by

$$\text{cap}(K) = \exp\left(\iint \log|\zeta \wedge \xi| d\mu_K(\zeta) d\mu_K(\xi)\right).$$

This capacity was introduced in [De2] and shown in [BR1] to satisfy $\text{cap}(K) = (d_\infty(K))^2$, where d_∞ is the transfinite diameter in \mathbb{C}^2 .

To compute the capacity, suppose that

$$F_n : \mathbb{C}^2 \rightarrow \mathbb{C}^2$$

is a sequence of homogeneous polynomial maps such that the resultant $\text{Res}(F_n) \neq 0$ for all n and that

$$\lim_{n \rightarrow \infty} \frac{1}{\deg(F_n)} \log \|F_n\|$$

converges locally uniformly in $\mathbb{C}^2 \setminus \{(0, 0)\}$ to the function G_K . The capacity $\text{cap}(K)$ may be computed as

$$(6.1) \quad \text{cap}(K) = \lim_{n \rightarrow \infty} |\text{Res}(F_n)|^{-1/\deg(F_n)^2}.$$

If, in addition, the maps F_n are defined over a number field k , and if the convergence holds for all absolute values v in \mathcal{M}_k , with limiting function $G_v : (\mathbb{C}_v)^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$, then the same computation works at all places v . That is,

$$(6.2) \quad \text{cap}(K_v) = \lim_{n \rightarrow \infty} |\text{Res}(F_n)|_v^{-1/\deg(F_n)^2}.$$

See [DWY] for a proof; similar statements appear in [DR].

6.2. Homogeneous capacity of the bifurcation sets. Recall the definition of $\gamma(\lambda)$ from Theorem 4.1.

Theorem 6.1. *For all $\lambda \in \mathbb{C} \setminus \{0\}$ such that $|\gamma(\lambda)| < \infty$, the set*

$$K_\lambda^\pm = \{z \in \mathbb{C}^2 : G_\lambda^\pm(z) \leq 0\}$$

is compact, circled, and pseudoconvex; its homogeneous capacity is

$$\text{cap}(K_\lambda^+) = \text{cap}(K_\lambda^-) = \frac{1}{|\lambda|^2} \prod_{j=1}^{+\infty} |1 + \lambda + \dots + \lambda^j|^{-3 \cdot 4^{-j-1}}.$$

For all $\lambda \in \overline{\mathbb{Q}} \setminus \{0\}$, not a root of unity, or for $\lambda = 1$, we have

$$\text{cap}(K_{\lambda,v}^+) = \text{cap}(K_{\lambda,v}^-) = \frac{1}{|\lambda|_v^2} \prod_{j=1}^{+\infty} |1 + \lambda + \dots + \lambda^j|_v^{-3 \cdot 4^{-j-1}}$$

for all places v of the number field $k = \mathbb{Q}(\lambda)$.

Proof. Fix $\lambda \in \mathbb{C} \setminus \{0\}$ so that $|\gamma(\lambda)| < \infty$. We will provide the proof for K_λ^+ ; the result for K_λ^- follows by symmetry. Recall the definition of F_n^+ from (4.1). That K_λ^+ is compact, circled and pseudoconvex follows immediately from the definition and continuity of G_λ^+ , stated in Theorem 4.1.

To compute the capacity of K_λ^+ , we give a recursive relation between $\text{Res}(F_n)$ and $\text{Res}(F_{n+1})$. Consider the transformation $A_\lambda(t_1, t_2) = (\lambda t_1 t_2, t_1^2 + t_2^2)$. Then $\text{Res}(A_\lambda \circ F_n)$ takes the form

$$\begin{vmatrix} 0 & a_1 & \cdots & a_{d-1} & a_d & 0 & \cdots & 0 \\ 0 & 0 & a_1 & \cdots & a_{d-1} & a_d & \cdots & 0 \\ & & & \cdots & & & & \\ 0 & 0 & \cdots & 0 & a_1 & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & \cdots & b_{d-1} & b_d & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & b_{d-1} & b_d & \cdots & 0 \\ & & & \cdots & & & & \\ 0 & 0 & \cdots & b_0 & b_1 & \cdots & b_{d-1} & b_d \end{vmatrix}$$

and $\text{Res}(F_{n+1})$ takes the form

$$\begin{vmatrix} 0 & a_1 & \cdots & a_{d-1} & a_d & 0 & \cdots & 0 \\ 0 & 0 & a_1 & \cdots & a_{d-1} & a_d & \cdots & 0 \\ & & & \cdots & & & & \\ 0 & 0 & \cdots & 0 & a_1 & \cdots & a_{d-1} & a_d \\ b_0 + \frac{a_1}{\lambda} & b_1 + \frac{a_2}{\lambda} & \cdots & b_{d-1} + \frac{a_d}{\lambda} & b_d & 0 & \cdots & 0 \\ 0 & b_0 + \frac{a_1}{\lambda} & b_1 + \frac{a_2}{\lambda} & \cdots & b_{d-1} + \frac{a_d}{\lambda} & b_d & \cdots & 0 \\ & & & \cdots & & & & \\ 0 & 0 & \cdots & b_0 + \frac{a_1}{\lambda} & b_1 + \frac{a_2}{\lambda} & \cdots & b_{d-1} + \frac{a_d}{\lambda} & b_d \end{vmatrix},$$

where $b_0 = C_n^2(\lambda)$ and $b_0 + a_1/\lambda = C_{n+1}(\lambda)$, the coefficients defined in (4.3). Therefore the resultants $\text{Res}(F_n)$ satisfy:

$$\begin{aligned} \text{Res}(F_{n+1}) &= \frac{C_{n+1}(\lambda)}{C_n^2(\lambda)} \text{Res}(A_\lambda \circ F_n) \\ &= \frac{C_{n+1}(\lambda)}{C_n^2(\lambda)} \text{Res}(A_\lambda)^{\deg(F_n)} \text{Res}(F_n)^{2 \deg(A_\lambda)} \\ &= \frac{C_{n+1}(\lambda)}{C_n^2(\lambda)} \text{Res}(A_\lambda)^{2^{n-1}} \text{Res}(F_n)^4 \end{aligned}$$

where the second equality follows from the decomposition property of resultants (see [De2, Proposition 6.1]).

We may compute from (4.5) that $C_1(\lambda) = 1$, $C_2(\lambda) = 1 + \lambda$, while for $n \geq 3$,

$$(6.3) \quad C_n(\lambda) = (1 + \lambda)^{2^{n-3}} (1 + \lambda + \lambda^2)^{2^{n-4}} \cdots (1 + \lambda + \cdots + \lambda^{n-2})^{2^0} (1 + \lambda + \cdots + \lambda^{n-1}).$$

From the definition of resultant, we have $\text{Res}(A_\lambda) = \lambda^2$ and $\text{Res}(F_1) = -\lambda$. Thus the recursive relation becomes

$$\text{Res}(F_{n+1}) = \frac{1 + \lambda + \cdots + \lambda^n}{1 + \lambda + \cdots + \lambda^{n-1}} \lambda^{2^n} \text{Res}(F_n)^4.$$

By induction, $\text{Res}(F_2) = \lambda^6(1 + \lambda)$, and for $n \geq 3$, we have

$$(6.4) \quad \text{Res}(F_n) = \lambda^{2 \cdot 4^{n-1} - 2^{n-1}} (1 + \lambda + \cdots + \lambda^{n-1}) \prod_{j=1}^{n-2} (1 + \lambda + \cdots + \lambda^j)^{3 \cdot 4^{n-2-j}}.$$

From equation (6.1), we conclude that

$$\begin{aligned} \text{cap}(K_\lambda^+) &= \lim_{n \rightarrow \infty} |\text{Res}(F_n)|^{-1/\deg(F_n)^2} \\ &= \lim_{n \rightarrow \infty} \left(\frac{|\lambda|^{-2+2^{1-n}}}{|1 + \lambda + \cdots + \lambda^{n-1}|^{4^{1-n}}} \prod_{j=1}^{n-2} |1 + \lambda + \cdots + \lambda^j|^{-3 \cdot 4^{-j-1}} \right) \\ &= \frac{1}{|\lambda|^2} \prod_{j=1}^{+\infty} |1 + \lambda + \cdots + \lambda^j|^{-3 \cdot 4^{-j-1}}. \end{aligned}$$

The proof for $\lambda \in \overline{\mathbb{Q}}$ and non-archimedean absolute values is identical, using (6.2). \square

6.3. Bounds for the homogeneous sets. Fix $\lambda \in \overline{\mathbb{Q}}$, not a root of unity (except possibly 1) and nonzero. In our proof of Theorem 1.3, we will need control over the diameter of $K_{\lambda,v}^\pm$ at most places v of the number field $k = \mathbb{Q}(\lambda)$. We define subsets of the set of places \mathcal{M}_k :

- for each $n \geq 1$, let $\mathcal{M}_{k,n}$ be the set of all non-archimedean places v for which $|\lambda|_v = 1$, $|1 + \lambda + \cdots + \lambda^i|_v = 1$ for all $i < n$, and $|1 + \lambda + \cdots + \lambda^n|_v < 1$; and
- let $\mathcal{M}_{k,0}$ be the set of all non-archimedean places v for which $|\lambda|_v = 1$ and $|1 + \lambda + \cdots + \lambda^i|_v = 1$ for all i .

Note that the set $\mathcal{M}_k \setminus \bigcup_{n \geq 0} \mathcal{M}_{k,n}$ is finite. (Indeed, for non-archimedean v , there is an integer $i > 0$ with $|1 + \lambda + \cdots + \lambda^i|_v > 1$ if and only if $|\lambda|_v > 1$. There are only finitely many such non-archimedean places. And there are only finitely many archimedean places.) Also, the set $\mathcal{M}_{k,0}$ might be empty, as will be the case for $\lambda = 1$.

Lemma 6.2. *For all v in $\mathcal{M}_{k,0}$, the sets K_v^+ and K_v^- are trivial; that is,*

$$G_v^\pm(t_1, t_2) = \log \|(t_1, t_2)\|_v$$

and $K_v^\pm = \bar{D}^2(0, 1)$.

Proof. For each $n \geq 1$, the coefficients of F_n will have absolute value ≤ 1 . From the formula for the resultant of F_n in (6.4), we see that $|\text{Res}(F_n)|_v = 1$ for all n . Applying [BR2, Lemma 10.1] to F_n , setting $B_1 = B_2 = 1$, we conclude that

$$\|F_n(t_1, t_2)\| = \|(t_1, t_2)\|^{\deg F_n}$$

for all $n \geq 1$. The conclusion follows immediately. \square

Proposition 6.3. Fix $\lambda \in \overline{\mathbb{Q}} \setminus \{0\}$ not a root of unity, or set $\lambda = 1$. There exists a constant $c = c(\lambda) > 0$ so that

$$\bar{D}^2(0, 1) \subset K_v^+ \subset \bar{D}^2(0, e^{-2\gamma_v(\lambda) - \frac{\log c}{2^{n-1}}})$$

for all $v \in \mathcal{M}_{k,n}$ and all $n \geq 1$.

Proof. Fix $n \geq 1$ and $v \in \mathcal{M}_{k,n}$. For each $m \geq 1$, the coefficients of F_m lie in the valuation ring of k (i.e. have absolute value ≤ 1). It follows immediately that $\|F_m(t, s)\|_v \leq 1$ for all m and all $\|(t, s)\|_v \leq 1$, and therefore, $G_v^+(t, s) \leq 0$ on $\bar{D}^2(0, 1)$. Consequently, $\bar{D}^2(0, 1) \subset K_v^+$.

Suppose $|s|_v = 1$ and $|t|_v \leq 1$. The resultant of the polynomial map $F_{\lambda,t}$ is λ^2 , so it has v -adic absolute value $= 1$. The non-archimedean estimates of [BR2, Lemma 10.1] imply that every iterate of $(z, w) = (1, 1)$ under $F_{\lambda,t/s}$ will have norm 1. Therefore, the v -adic norm of

$$F_m(t, s) = s^{2^{m-1}} F_{\lambda,t/s}^m(1, 1)$$

is also equal to 1 for all m . Consequently, $G_v^+(t, s) = 0$ whenever $|t|_v \leq 1$ and $|s|_v = 1$.

The coefficients C_i of F_i , defined in (4.5) with formulas in (6.3), satisfy $|C_i|_v = 1$ for $i \leq n$ and $|C_i|_v < 1$ for $i > n$. In fact, from the explicit expressions for C_i , we see that $|C_i|_v$ form a non-increasing sequence with

$$\lim_{i \rightarrow \infty} |C_i|_v = 0$$

and the sequence of expressions $\frac{\log |C_i|_v}{2^{i-1}}$ also form a non-increasing sequence with

$$\lim_{i \rightarrow \infty} \frac{\log |C_i|_v}{2^{i-1}} = \gamma_v(\lambda).$$

Fix $s \in \mathbb{C}_v$ with $|s|_v < 1$, and choose $j \geq n$ so that

$$|C_{j+1}|_v \leq |s|_v < |C_j|_v.$$

Then by Lemma 5.3 (applied exactly as in (4.10) in the proof of Theorem 4.1), we have

$$\frac{\log \|F_m^+(1, s)\|_v}{2^{m-1}} \geq \frac{\log \|F_j^+(1, s)\|_v}{2^{j-1}} + \frac{\log c|s|_v}{2^{j-1}}$$

for all $m \geq j$. Since $|C_{j+1}|_v \leq |s|_v < |C_j|_v$ and all the coefficients of $F_j(1, s)$ are bounded by 1, the constant term C_j dominates the norm of $F_j(1, s)$ (that is, $\|F_j(1, s)\|_v = |C_j|_v$); hence

$$\frac{\log \|F_m^+(1, s)\|_v}{2^{m-1}} \geq \frac{\log |C_j|_v}{2^{j-1}} + \frac{\log(c|C_{j+1}|_v)}{2^{j-1}}.$$

Letting $m \rightarrow \infty$, and since $\frac{\log |C_j|_v}{2^{j-1}} \geq \gamma_v(\lambda)$ for all j , we have

$$G_v^+(1, s) \geq 2\gamma_v(\lambda) + \frac{\log c}{2^{j-1}}.$$

It follows that

$$G_v^+(1, s) \geq 2\gamma_v(\lambda) + \frac{\log c}{2^{n-1}}$$

for all s with $|s|_v < 1$, and we conclude that

$$K_v^+ \subset \bar{D}^2(0, e^{-2\gamma_v(\lambda) - \frac{\log c}{2n-1}}).$$

□

7. THE EQUIDISTRIBUTION THEOREM

Throughout this section, we fix $\lambda \neq 0$ in $\overline{\mathbb{Q}}$, and fix a number field k containing λ . As in [CS], we define canonical height functions \hat{h}^+ and \hat{h}^- on parameters $t \in \text{Per}_1(\lambda)^{cm}(\overline{\mathbb{Q}})$, by

$$\hat{h}^\pm(t) := \hat{h}_{f_t}(\pm 1) = \lim_{n \rightarrow \infty} \frac{1}{2^n} h(f_t^n(\pm 1)),$$

where h is the logarithmic Weil height on $\mathbb{P}^1(\overline{\mathbb{Q}})$ and \hat{h}_{f_t} is the canonical height of the morphism f_t .

Theorem 7.1. *Assume that $\lambda \in \overline{\mathbb{Q}} \setminus \{0\}$ is not a root of unity, or set $\lambda = 1$. Let $\{S_n\}$ be any non-repeating sequence of $\text{Gal}(\overline{k}/k)$ -invariant finite sets in $\text{Per}_1(\lambda)^{cm}$ for which*

$$\hat{h}^+(S_n) \rightarrow 0$$

as $n \rightarrow \infty$. Then the sets S_n are equidistributed with respect to the measure μ_λ^+ . In fact, for each place v of k , the discrete measures

$$\mu_n = \frac{1}{|S_n|} \sum_{t \in S_n} \delta_t$$

converge weakly to the measure μ_v^+ on the Berkovich projective line $\mathbb{P}_v^{1,an}$. Similarly for \hat{h}^- and the measures $\{\mu_v^-\}$.

The main idea of the proof is to show that \hat{h}^+ and \hat{h}^- are canonically associated to the “quasi-adelic” measures $\{\mu_v^\pm\}$. We use Theorems 4.1 and 5.1. Then we may apply the arithmetic equidistribution theorem (as appearing in [Ye], modified from the original treatments in [BR2, FRL]) to obtain the theorem.

7.1. Quasi-adelic measures and equidistribution. For each $v \in \mathcal{M}_k$ there is a distribution-valued Laplacian operator Δ on $\mathbb{P}_{\text{Berk},v}^1$. For example, the function $\log^+ |z|_v$ on $\mathbb{P}^1(\mathbb{C}_v)$ extends naturally to a continuous real valued function $\mathbb{P}_{\text{Berk},v}^1 \setminus \{\infty\} \rightarrow \mathbb{R}$ and

$$\Delta \log^+ |z|_v = \lambda_v - \delta_\infty,$$

where λ_v is the uniform probability measure on the complex unit circle $\{|z| = 1\}$ when v is archimedean and λ_v is a point mass at the Gauss point of $\mathbb{P}_{\text{Berk},v}^1$ when v is non-archimedean. (The sign of the Laplacian Δ is reversed from that of [BR2] or the presentation in [BD2], to match the sign convention from complex analysis.)

A probability measure μ_v on $\mathbb{P}_{\text{Berk},v}^1$ is said to have a *continuous potential* if $\mu_v - \lambda_v = \Delta g$ with $g : \mathbb{P}_{\text{Berk},v}^1 \rightarrow \mathbb{R}$ continuous. If μ_v has a continuous potential then there is a corresponding *Arakelov-Green function* $g_{\mu_v} : \mathbb{P}_{\text{Berk},v}^1 \times \mathbb{P}_{\text{Berk},v}^1 \rightarrow \mathbb{R} \cup \{+\infty\}$ which is characterized by the differential equation $\Delta_x g_{\mu_v}(x, y) = \mu - \delta_y$ and the normalization

$$(7.1) \quad \iint g_{\mu_v}(x, y) d\mu(x) d\mu(y) = 0.$$

Working with homogeneous coordinates, g_{μ_v} may be computed in terms of a continuous potential function for μ_v ,

$$G_v : (\mathbb{C}_v)^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$$

satisfying $G_v(z_1, z_2) = g(z_1/z_2) + \log^+ |z_1/z_2|_v + \log |z_2|_v$ for some continuous potential g as described above. For $x, y \in \mathbb{P}^1(\mathbb{C}_v)$, the Arakelov-Green function for μ_v is given by

$$(7.2) \quad g_{\mu_v}(x, y) = -\log |\tilde{x} \wedge \tilde{y}|_v + G_v(\tilde{x}) + G_v(\tilde{y}) + \log \text{cap}(K_v),$$

for any choice of lifts \tilde{x} of x and \tilde{y} of y to $(\mathbb{C}_v)^2$. Here

$$K_v = \{(a, b) \in (\mathbb{C}_v)^2 : G_v(a, b) \leq 0\}.$$

The homogeneous capacity $\text{cap}(K_v)$ is exactly what is needed to normalize g_{μ_v} according to (7.1). See [BR2, §10.2] for details, in the setting where K_v is the filled Julia set of a homogeneous polynomial lift of a rational function defined over k .

A *quasi-adelic measure* on \mathbb{P}^1 (with respect to the field k) is a collection $\mu = \{\mu_v\}_{v \in M_k}$ of probability measures on $\mathbb{P}_v^{1,an}$ with continuous potentials for which the product

$$\prod_{v \in M_k} (r(K_v) / \text{cap}(K_v)^{1/2})^{N_v}$$

converges strongly to a positive real number, where

$$r(K_v) = \inf\{r > 0 : K_v \subset \bar{D}_v^2(0, r)\}$$

is the outer radius of K_v . (Strong convergence of a product is, by definition, absolute convergence of the sum of logarithms of the entries.)

If ρ, ρ' are measures on $\mathbb{P}_v^{1,an}$, we define the μ_v -energy of ρ and ρ' by

$$(\rho, \rho')_{\mu_v} := \frac{1}{2} \iint_{\mathbb{P}_{\text{Berk},v}^1 \times \mathbb{P}_{\text{Berk},v}^1 \setminus \text{Diag}} g_{\mu_v}(x, y) d\rho(x) d\rho'(y).$$

Let S be a finite, $\text{Gal}(\bar{k}/k)$ -invariant subset of $\mathbb{P}^1(\bar{k})$ with $|S| > 1$. For each $v \in \mathcal{M}_k$, we denote by $[S]_v$ the discrete probability measure on $\mathbb{P}_v^{1,an}$ supported equally on the elements of S . For a quasi-adelic measure $\mu = \{\mu_v\}$, the μ -canonical height of S is defined by

$$(7.3) \quad \hat{h}_\mu(S) := \frac{|S|}{|S| - 1} \sum_{v \in \mathcal{M}_k} N_v \cdot ([S]_v, [S]_v)_{\mu_v}.$$

The constants N_v are the same as those appearing in the product formula (5.1).

Remark. The definition of \hat{h}_μ differs slightly from that given in [BD2] or [FRL], but agrees with the definition in [DWY]; the factor of $|S|/(|S| - 1)$ is included to match the usual definition of canonical height. See Proposition 7.6 and [BR2, Lemma 10.27]. With this normalization, the function \hat{h}_μ will extend naturally to sets with $|S| = 1$, to define a function on $\mathbb{P}^1(\bar{k})$.

The following equidistribution theorem is a modification of the ones appearing in [BR2, FRL]; the proof is given in [Ye].

Theorem 7.2. *Let \hat{h}_μ be the canonical height associated to a quasi-adelic measure μ . Let $\{S_n\}_{n \geq 0}$ be any non-repeating sequence of $\text{Gal}(\bar{k}/k)$ -invariant finite subsets of $\mathbb{P}^1(\bar{k})$ for which $\hat{h}_\mu(S_n) \rightarrow 0$ as $n \rightarrow \infty$. Then $[S_n]_v$ converges weakly to μ_v on $\mathbb{P}_{\text{Berk},v}^1$ as $n \rightarrow \infty$ for all $v \in \mathcal{M}_k$.*

7.2. Bifurcation measures are quasi-adelic. Now we prove that the escape-rate functions G_v^\pm from Theorems 4.1 and 5.1 are potential functions for a quasi-adelic measure. For each $n \geq 1$, recall from §6.3 that $\mathcal{M}_{k,n}$ denotes the set of all non-archimedean places in \mathcal{M}_k such that $|\lambda|_v = 1$, $|1 + \lambda + \cdots + \lambda^i|_v = 1$ for all $i < n$, and $|1 + \lambda + \cdots + \lambda^n|_v < 1$.

Lemma 7.3. *For any $\lambda \in \overline{\mathbb{Q}} \setminus \{0\}$ not a root of unity, or for $\lambda = 1$, there exists a constant $C = C(\lambda, k)$ so that*

$$|\mathcal{M}_{k,n}| \leq C n$$

for all $n \geq 1$, where the places $v \in \mathcal{M}_{k,n}$ are counted with multiplicity N_v .

Proof. We begin with a basic observation from algebraic number theory. Let $m = [k : \mathbb{Q}]$, the degree of the field extension. Suppose that v is a place of k extending the p -adic absolute value on \mathbb{Q} for a prime p . Then

$$|x|_v < 1 \implies |x|_v \leq \frac{1}{p^{1/m}}$$

for all $x \in k$. Indeed, the absolute value will be bounded by $p^{-1/e}$ where e is the index of ramification of the field k at the prime p ; and $e \leq m$.

The proof of the lemma follows from the product formula and the above control on the absolute values. There are only finitely many places $v \in \mathcal{M}_k$ for which $|1 + \lambda + \cdots + \lambda^i|_v > 1$ for some $i \geq 1$. Then there is an $M > 1$ such that for any $n \geq 1$,

$$\prod_{v \in \mathcal{M}_k, |1 + \lambda + \cdots + \lambda^n|_v > 1} |1 + \lambda + \cdots + \lambda^n|_v^{N_v} \leq M^n.$$

For each $v \in \mathcal{M}_{k,n}$, we have

$$|1 + \lambda + \cdots + \lambda^n|_v^{N_v} \leq \frac{1}{2^{\frac{N_v}{m}}}.$$

By the product formula, we see that

$$\begin{aligned} \prod_{v \in \mathcal{M}_{k,n}} |1 + \lambda + \cdots + \lambda^n|_v^{N_v} &\geq \prod_{v, |1 + \lambda + \cdots + \lambda^n|_v < 1} |1 + \lambda + \cdots + \lambda^n|_v^{N_v} \\ &= \prod_{v, |1 + \lambda + \cdots + \lambda^n|_v > 1} |1 + \lambda + \cdots + \lambda^n|_v^{-N_v} \\ &\geq \frac{1}{M^n}. \end{aligned}$$

Therefore,

$$\frac{M^n}{2^{\frac{|\mathcal{M}_{k,n}|}{m}}} \geq 1.$$

and we conclude that

$$|\mathcal{M}_{k,n}| \leq nm \log_2 M,$$

for any n . Set $C = m \log_2 M$. □

Proposition 7.4. *For each $\lambda \in \overline{\mathbb{Q}} \setminus \{0\}$ not a root of unity, or for $\lambda = 1$, the bifurcation measures $\{\mu_v^+\}$ and $\{\mu_v^-\}$ are quasi-adelic.*

Proof. As continuity of the potentials has already been established (Theorem 5.1), we need only show that the product

$$\prod_{v \in \mathcal{M}_k} (r(K_v) / \text{cap}(K_v)^{1/2})^{N_v}$$

converges strongly to a positive real number. That is, we need to show the absolute convergence of the sum

$$\sum_{v \in \mathcal{M}_k} N_v \log |r(K_v) / \text{cap}(K_v)^{1/2}|.$$

Lemma 7.5 implies that $\prod \text{cap}(K_v)^{N_v}$ converges strongly to 1. It remains to show the strong convergence of $\prod_v r(K_v)^{N_v}$. Recall the definitions of $\mathcal{M}_{k,n}$ and $\mathcal{M}_{k,0}$ from §6.3, and recall that the set of places not in $\mathcal{M}_{k,0}$ or $\mathcal{M}_{k,n}$ for any n is finite. From Lemma 6.2, we have $K_v = \overline{D}^2(0, 1)$ for all $v \in \mathcal{M}_{k,0}$ so that $r(K_v) = 1$. For $v \in \mathcal{M}_{k,n}$, Proposition 6.3 shows that

$$1 \leq r(K_v) \leq e^{-2\gamma_v(\lambda) - (\log c)/2^{n-1}}.$$

Strong convergence of $\prod_v r(K_v)^{N_v}$ will then follow from convergence of

$$\sum_{n=1}^{\infty} \sum_{v \in \mathcal{M}_{k,n}} N_v \left(-2\gamma_v(\lambda) - \frac{\log c}{2^{n-1}} \right).$$

Lemma 7.5 implies that the sum of the $N_v \gamma_v(\lambda)$ terms will converge. Lemma 7.3 shows that $|\mathcal{M}_{k,n}| \leq Cn$ when counted with multiplicities N_v , showing that the sum of the $(\log c)/2^{n-1}$ terms will also converge. □

7.3. Equivalence of two canonical heights. Now we show that the Call-Silverman heights \hat{h}^\pm , defined at the beginning of §7, coincide with the μ^\pm -canonical heights associated to the quasi-adelic measures $\{\mu_v^\pm\}$, defined in (7.3). We begin with a lemma. In other settings (e.g. those in [BR2, Chapter 10], [BD2] or [DWY]), the analogous conclusion of Lemma 7.5 would be immediate from the product formula, since all but finitely many terms would be 0. Recall that the definition of $\gamma_v(\lambda)$ appears in Lemma 5.2 and the formula for the capacity is given in Theorem 6.1.

Lemma 7.5. *Fix $\lambda \in \overline{\mathbb{Q}} \setminus \{0\}$ not a root of unity, or set $\lambda = 1$. The sums*

$$\sum_{v \in \mathcal{M}_k} N_v \gamma_v(\lambda) \quad \text{and} \quad \sum_{v \in \mathcal{M}_k} N_v \log \text{cap}(K_v^\pm)$$

converge absolutely, and the sum is equal to 0.

Proof. There are only finitely many places v for which there is an integer $i > 0$ with $|1 + \lambda + \cdots + \lambda^i|_v > 1$. (For non-Archimedean v , this condition is equivalent to $|\lambda|_v > 1$; there are only finitely many such non-Archimedean places.) Therefore, there exists $M > 1$ such that

$$\prod_{v \in \mathcal{M}_k, |1 + \lambda + \cdots + \lambda^n|_v > 1} |1 + \lambda + \cdots + \lambda^n|_v^{N_v} \leq M^n$$

for all $n \geq 1$. From the product formula, we see that

$$\prod_{v \in \mathcal{M}_k, |1 + \lambda + \cdots + \lambda^n|_v < 1} |1 + \lambda + \cdots + \lambda^n|_v^{N_v} \geq \frac{1}{M^n},$$

though the number of such places grows with n . Consequently,

$$\sum_{n=1}^{\infty} \frac{1}{r^n} \sum_{v \in \mathcal{M}_k} N_v |\log |1 + \lambda + \cdots + \lambda^n|_v| < \infty$$

for any constant $r > 1$. Therefore, we can interchange the order of summation and deduce that

$$\sum_v N_v \sum_{n=1}^{\infty} \frac{1}{r^n} \log |1 + \lambda + \cdots + \lambda^n|_v = \sum_{n=1}^{\infty} \frac{1}{r^n} \sum_v N_v \log |1 + \lambda + \cdots + \lambda^n|_v = 0.$$

For $r = 2$, we obtained the desired convergence for $\gamma_v(\lambda)$, and for $r = 4$ the sum of the capacities. \square

Proposition 7.6. *For each $\lambda \in \overline{\mathbb{Q}} \setminus \{0\}$, not a root of unity, or for $\lambda = 1$, the Call-Silverman canonical height \hat{h}^+ and the $\{\mu_v^+\}$ -canonical height \hat{h}_μ are related by*

$$\hat{h}_\mu(S) = \frac{2[k : \mathbb{Q}]}{|S|} \sum_{t \in S} \hat{h}^+(t)$$

for any $\text{Gal}(\overline{k}/k)$ -invariant, finite set S with $|S| > 1$. Similarly for \hat{h}^- and the $\{\mu_v^-\}$ -canonical height.

Proof. Fix a finite set $S \subset \bar{k}$ which is $\text{Gal}(\bar{k}/k)$ -invariant and has at least two elements. We begin by computing the $\{\mu_v^+\}$ -canonical height of S , from the definition given in (7.3). For each $t \in S$, we choose a lift $\tilde{t} \in \bar{k}^2$ of t .

$$\begin{aligned}
\hat{h}_\mu(S) &= \frac{|S|}{|S|-1} \sum_{v \in \mathcal{M}_k} N_v \cdot ([S]_v, [S]_v)_{\mu_v} \\
&= \frac{|S|}{|S|-1} \sum_{v \in \mathcal{M}_k} \frac{N_v}{2|S|^2} \sum_{x, y \in S, x \neq y} g_{\mu_v}(x, y) \\
&= \frac{1}{2|S|(|S|-1)} \sum_{x, y \in S, x \neq y} \sum_{v \in \mathcal{M}_k} N_v (-\log |\tilde{x} \wedge \tilde{y}|_v + G_v^+(\tilde{x}) + G_v^+(\tilde{y}) + \log \text{cap}(K_{\lambda, v}^+)) \\
&= \frac{1}{2|S|(|S|-1)} \sum_{x, y \in S, x \neq y} \sum_{v \in \mathcal{M}_k} N_v (G_v^+(\tilde{x}) + G_v^+(\tilde{y})) \quad \text{by (5.1) and Lemma 7.5} \\
&= \frac{1}{2|S|(|S|-1)} \sum_{v \in \mathcal{M}_k} N_v \cdot 2(|S|-1) \sum_{x \in S} G_v^+(\tilde{x}) \\
&= \frac{1}{|S|} \sum_{v \in \mathcal{M}_k} N_v \cdot \sum_{x \in S} G_v^+(\tilde{x}) \\
&= \frac{1}{|S|} \cdot \sum_{x \in S} \sum_{v \in \mathcal{M}_k} N_v G_v^+(\tilde{x})
\end{aligned}$$

Note that the product formula (and homogeneity of G_v^+) implies that $\sum_v G_v^+(\tilde{x})$ depends on x but is independent of the choice of \tilde{x} . This formula for \hat{h}_μ also shows that it extends to define a function on points of $\mathbb{P}^1(k)$, as mentioned in the remark after equation (7.3).

Recall that the Weil height of $\alpha \in \bar{k}$ may be computed as

$$h(\alpha) = \frac{1}{[k(\alpha) : \mathbb{Q}]} \sum_{v \in \mathcal{M}_{k(\alpha)}} N_v \log \|(\alpha_1, \alpha_2)\|_v = \frac{1}{[k(\alpha) : \mathbb{Q}]} \sum_{v \in \mathcal{M}_{k(\alpha)}} N_v \log \max\{|\alpha_1|_v, |\alpha_2|_v\}$$

for any homogeneous presentation $(\alpha_1, \alpha_2) \in k(\alpha)^2$ of α (so $\alpha = \alpha_1/\alpha_2$). Further, if A is a $\text{Gal}(\bar{k}/k)$ -invariant and finite set, then

$$h(A) := \frac{1}{|A|} \sum_{\alpha \in A} h(\alpha) = \frac{1}{[k : \mathbb{Q}]} \frac{1}{|A|} \sum_{\alpha \in A} \sum_{v \in \mathcal{M}_k} N_v \log \max\{|\alpha_1|_v, |\alpha_2|_v\},$$

where $|\cdot|_v$ is a fixed choice of extension to \bar{k} of the absolute value $|\cdot|_v$ on k .

For each $t \in S$, from the definition in (4.1), the point $F_n^+(\tilde{t}) \in \bar{k}^2$ is a homogeneous presentation of $f_t^n(+1) \in \bar{k}$. The Galois invariance of S implies that

$$(7.4) \quad \hat{h}^+(S) = \frac{1}{|S|} \sum_{t \in S} \hat{h}^+(t) = \frac{1}{[k : \mathbb{Q}]} \frac{1}{|S|} \sum_{t \in S} \lim_{n \rightarrow \infty} \frac{1}{2^n} \sum_{v \in \mathcal{M}_k} N_v \log \|F_n(\tilde{t})\|_v.$$

We need to show that we can interchange the limit and the infinite sum over \mathcal{M}_k . Then Theorems 4.1 and 5.1 will imply that

$$\hat{h}^+(S) = \frac{1}{2|S|[k:\mathbb{Q}]}\sum_{t \in S} \sum_{v \in \mathcal{M}_k} N_v G_v^+(\tilde{t}) = \frac{1}{2[k:\mathbb{Q}]} \hat{h}_\mu(S),$$

completing the proof of the theorem.

To see that we may interchange the limit and the sum in (7.4), we can use Lemma 6.2 and Proposition 6.3. Outside of a finite number of places, we have $\|\tilde{t}\|_v = 1$ for all $t \in S$. For $v \in \mathcal{M}_{k,0}$, Lemma 6.2 states that $\|F_n(\tilde{t})\|_v = 1$ for all n when $\|\tilde{t}\|_v = 1$, and so these terms do not contribute to the sum of (7.4). For $v \in \mathcal{M}_{k,m}$ with $m \geq 1$, Proposition 6.3 implies that

$$e^{2^{n-1}(2\gamma_v(\lambda) + (\log c)/2^{m-1})} \leq \|F_n(\tilde{t})\|_v \leq 1$$

for all $n \geq 1$, when $\|\tilde{t}\|_v = 1$. Since the sum of the $\gamma_v(\lambda)$ converges absolutely (Lemma 7.5) and there aren't too many elements in each $\mathcal{M}_{k,m}$ (Lemma 7.3), we deduce that for any $\varepsilon > 0$, there is a finite set $\mathcal{M}(\varepsilon)$ of places so that

$$\frac{1}{2^n} \sum_{v \in \mathcal{M}_k \setminus \mathcal{M}(\varepsilon)} N_v |\log \|F_n(\tilde{t})\|_v| < \varepsilon$$

for every $n \geq 1$. This is enough to allow the exchange of the limit with the sum in (7.4).

The proof for \hat{h}^- and the $\{\mu_v^-\}$ -canonical height is identical. \square

7.4. Proofs of the equidistribution theorems. We are ready to complete the proofs of the equidistribution theorems.

Proof of Theorem 7.1. The proof is immediate from Theorem 7.2, once we know that the canonical height functions \hat{h}^+ and \hat{h}^- are associated to the quasi-adelic measures $\{\mu_v^\pm\}$. That is the content of Propositions 7.4 and 7.6. \square

Proof of Theorem 1.3. This theorem is an immediate corollary of Theorem 7.1, because the parameters where the critical point ± 1 has finite orbit coincide with the points of canonical height 0, by [CS, Corollary 1.1.1]. \square

8. PROOF OF THEOREM 1.1

In this section, we complete the proof of Theorem 1.1. As discussed in the introduction, one implication is well known; namely, the family of quadratic polynomials $\text{Per}_1(0)$ contains infinitely many postcritically-finite maps. Indeed, as explained in Example 3.1, the bifurcation locus for the (non-fixed) critical point is nonempty. The conclusion then follows from Lemma 2.1.

For dynamical reasons, there can be no postcritically-finite maps in $\text{Per}_1(\lambda)$ for $0 < |\lambda| \leq 1$. Indeed, if f has a fixed point of multiplier λ , at least one critical point must have infinite forward orbit, as it is attracted to (or accumulates upon) the fixed point (or on the boundary of the Siegel disk in case the fixed point is of Siegel type). See [Mi2, Corollary

14.5]. (Actually, for our proof of Theorem 1.1, we only need the easier fact that parabolic cycles always attract a critical point with infinite forward orbit [Mi2, §10], since all other cases follow from the arguments below.)

Now assume that $\lambda \in \mathbb{C}$, $|\lambda| > 1$, is chosen so that $\text{Per}_1(\lambda)$ contains infinitely many postcritically-finite maps. We shall derive a contradiction. We first observe that $\lambda \in \overline{\mathbb{Q}}$, as a consequence of Thurston Rigidity.

Proposition 8.1. *If $f_{\lambda,t}(z) = \lambda z/(z^2 + tz + 1)$ is postcritically finite, then $\lambda, t \in \overline{\mathbb{Q}}$.*

Proof. The critical points of a postcritically-finite map satisfy two equations

$$f_{\lambda,t}^n(+1) = f_{\lambda,t}^m(+1) \quad \text{and} \quad f_{\lambda,t}^r(-1) = f_{\lambda,t}^s(-1)$$

for pairs of integers $n > m \geq 0$ and $r > s \geq 0$. Note that these two equations define polynomials in (λ, t) with coefficients in \mathbb{Q} . By Thurston Rigidity (see [Mc1, Theorem 2.2]), we know that the set of solutions must be finite (or empty), since there are no flexible Lattès maps in degree 2. Consequently, a solution (λ, t) must have coordinates in $\overline{\mathbb{Q}}$. \square

Let $k = \mathbb{Q}(\lambda)$. Then k is a number field, by Proposition 8.1. Construct the Call-Silverman canonical height functions \hat{h}_λ^+ and \hat{h}_λ^- on \bar{k} as in Section 7. Let $\{t_n\}_{n \in \mathbb{N}}$ denote a sequence of parameters in $\text{Per}_1(\lambda)^{cm}$ for which both critical points have finite orbit. Proposition 8.1 also shows that $t_n \in \bar{k}$ for all n . Let S_n denote the $\text{Gal}(\bar{k}/k)$ -orbit of t_n . From [CS, Corollary 1.1.1], we see that

$$\hat{h}_\lambda^+(S_n) = \hat{h}_\lambda^-(S_n) = 0$$

for all n . By Theorem 7.1, the sequence $\{S_n\}$ is equidistributed with respect to the bifurcation measures μ_v^+ and μ_v^- at all places v of k . It follows that $\mu_v^+ = \mu_v^-$ for all v , and therefore the quasi-adelic height functions (defined in (7.3)) must coincide. From Proposition 7.6, we conclude that

$$\hat{h}_\lambda^+ = \hat{h}_\lambda^-$$

on $\text{Per}_1(\lambda)^{cm}$. Again appealing to [CS, Corollary 1.1.1], we find that the critical point $+1$ will have finite orbit for f_t if and only if -1 has finite orbit for f_t . This conclusion contradicts Proposition 2.5. The proof is complete.

Remark. In our proof, we have used the full strength of Theorem 7.1, with the equidistribution at all places of the number field $k = \mathbb{Q}(\lambda)$ to deduce that $\hat{h}^+ = \hat{h}^-$. Alternatively, we could have used only Theorem 1.3, the equidistribution to the (complex) bifurcation measure μ_λ^+ , and deduced that $\mu_\lambda^+ = \mu_\lambda^-$. This conclusion would contradict Theorem 1.2.

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