DISCONTINUITY OF A DEGENERATING ESCAPE RATE

LAURA DEMARCO AND YÛSUKE OKUYAMA

Abstract. We look at degenerating meromorphic families of rational maps on $\mathbb{P}^1$ – holomorphically parameterized by a punctured disk – and we provide examples where the bifurcation current fails to have a bounded potential in a neighborhood of the puncture. This is in contrast to the recent result of Favre-Gauthier that we always have continuity across the puncture for families of polynomials; and it provides a counterexample to a conjecture posed by Favre in 2016. We explain why our construction fails for polynomial families and for families of rational maps defined over finite extensions of the rationals $\mathbb{Q}$.

1. Introduction

Let f_t be a holomorphic family of rational maps on $\mathbb{P}^1$ of degree $d > 1$, parameterized by the punctured unit (open) disk $\mathbb{D}^* = \{t \in \mathbb{C} : 0 < |t| < 1\}$, and assume that the coefficients of f_t extend to meromorphic functions on the unit disk $\mathbb{D} = \{t \in \mathbb{C} : |t| < 1\}$. Let $a : \mathbb{D} \to \mathbb{P}^1$ be a holomorphic map. In this article, we examine the potential function $g_{f,a}$ on $\mathbb{D}^*$ (having the order $o(\log|t|)$ as $t \to 0$) for the bifurcation measure associated to the pair (f, a). Our main result is that this potential function does not necessarily extend continuously across the puncture at $t = 0$.

The question of continuous extendability of $g_{f,a}$ across $t = 0$ arose naturally in the study of degenerating families of rational maps, and specifically in the context of equidistribution questions and height functions associated to the family f_t; see, e.g., [BD, Fa2]. Continuity of the potential at $t = 0$ was required to apply certain equidistribution theorems on arithmetic varieties (as in the proofs of the main results of [BD, DM, FG1, GY], and others). Moreover, when $a(t)$ parameterizes a critical point of f_t, the bifurcation measure and its potential are related to the structural stability of the family f_t [De1, DF]. It is well known that continuity holds when f_t has a uniform limit on the whole $\mathbb{P}^1$ as $t \to 0$, for any choice of a. It is also true when f_t is any family of polynomials with coefficients meromorphic in t, again for any choice of a [FG2]

To formulate the problem and our construction more precisely, we will work with f_t in homogeneous coordinates: assume that we are given a family of homogeneous polynomial maps $\tilde{f}_t : \mathbb{C}^2 \to \mathbb{C}^2$ of degree d, where the coefficients are holomorphic functions on the

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entire disk $\mathbb{D}$, such that for every $t \in \mathbb{D}^*$, $\tilde{f}_t^{-1}(0,0) = \{(0,0)\}$ and $\tilde{f}_t$ projects to f_t on $\mathbb{P}^1$. There exists a continuous plurisubharmonic escape rate

$$G_{\tilde{f}} : \mathbb{D}^* \times (\mathbb{C}^2 \setminus \{(0,0)\}) \to \mathbb{R}$$

such that for each fixed $t \in \mathbb{D}^*$, the current $dd^c G_{\tilde{f}}(t, \cdot)$ on $\mathbb{C}^2 \setminus \{(0,0)\}$ projects to the measure of maximal entropy of f_t on $\mathbb{P}^1$ [HP, FS]. Given a holomorphic lift of a to $\tilde{a} : \mathbb{D} \to \mathbb{C}^2 \setminus \{(0,0)\}$, we may write

$$G_{\tilde{f}}(t, \tilde{a}(t)) = \eta \log |t| + g_{f,a}(t), \tag{1.1}$$

where $\eta \in \mathbb{R}$ represents a "local height" for the pair (f, a), and the function $g_{f,a}$ on $\mathbb{D}^*$ satisfies

$$g_{f,a}(t) = o(\log |t|)$$

as $t \to 0$ [De4]; see §2.1. The value of η and the subharmonic function $g_{f,a}$ depend on the choices of $\tilde{f}$ and $\tilde{a}$, but $g_{f,a}$ is uniquely determined up to the addition of a harmonic function on $\mathbb{D}^*$ which is bounded near $t = 0$. The Laplacian $\mu_{f,a} = \frac{1}{2\pi} \Delta g_{f,a}$ on $\mathbb{D}^*$ is the *bifurcation measure associated to the pair* (f, a) [DF, §3].

It turns out that the function $g_{f,a}$ is always bounded from above near $t = 0$ (Lemma 2.1). In this article, we construct examples of pairs (f, a) that satisfy

$$\limsup_{t \to 0} g_{f,a}(t) = -\infty$$

to show that it need not be bounded from below, so in particular does not extend continuously across $t = 0$. In our examples, the maps f_t will converge to a rational map φ on $\mathbb{P}^1$ of degree $< d$ as $t \to 0$ locally uniformly on $\mathbb{P}^1 \setminus H$, where H is a non-empty finite set. The idea of the construction is to choose φ and a so that some sequence of iterates $\varphi^{n_j}(a(0))$ accumulates fast on H as $n_j \to \infty$.

Furthermore, choosing $a(t)$ to parameterize a critical point of the family f_t, we obtain a counterexample to the continuity statement in [Fa2, Conjecture 1], in proving:

Theorem 1.1. *For every integer $d > 1$, there exists a holomorphic family f_t of rational maps on $\mathbb{P}^1$ of degree d, parameterized by $t \in \mathbb{D}^*$, whose coefficients extend to meromorphic functions on $\mathbb{D}$ but for which the bifurcation current associated to the family f_t fails to have a bounded potential in any punctured neighborhood of $t = 0$.*

Remark. It will be clear from the proof that the family f_t can be chosen to be algebraic, in the sense that it extends to define a holomorphic family parameterized by t in a quasiprojective curve X, with coefficients that are meromorphic on a compactification of X.

The *bifurcation current* associated to the family f_t is equal to the Laplacian of the continuous and subharmonic function $t \mapsto L(f_t)$ on $\mathbb{D}^*$, where for each t, $L(f_t)$ is the Lyapunov exponent of f_t with respect to its unique measure of maximal entropy. For more details on $L(f_t)$ and its relationship to $g_{f,a}$, see Section 3.

The construction of examples of pairs (f, a) for which $g_{f,a}$ fails to extend continuously across $t = 0$ is laid out in Section 2. Our use of the Baire category theorem in the construction is similar to that of [Fa1, Example 4], [Bu], or [DG] in the context of higher-dimensional (bi)rational maps. In Section 3, we give the proof of Theorem 1.1. In Section 4, we comment on why the strategy for producing these examples fails for families of polynomials and for rational maps on $\mathbb{P}^1$ defined over $\overline{\mathbb{Q}}$. We expect that a continuous extension of $g_{f,a}$ to $\mathbb{D}$ always exists when the pair (f, a) is algebraic and defined over $\overline{\mathbb{Q}}$, as is known for algebraic families of elliptic curves [Si2, Theorem II.0.1] and therefore also for Lattès maps on $\mathbb{P}^1$ [DM, Proposition 3.4]; see also [JR, Theorem A] in the context of (bi)rational maps in dimension 2. The bifurcation current associated to a family f was introduced in [De1]; its properties at infinity in the moduli space of quadratic rational maps (related to our Theorem 1.1) were studied in [BG].

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2. A recipe for discontinuity

In this section, we construct the examples for which $g_{f,a}$ fails to extend continuously to the disk $\mathbb{D}$.

2.1. **The potential is bounded from above.** Suppose we are given a family of homogeneous polynomial maps $\tilde{f}_t : \mathbb{C}^2 \to \mathbb{C}^2$ of degree $d > 1$, where the coefficients are holomorphic functions on the entire disk $\mathbb{D}$, and such that for every $t \in \mathbb{D}^*$, we have $\tilde{f}_t^{-1}(0,0) = \{(0,0)\}$ so $\tilde{f}_t$ projects to a rational map f_t on $\mathbb{P}^1$ of degree d. We let $\tilde{a} : \mathbb{D} \to \mathbb{C}^2 \setminus \{(0,0)\}$ be any holomorphic map, and let $a : \mathbb{D} \to \mathbb{P}^1$ be its projection. For each $n \in \mathbb{N}$, there is a unique non-negative integer o_n so that

$$F_n(t) := t^{-o_n} \tilde{f}_t^n(\tilde{a}(t))$$

is a holomorphic map from $\mathbb{D}$ to $\mathbb{C}^2 \setminus \{(0,0)\}$. Choose any norm $\|\cdot\|$ on $\mathbb{C}^2$. The function $g_{f,a}$ on $\mathbb{D}^*$ defined by (1.1) is the locally uniform limit on $\mathbb{D}^*$ of the sequence of continuous and subharmonic functions

$$g_n(t) := \frac{1}{d^n} \log \|F_n(t)\| \quad \text{on } \mathbb{D},$$

as $n \to \infty$, and the value η of (1.1) is given by

$$\eta = \lim_{n\to\infty} \frac{o_n}{d^n},$$

as explained in [De4, §3]. Note, in particular, that the function $g_{f,a}$ is continuous and subharmonic on $\mathbb{D}^*$.

The following observation is not required in this section, but it will be useful in Section 3.

Lemma 2.1. *The function $g_{f,a}$ is bounded from above on $\{0 < |t| \le r\}$ for every $r \in (0, 1)$.*

Proof. Fix any $r \in (0,1)$. We know that $\lim_{n\to\infty} g_n = g_{f,a}$ uniformly on the circle $C_r = \{|t| = r\}$. Let $M_r = \max_{C_r} g_{f,a}$. Then, for all n large enough, we have $g_n \leq M_r + 1$ on C_r. As the functions g_n are subharmonic on $\mathbb{D}$, we also have $g_n \leq M_r + 1$ on the disk $\{|t| \leq r\}$ for all n large enough. It follows that $g_{f,a} \leq M_r + 1$ on the punctured disk $\{0 < |t| \leq r\}$. □

Remark. As an immediate consequence of Lemma 2.1, the function $g_{f,a}$ extends to a subharmonic function on the disk $\mathbb{D}$, though its value at $t = 0$ may be $-\infty$. In this article, however, we will keep the domain of definition of $g_{f,a}$ as the punctured disk $\mathbb{D}^*$.

2.2. **The ingredients for discontinuity.** Let $\varphi \in \mathbb{C}(z)$ be a rational map on $\mathbb{P}^1$ of degree $e \geq 1$, and suppose that there is a point $a_0 \in \mathbb{C}$ such that $\#\{\varphi^n(a_0) : n \in \mathbb{N}\} = \infty$ and that $\omega_\varphi(a_0) \cap \{\varphi^n(a_0) : n \in \mathbb{N}\} \neq \emptyset$, where

$$\omega_\varphi(a_0) := \bigcap_{N\in\mathbb{N}} \overline{\{\varphi^n(a_0) : n > N\}}$$

is the ω-limit set of a_0 under φ. Then there exists $N_0 \in \mathbb{N}$ so that $\{\varphi^n(a_0) : n \geq N\}$ is dense in $\omega_\varphi(a_0)$ for all $N \geq N_0$.

Let $\{r_n\}$ be any sequence in $\mathbb{R}_{>0}$ decreasing to 0 as $n \to \infty$, which will be chosen appropriately later. It follows that the set

$$U_N(a_0, \{r_n\}) := \left(\bigcup_{n\geq N} \{z \in \omega_\varphi(a_0) : [z, \varphi^n(a)] < r_n\} \right) \setminus \{a_0, \varphi(a_0), \ldots, \varphi^{N-1}(a_0)\}$$

is open and dense in $\omega_\varphi(a_0)$ for all $N \geq N_0$. Here $[\cdot,\cdot]$ denotes the chordal distance on $\mathbb{P}^1$. Therefore, by the Baire category theorem,

$$B_\varphi(a_0, \{r_n\}) := \bigcap_{N\geq N_0} U_N(a_0, \{r_n\})$$

is dense in $\omega_\varphi(a_0)$.

Fix any $h \in B_\varphi(a_0, \{r_n\}) \cap \mathbb{C}$. Then $\varphi^n(a_0) \neq h$ for all $n \in \mathbb{N} \cup \{0\}$, and there is a sequence $n_j \to \infty$ such that

$$0 < [\varphi^{n_j}(a_0), h] < r_{n_j} \tag{2.1}$$

for all $j \in \mathbb{N}$.

We consider the family

$$f_t(z) := \varphi(z) \cdot \frac{z - h - \varepsilon t}{z - h + \varepsilon t} \tag{2.2}$$

parameterized by $t \in \mathbb{D}^*$, where $\varepsilon > 0$ is chosen so that φ has neither zeros nor poles in the set $\{z : 0 < |z - h| < \varepsilon\}$. Thus, f_t defines a holomorphic family of rational maps of degree $d := e + 1 > 1$. As $t \to 0$, the maps f_t converge locally uniformly to φ on $\mathbb{P}^1 \setminus \{h\}$.

2.3. **An unbounded escape rate.** Set now

$$r_n = \exp(-n\,d^{n+1})$$

for each $n \in \mathbb{N}$. Working on $\mathbb{C}^2$, we define

$$\tilde{f}_t(z,w) := (\,P(z,w)(z-(h+\varepsilon t)w),\; Q(z,w)(z-(h-\varepsilon t)w)\,)$$

for all $t \in \mathbb{D}$, where P and Q are homogeneous polynomials of degree $e = \deg\varphi$ such that $\varphi(z) = P(z,1)/Q(z,1)$. Let $\tilde{a} : \mathbb{D} \to \mathbb{C}^2 \setminus \{(0,0)\}$ be any holomorphic map such that $\tilde{a}(0) = (a_0, 1)$ and let $a : \mathbb{D} \to \mathbb{P}^1$ be its projection to $\mathbb{P}^1$.

Choose any norm $\|\cdot\|$ on $\mathbb{C}^2$. As $\varphi^n(a_0) \neq h$ for all $n \geq 0$, we see that $\tilde{f}_0^n(\tilde{a}(0)) \neq (0,0)$ for all $n \geq 0$. Therefore, as described in §2.1, we have $\eta = 0$ and the function $g_{f,a}$ is given by the formula

$$g_{f,a}(t) = \lim_{n\to\infty} \frac{1}{d^n} \log \|\tilde{f}_t^n(\tilde{a}(t))\| \tag{2.3}$$

for $t \in \mathbb{D}^*$ [De4, Proposition 3.1].

Set

$$\Phi := (P,Q) \quad \text{and} \quad H(z,w) := z - hw$$

so that $\tilde{f}_0 = (HP, HQ)$. For all $n \geq 0$, as $\deg\Phi = e > 0$, the iteration formula of [De3, Lemma 2.2] states that

$$\tilde{f}_0^n = \left(P_n \cdot \prod_{k=0}^{n-1} ((\Phi^k)^* H)^{d^{n-k-1}},\; Q_n \cdot \prod_{k=0}^{n-1} ((\Phi^k)^* H)^{d^{n-k-1}}\right),$$

where $\Phi^n = (P_n, Q_n)$, so that

$$\frac{\log\|\tilde{f}_0^n\|}{d^n} = \sum_{k=0}^{n-1} \frac{\log|(\Phi^k)^* H|}{d^{k+1}} + \frac{\log\|\Phi^n\|}{d^n} \quad \text{on } \mathbb{C}^2 \setminus \{(0,0)\},$$

and consequently,

$$\log\frac{\|\tilde{f}_0 \circ \tilde{f}_0^n\|}{\|\tilde{f}_0^n\|^d} = \log\frac{|(\Phi^n)^* H|}{\|\Phi^n\|} + \log\frac{\|\Phi\circ\Phi^n\|}{\|\Phi^n\|^e} \quad \text{on } \mathbb{C}^2 \setminus \tilde{f}_0^{-n}(0,0). \tag{2.4}$$

Note that $\log\|\Phi\|$ is bounded on the unit sphere in $\mathbb{C}^2$, so the last term on the right-hand side of (2.4) is bounded on $\mathbb{C}^2 \setminus \{(0,0)\}$ uniformly in $n \geq 0$. The first term on the right-hand side of (2.4) is the log of $[\varphi^n(\cdot), h]$, up to scaling of the metric $[\cdot,\cdot]$; therefore, combined with (2.1), we see that there is a constant C so that

$$\log\frac{\|\tilde{f}_0(\tilde{f}_0^{n_j}(\tilde{a}(0)))\|}{\|\tilde{f}_0^{n_j}(\tilde{a}(0))\|^d} < C + \log(r_{n_j}) = C - n_j d^{n_j+1}$$

for all j. For all j, by continuity of $\tilde{f}_t^{n_j}(\tilde{a}(t))$ as a map from $\mathbb{D}$ to $\mathbb{C}^2 \setminus \{(0,0)\}$, there is a radius $\delta_j \in (0, 1/2)$ such that

$$\sup_{|t|\leq\delta_j} \log\frac{\|\tilde{f}_t(\tilde{f}_t^{n_j}(\tilde{a}(t)))\|}{\|\tilde{f}_t^{n_j}(\tilde{a}(t))\|^d} \leq C - n_j d^{n_j+1}. \tag{2.5}$$

On the other hand, we also have from (2.3) that

$$\begin{aligned} g_{f,a}(t) &= \log\|\tilde{a}(t)\| + \sum_{k=0}^{\infty} \frac{1}{d^{k+1}} \log \frac{\|\tilde{f}_t(\tilde{f}_t^k(\tilde{a}(t)))\|}{\|\tilde{f}_t^k(\tilde{a}(t))\|^d} \\ &= \log\|\tilde{a}(t)\| + \frac{1}{d^{n_j+1}} \log \frac{\|\tilde{f}_t(\tilde{f}_t^{n_j}(\tilde{a}(t)))\|}{\|\tilde{f}_t^{n_j}(\tilde{a}(t))\|^d} + \sum_{k\neq n_j} \frac{1}{d^{k+1}} \log \frac{\|\tilde{f}_t(\tilde{f}_t^k(\tilde{a}(t)))\|}{\|\tilde{f}_t^k(\tilde{a}(t))\|^d} \end{aligned} \tag{2.6}$$

for each j.

The following is elementary but useful:

Lemma 2.2. *Let $F_t = (P_t, Q_t)$ be any family of homogeneous polynomial maps of degree $d \geq 2$, with coefficients that are bounded holomorphic functions of t in $\mathbb{D}$. Then there is a constant C so that*

$$\frac{\|F_t(z,w)\|}{\|(z,w)\|^d} \leq C$$

for all $(z,w) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and all $t \in \mathbb{D}$.

Proof. As F_t is homogeneous, it suffices to bound its values on the unit sphere in $\mathbb{C}^2$. The result follows because the coefficients are bounded uniformly on $\mathbb{D}$. □

As a consequence of Lemma 2.2, we can bound all the terms in the final sum of (2.6) from above, uniformly on the disk $\{|t| \leq 1/2\}$, and therefore there is a constant C' so that

$$\sup_{|t|\leq 1/2} g_{f,a}(t) \leq C' + \frac{1}{d^{n_j+1}} \log \frac{\|\tilde{f}_t(\tilde{f}_t^{n_j}(\tilde{a}(t)))\|}{\|\tilde{f}_t^{n_j}(\tilde{a}(t))\|^d} \tag{2.7}$$

for every j. Combined with (2.5), we conclude that there is another constant C so that

$$\sup_{|t|\leq \delta_j} g_{f,a}(t) \leq C - n_j$$

for every j. Letting $j \to \infty$ shows that

$$\limsup_{t\to 0} g_{f,a}(t) = -\infty.$$

2.4. **Examples with degree $d = 2$.** Fix any $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let

$$\varphi(z) = e^{2\pi i\theta} z.$$

Set $a_0 = 1$; the ω-limit set $\omega_\varphi(a_0)$ is the unit circle in $\mathbb{C}$. Set $r_n = \exp(-n\,2^{n+1})$ for each $n \in \mathbb{N}$, and define $B_\varphi(1, \{r_n\})$ as above. Taking any $h \in B_\varphi(1, \{r_n\})$ and setting $\varepsilon = 1$, we define the family f_t as in (2.2). Then the potential function $g_{f,a}$ fails to be bounded around $t = 0$ for any holomorphic map $a : \mathbb{D} \to \mathbb{P}^1$ with $a(0) = 1$.

Note that only the Möbius transformations φ which are Möbius (i.e., $\mathrm{PSL}(2,\mathbb{C})$-) conjugate to an irrational rotation have recurrent orbits, as needed for the construction described above.

2.5. **Examples in degree > 2, with a marked critical point.** Fix an integer $d > 2$. For every $\theta \in \mathbb{R}$, the polynomial

$$\varphi(z) = e^{2\pi i\theta}\left(z - \frac{e^{2\pi i\theta} - (d-1)}{(d-1)^{(d-1)/(d-2)}}\right)^{d-1} \tag{2.8}$$

of degree $d-1$ has a fixed point with multiplier $e^{2\pi i\theta}$ and its unique finite critical value at $z = 0$. Now fix θ to be irrational; the critical point

$$a_0 := \frac{e^{2\pi i\theta} - (d-1)}{(d-1)^{(d-1)/(d-2)}}$$

of φ satisfies $\#\{\varphi^n(a_0) : n \in \mathbb{N}\} = \infty$ and $a_0 \in \omega_\varphi(a_0)$ [Ma]. Let

$$r_n = \exp(-n\, d^{n+1})$$

for each $n \in \mathbb{N}$ and fix any point $h \in B_\varphi(a_0, \{r_n\})$. Choose any $\varepsilon \in (0, |a_0 - h|]$, and set

$$f_t(z) = \varphi(z) \cdot \frac{z - h - \varepsilon\, t}{z - h + \varepsilon\, t},$$

which is a rational map on $\mathbb{P}^1$ of degree d for all $t \in \mathbb{D}^*$. We let

$$a(t) = a_0$$

for all $t \in \mathbb{D}$, which satisfies $f_t'(a(t)) = 0$ for all $t \in \mathbb{D}^*$. (This is the reason for requiring the unique finite critical value $\varphi(a_0)$ of φ to be 0.) It follows that $g_{f,a}$ fails to be bounded around $t = 0$.

2.6. **Example in degree 2, with a marked critical point.** We can produce examples of (f, a) also for quadratic rational maps f_t where $a(t)$ parameterizes a critical point of f_t, though we do not have as much flexibility as in higher degrees. For example, we have:

Lemma 2.3. *Suppose $f_t(z) = \varphi(z)(z-h-t^n)/(z-h+t^n)$ is a family of quadratic rational maps, for some $h \in \mathbb{C}$, $n \in \mathbb{N}$, and a rational map φ on $\mathbb{P}^1$ of degree 1, and suppose that $c_1, c_2 : \mathbb{D} \to \mathbb{P}^1$ are holomorphic maps parameterizing the two critical points of f_t. Then*

$$\lim_{t\to 0} c_1(t) = \lim_{t\to 0} c_2(t) = h.$$

Proof. As φ has degree 1, it has no critical points of its own. On the other hand, $\lim_{t\to 0} f_t = \varphi$ locally uniformly on $\mathbb{P}^1 \setminus \{h\}$, so it must be that $c_1(t), c_2(t) \to h$ as $t \to 0$. ☐

In particular, if we wish to let $a(t)$ parameterize a critical point of f_t, then necessarily we will have $a(0) = h$, which was not allowed by the construction above.

However, let us fix our decreasing sequence as

$$r_n = \exp(-(n-1)\, 2^n),$$

for each $n \in \mathbb{N}$, and apply the Baire Category Theorem now to the space of rotations $z \mapsto e^{2i\pi\theta} z$ to find

$$\theta_0 \in \left(\bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} \{\theta \in \mathbb{R} : [1, e^{2\pi i n \theta}] < r_n\} \right) \setminus \mathbb{Q}.$$

Then $e^{2\pi i n \theta_0} \neq 1$ for all $n \in \mathbb{N}$, and there is a sequence $n_j \to \infty$ as $j \to \infty$ such that

$$0 < [1, e^{2\pi i n_j \theta_0}] < r_{n_j}$$

for all j.

Now set $\varphi_0(z) = e^{2\pi i \theta_0} z$, and

$$f_t(z) = \varphi_0(z) \cdot \frac{z - 1 - t^2}{z - 1 + t^2}$$

for $t \in \mathbb{D}^*$. Note that we have used t^2 here rather than t in (2.2); this is so that we can holomorphically parameterize the critical points of f_t. Indeed, the critical points of f_t are

$$c_\pm(t) = 1 - t^2 \pm \sqrt{2t^4 - 2t^2} = 1 - t^2 \pm i\sqrt{2}\, t\sqrt{1 - t^2},$$

which extend holomorphically on $\mathbb{D}$ by setting $c_\pm(0) = 1$. Define the function $a : \mathbb{D} \to \mathbb{C}$ by either c_+ or c_- so that f_t has the critical value

$$v(t) := f_t(a(t)) = e^{2\pi i \theta_0} + O(t) \quad \text{as } t \to 0,$$

which also extends holomorphically to $\mathbb{D}$ by setting $v(0) := v_0 := e^{2\pi i \theta_0} = \varphi_0(1)$. We also set

$$\tilde{f}_t(z, w) := (e^{2\pi i \theta_0} z(z - (1 + t^2)w), (z - (1 - t^2)w)w)$$

for all $t \in \mathbb{D}$.

We will work with the pair (f, v). Then, since $\varphi_0^n(v_0) \neq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and since we also have

$$0 < [1, \varphi_0^{n_j - 1}(v_0)] < r_{n_j} = \exp(-(n_j - 1)\, 2^{(n_j - 1) + 1})$$

for all j, the arguments above go through exactly as before – applied to the sequence $\{n_j - 1\}_j$ – to show that $g_{f,v}$ fails to be bounded around $t = 0$.

Finally, if we set $\tilde{v}(t) = (v(t), 1)$ and $\tilde{a}(t) = (a(t), 1)$, then

$$\tilde{f}_t(\tilde{a}(t)) = (a(t) - 1 + t^2)\, \tilde{v}(t).$$

Note that $a(t) - 1 + t^2 = \pm i\sqrt{2}\, t\sqrt{1 - t^2}$ on $\mathbb{D}$, so that the function

$$h(t) = \log|a(t) - 1 + t^2| - \log|t|$$

on $\mathbb{D}^*$ extends to a harmonic function on the disk $\mathbb{D}$. We have

$$G_{\tilde{f}}(t, \tilde{v}(t)) = g_{f,v}(t)$$

on $\mathbb{D}^*$ from the definitions given in (1.1) and because the pair satisfies the hypotheses for (2.3). Consequently,

$$\begin{aligned} G_{\tilde f}(t, \tilde a(t)) &= \frac{1}{2}\, G_{\tilde f}(t, \tilde f_t(\tilde a(t))) \\ &= \frac{1}{2}\left(G_{\tilde f}(t, \tilde v(t)) + \log|a(t) - 1 + t^2|\right) \\ &= \frac{1}{2}\, g_{f,v}(t) + \frac{1}{2}\, h(t) + \frac{1}{2}\, \log|t| \end{aligned}$$

so that, by the definition of $g_{f,a}$ in (1.1), we have $\eta = 1/2$ and

$$g_{f,a}(t) = \frac{1}{2}\, g_{f,v}(t) + \frac{1}{2}\, h(t)$$

on $\mathbb{D}^*$. We conclude that the function $g_{f,a}$ also fails to be bounded near $t = 0$.

3. Lyapunov exponents and the bifurcation current

The Lyapunov exponent of an individual rational map f on $\mathbb{P}^1$ of degree > 1, with respect to its unique measure μ_f of maximal entropy on $\mathbb{P}^1$, is the positive and finite quantity

$$L(f) = \int_{\mathbb{P}^1} \log|f'|\, d\mu_f,$$

where $|\cdot|$ is any choice of metric on the tangent bundle of $\mathbb{P}^1$.

Let f_t be a holomorphic family of rational maps on $\mathbb{P}^1$ of degree $d > 1$ parameterized by $\mathbb{D}^*$ whose coefficients extend to meromorphic functions on $\mathbb{D}$. If all the critical points of f_t are parameterized by holomorphic maps $c_1, \ldots, c_{2d-2} : \mathbb{D} \to \mathbb{P}^1$, then

$$L(f_t) = h(t) + \sum_{j=1}^{2d-2} g_{f,c_j}(t) \tag{3.1}$$

on $\mathbb{D}^*$, for a harmonic function h on $\mathbb{D}^*$ satisfying $h(t) = O(\log|t|)$ as $t \to 0$ [De2, Theorem 1.4], [Fa2, Theorem C]. By the symmetry in the critical points in (3.1), this formula holds even if the critical points cannot be holomorphically parameterized on $\mathbb{D}^*$. The *bifurcation current* associated to the family f_t can be given by

$$T_{\mathrm{bif}} := \frac{1}{2\pi}\, \Delta L(f_t)$$

on $\mathbb{D}^*$, in the sense of distributions; the original definition of T_{bif} in [De1] was based on the right hand side of (3.1). From [De2, Theorem 1.1], the support of T_{bif} is equal to the bifurcation locus of the family f_t in the sense of [MSS, Ly].

In particular, because the sum $\sum_j g_{f,c_j}$ in (3.1) is $o(\log|t|)$ near $t = 0$, we see that the bifurcation current T_{bif} has a bounded potential if and only if the sum $\sum_j g_{f,c_j}$ is bounded near $t = 0$.

Proof of Theorem 1.1. We give examples in an arbitrary degree > 1. First, let f_t be the holomorphic family of quadratic rational maps on $\mathbb{P}^1$ parameterized by $\mathbb{D}^*$ described in §2.6. As we have seen, neither of the functions $g_{f,c_\pm}$ extend continuously to $\mathbb{D}$; indeed, both tend to $-\infty$ as $t \to 0$. Hence by (3.1), the bifurcation current for the family f_t fails to have a potential bounded around $t = 0$.

Next, let f_t be the holomorphic family of rational maps on $\mathbb{P}^1$ of degree $d > 2$ parameterized by $\mathbb{D}^*$, described in §2.5. As we have seen, the constant map $a(t) \equiv a_0$ on $\mathbb{D}$ satisfies $f_t'(a(t)) = 0$ for every $t \in \mathbb{D}^*$, and the function $g_{f,a}$ tends to $-\infty$ as $t \to 0$. Taking an at most finitely ramified holomorphic covering $\pi : \mathbb{D}^* \to \mathbb{D}^*$ if necessary, all the critical points of $f_{\pi(s)}$ are parameterized by holomorphic maps $c_1, \dots, c_{2d-2} : \mathbb{D} \to \mathbb{P}^1$. We may assume the points are labeled so that $c_1 = a \circ \pi$ on $\mathbb{D}$. By the formula (2.3) for g_{f,c_1}, we have $g_{f_{\pi(\cdot)},c_1}(s) = g_{f,a}(\pi(s))$ on $\mathbb{D}^*$. On the other hand, for every $j \in \{2, \dots, 2d-2\}$, the function $g_{f_{\pi(\cdot)},c_j}$ is bounded from above on $\{0 < |s| \le r\}$ for every $r \in (0,1)$, by Lemma 2.1. Hence the sum $\sum_j g_{f_{\pi(\cdot)},c_j}(s)$ tends to $-\infty$ as $s \to 0$. Therefore, the bifurcation current associated to the family f_t fails to have a potential bounded around $t = 0$. □

4. Limitations of the construction

To find the examples of Section 2, we used a rational map $\varphi \in \mathbb{C}(z)$ of degree ≥ 1 and points $a_0, h \in \mathbb{P}^1(\mathbb{C})$ such that

$$0 < [\varphi^{n_j}(a_0), h] < r_{n_j}$$

in the chordal metric $[\cdot,\cdot]$, along a sequence $n_j \to \infty$, with (r_n) chosen so that

$$\lim_{n\to\infty} \frac{\log r_n}{d^n} = -\infty.$$

Combining (2.7) with (2.5) guaranteed that $\lim_{t\to 0} g_{f,a}(t) = -\infty$. Looking carefully at the estimates, we see that the orbit $\{\varphi^n(a_0)\}$ needs only to satisfy a weaker divergence condition

$$\sum_{n=0}^{\infty} \frac{\log [\varphi^n(a_0), h]}{d^n} = -\infty, \tag{4.1}$$

with $\varphi^n(a_0) \ne h$ for all $n \in \mathbb{N} \cup \{0\}$, to achieve our conclusion with this method.

As observed in the Introduction, the function $g_{f,a}$ will always extend continuously to $\mathbb{D}$ when f_t is a family of polynomials, by [FG2, Main Theorem]. Here we explain explicitly why our construction breaks down for polynomials.

Proposition 4.1. *The construction of Section 2 cannot produce any pair (f, a) such that for every $t \in \mathbb{D}^*$, f_t is Möbius conjugate to a polynomial.*

Proof. Suppose that f_t is a holomorphic family of rational maps of degree $d > 1$ parameterized by $\mathbb{D}^*$, that for every $t \in \mathbb{D}^*$, there exists $A_t \in \mathrm{PSL}(2, \mathbb{C})$ such that $A_t \circ f_t \circ A_t^{-1}$ is a polynomial, and that $\lim_{t\to 0} f_t = \varphi$ locally uniformly on $\mathbb{P}^1 \setminus \{h\}$ for some $h \in \mathbb{P}^1$ and some $\varphi \in \mathbb{C}(z)$ of degree $d-1$ (> 0). For every $t \in \mathbb{D}^*$, the point $p_t := A_t^{-1}(\infty)$ is a superattracting fixed point of f_t for which $\deg_{p_t} f_t = d$. We first claim that $\lim_{t\to 0} p_t = h$; otherwise, there is a sequence (t_j) in $\mathbb{D}^*$ tending to 0 as $j \to \infty$ such that there is the limit $p := \lim_{j\to\infty} p_{t_j} \in \mathbb{P}^1 \setminus \{h\}$. By the locally uniform convergence $\lim_{t\to 0} f_t = \varphi$ on $\mathbb{P}^1 \setminus \{h\}$, $\deg \varphi > 0$, and the Argument Principle, this p must be a superattracting fixed point of φ for which $\deg_p \varphi = d$, contradicting $\deg \varphi = d - 1$. We next claim that $\varphi^{-1}(h) = \{h\}$; for, if there is a point $q \in \mathbb{P}^1 \setminus \{h\}$ for which $\varphi(q) = h$, then by the first claim, the locally uniform convergence $\lim_{t\to 0} f_t = \varphi$ on $\mathbb{P}^1 \setminus \{h\}$, $\deg \varphi > 0$, and the Argument Principle, for any $t \in \mathbb{D}^*$ close enough to 0, there must exist a point $q_t \in \mathbb{P}^1 \setminus \{p_t\}$ (near q) for which $f_t(q_t) = p_t$, contradicting $\deg f_t = d$.

Suppose $a : \mathbb{D} \to \mathbb{P}^1$ is any holomorphic map with $a(0) =: a_0 \neq h$. If $d > 2$ so that $\deg \varphi = d - 1 > 1$, then by the second claim, we have a constant $C < 0$ so that

$$\log [\varphi^n(a_0), h] \geq C \cdot (d-1)^n$$

for all $n \in \mathbb{N}$. If $d = 2$ so that $\deg \varphi = d - 1 = 1$, then by the second claim above, the orbit $\{\varphi^n(a_0)\}$ can accumulate to h and satisfy $\varphi^n(a_0) \neq h$ for any $n \geq 0$ only if h is an attracting or parabolic fixed point of φ. Hence we still have a constant $C < 0$ so that

$$\log [\varphi^n(a_0), h] \geq C \cdot n$$

for all $n \in \mathbb{N}$. Therefore in both cases, the points $a_0, h \in \mathbb{P}^1$ cannot satisfy (4.1). ☐

Working over the field $\mathbb{C}$ of complex numbers allowed us to exploit the Baire Category Theorem in our construction. In fact, the construction is impossible over a field such as $\overline{\mathbb{Q}}$.

Proposition 4.2. *The construction of Section 2 cannot produce any pair (f, a) such that the map φ and points a_0 and h are simultaneously defined over $\overline{\mathbb{Q}}$.*

Proof. Suppose f_t is any holomorphic family of rational maps of degree $d > 1$ parameterized by $\mathbb{D}^*$ such that $\lim_{t\to 0} f_t = \varphi$ locally uniformly on $\mathbb{P}^1 \setminus \{h\}$, for some $\varphi \in \overline{\mathbb{Q}}(z)$ of degree $d-1$ and some $h \in \mathbb{P}^1(\overline{\mathbb{Q}})$. Fix any point $a_0 \in \mathbb{P}^1(\overline{\mathbb{Q}})$ such that $\varphi^n(a_0) \neq h$ for all $n \geq 0$.

Suppose first that $d > 2$, so that $\deg \varphi = d - 1 > 1$. If there is $A \in \mathrm{PSL}(2, \overline{\mathbb{Q}})$ such that either $A \circ \varphi \circ A^{-1}$ or $A \circ \varphi^2 \circ A^{-1}$ is a polynomial and that $A(h) = \infty$, then we have a constant $C < 0$ so that

$$\log [\varphi^n(a_0), h] \geq C(d-1)^n$$

for all $n \in \mathbb{N}$. Otherwise, by [Si1, Theorem E], which uses the Roth theorem, we have the stronger result that

$$\log [\varphi^n(a_0), h] = o((d-1)^n)$$

as $n \to \infty$. Therefore, in both cases, φ, a_0, and h cannot satisfy (4.1).

Now suppose that $d = 2$ so that $\deg\varphi = d - 1 = 1$. Note that the orbit $\{\varphi^n(a_0)\}$ can accumulate to h and satisfy $\varphi^n(a_0) \neq h$ for any $n \geq 0$ only if either h is an attracting or parabolic fixed point of φ (in $\mathbb{P}^1(\overline{\mathbb{Q}})$) or there exists $A \in \mathrm{PSL}(2, \overline{\mathbb{Q}})$ such that $A \circ \varphi \circ A^{-1}$ is an irrational rotation $z \mapsto \lambda z$, where λ is not a root of unity, $\lambda \in \overline{\mathbb{Q}}$, and $|\lambda| = 1$, with $|A(h)| = |A(a_0)| = 1$. In the former case, we have a constant $C < 0$ so that $\log[\varphi^n(a_0), h] \geq C \cdot n$ for all $n \in \mathbb{N}$. So φ, a_0, and h cannot satisfy (4.1).

In the latter case, we claim that we still have a constant $C < 0$ such that

$$\log[\varphi^n(a_0), h] \geq C \cdot n$$

for all $n \in \mathbb{N}$; since $A \in \mathrm{PSL}(2, \overline{\mathbb{Q}})$ is biLipschitz with respect to $[\cdot, \cdot]$, we can assume that φ is an irrational rotation $z \mapsto \lambda z$, where λ is not a root of unity, $\lambda \in \overline{\mathbb{Q}}$, and $|\lambda| = 1$, with $h, a_0 \in \overline{\mathbb{Q}}$ and $|h| = |a_0| = 1$. Fix a number field K so that $\lambda, a_0, h \in K$, and denote by M_K the set of all places (i.e., equivalence classes of non-trivial either archimedean or non-archimedean absolute values) of K. Recall that there are a family $(N_v)_{v \in M_K}$ in $\mathbb{N}$ and a family $(|\cdot|_v)_{v \in M_K}$ of representatives $|\cdot|_v$ of places v such that for every $x \in K^*$, $|x|_v = 1$ for all but finitely many $v \in M_K$ and $\prod_{v \in M_K} |x|_v^{N_v} = 1$. Then by the (strong) triangle inequality, we can choose a family of real numbers $C_v \geq 1$, $v \in M_K$, such that $|\lambda^n - h/a_0|_v \leq C_v^n$ for any $v \in M_K$ and any $n \in \mathbb{N}$ and that $C_v = 1$ for all but finitely many $v \in M_K$. We also note that $\lambda^n a_0 = \varphi^n(a_0) \neq h$ for all $n \in \mathbb{N}$. Hence for every $v_0 \in M_K$ and every $n \in \mathbb{N}$, we have $|\lambda^n - h/a_0|_{v_0} \geq (\prod_{v \in M_K} C_v^{-N_v})^n$. In particular, recalling that $[z, w] = |z - w|[z, \infty][w, \infty]$ on $\mathbb{C} \times \mathbb{C}$, there is a constant $C < 0$ such that

$$\log[\varphi^n(a_0), h] = \log|\lambda^n - h/a_0| - \log 2 \geq C \cdot n$$

for all $n \in \mathbb{N}$. So the claim holds, and φ, a_0, and h cannot satisfy (4.1). ☐

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Department of Mathematics, Northwestern University, USA
E-mail address: demarco@math.northwestern.edu

Division of Mathematics, Kyoto Institute of Technology, Japan
E-mail address: okuyama@kit.ac.jp