

PREPERIODIC POINTS AND UNLIKELY INTERSECTIONS

MATTHEW BAKER AND LAURA DEMARCO

ABSTRACT. In this article, we combine complex-analytic and arithmetic tools to study the preperiodic points of one-dimensional complex dynamical systems. We show that for any fixed $a, b \in \mathbb{C}$, and any integer $d \geq 2$, the set of $c \in \mathbb{C}$ for which both a and b are preperiodic for $z^d + c$ is infinite if and only if $a^d = b^d$. This provides an affirmative answer to a question of Zannier, which itself arose from questions of Masser concerning simultaneous torsion sections on families of elliptic curves. Using similar techniques, we prove that if rational functions $\varphi, \psi \in \mathbb{C}(z)$ have infinitely many preperiodic points in common, then all of their preperiodic points coincide (and in particular the maps must have the same Julia set). This generalizes a theorem of Mimar, who established the same result assuming that φ and ψ are defined over $\bar{\mathbb{Q}}$. The main arithmetic ingredient in the proofs is an adelic equidistribution theorem for preperiodic points over number fields and function fields, with non-archimedean Berkovich spaces playing an essential role.

1. INTRODUCTION

1.1. Statement of main results. A complex number a is *preperiodic* for a polynomial map $f \in \mathbb{C}[z]$ if the forward orbit of a under iteration by f is finite. In this article, we examine preperiodic points for the unicritical polynomials, those of the form $z^d + c$ for a complex parameter c . The main result of this article is the following.

Theorem 1.1. *Let $d \geq 2$ be an integer, and fix $a, b \in \mathbb{C}$. The set of parameters $c \in \mathbb{C}$ such that both a and b are preperiodic for $z^d + c$ is infinite if and only if $a^d = b^d$.*

One direction of Theorem 1.1 follows easily from Montel's theorem: if $a^d = b^d$ then a is preperiodic for $z^d + c$ if and only if b is preperiodic, and the set of complex numbers $c \in \mathbb{C}$ such that a is preperiodic for $z^d + c$ is always infinite (see §3). The reverse implication combines ideas from number theory and complex analysis, the main arithmetic ingredient being an equidistribution theorem for points of *small*

Date: August 2, 2010.

Key words and phrases. Preperiodic points, canonical heights, arithmetic dynamics, complex dynamics, potential theory, equidistribution, Berkovich spaces.

The research was supported by the National Science Foundation and the Sloan Foundation. The authors would like to thank Rob Benedetto, Xander Faber, Joe Silverman, and Umberto Zannier for helpful discussions, Daniel Connelly for computations related to Conjecture 1.6, and the anonymous referees for useful suggestions. We also thank Sarah Koch for generating the images in Figure 1. Finally, we thank AIM for sponsoring the January 2008 workshop that inspired the results of this paper.

height with respect to an *adelic height function* (see §2.3 below for details). When a and b are algebraic, the equidistribution in question takes place over \mathbb{C} , but in the transcendental case, we require an equidistribution theorem which takes place on the Berkovich projective line¹ $\mathbb{P}_{\text{Berk},K}^1$ over some complete and algebraically closed non-archimedean field K . The idea is to think of the field $k = \bar{\mathbb{Q}}(a) \subset \mathbb{C}$ as the function field of $\mathbb{P}_{\bar{\mathbb{Q}}}^1$, and to consider the distribution of the preperiodic parameters $c \in \bar{k}$ inside a collection of non-archimedean Berkovich analytic spaces, one for each completion of k .

Our method of proof also provides an analog of Theorem 1.1 in the dynamical plane, which we state in the more general context of rational functions. For a rational function $\varphi \in \mathbb{C}(z)$, let $\text{Preper}(\varphi)$ denote its set of preperiodic points in $\mathbb{P}^1(\mathbb{C})$.

Theorem 1.2. *Let $\varphi, \psi \in \mathbb{C}(z)$ be rational functions of degrees at least 2. Then $\text{Preper}(\varphi) \cap \text{Preper}(\psi)$ is infinite if and only if $\text{Preper}(\varphi) = \text{Preper}(\psi)$.*

As a consequence, we have:

Corollary 1.3. *Let $\varphi, \psi \in \mathbb{C}(z)$ be rational functions of degrees at least 2 with Julia sets $J(\varphi) \neq J(\psi)$. Then the set of points which are preperiodic for both φ and ψ is finite.*

When φ and ψ are defined over $\bar{\mathbb{Q}}$, Corollary 1.3 is a special case of a result of Mimar [Mim97]. In this setting, the proof of Theorem 1.2 shows that infinite intersection of $\text{Preper}(\varphi)$ and $\text{Preper}(\psi)$ implies equality of the canonical height functions h_φ and h_ψ on $\mathbb{P}^1(\bar{\mathbb{Q}})$; this result is also proved in [PST09]. In particular, the measures of maximal entropy for the complex dynamical systems $\varphi, \psi : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ must coincide. For polynomials defined over a number field, we use a result of [KS07] to strengthen the conclusion of Theorem 1.2; see Theorem 5.7.

The transcendental case of Theorem 1.2, which is handled using Berkovich spaces, is the main new ingredient. We again deduce that canonical height functions h_φ and h_ψ are equal, only now they are defined over an appropriate function field. In the absence of an archimedean absolute value, we are not able to deduce automatically that the measures of maximal entropy coincide, only that φ and ψ have the same set of preperiodic points.

Yuan and Zhang recently (and independently) proved a generalization of Theorem 1.2 to polarized algebraic dynamical systems of any dimension [YZ09]. Their proof also makes use of Berkovich spaces and equidistribution.

¹The Berkovich projective line $\mathbb{P}_{\text{Berk},K}^1$ is a canonical compact, Hausdorff, path-connected space containing $\mathbb{P}^1(K)$ as a dense subspace. It is for many applications the “correct” setting for non-archimedean potential theory and dynamics, see e.g. [Bak08, BR10, Ben10, CCL10, CL06, FRL04, FRL06, RL03, Thu05].

The statements of Theorem 1.1 and Corollary 1.3 are false if “are preperiodic for” is replaced with “are in the Julia set of”. For example, for any two points $a, b \in (-2, 2)$, there is an infinite family of polynomials $z^2 + c$, with $c \in \mathbb{R}$ descending to -2 , with Julia sets containing both a and b (in fact, containing a closed interval in \mathbb{R} increasing to $[-2, 2]$ as $c \rightarrow -2$). These same examples show that distinct Julia sets can have infinite intersections. Because of such examples, it seems hard to prove results such as Theorems 1.1 and 1.2 using complex analysis alone, without making use of some arithmetic information about preperiodic points.

In principle, it should be possible to extend the methods of the present paper to other one-parameter families of rational maps on $\mathbb{P}^1(\mathbb{C})$ and to allow the points a, b to depend algebraically on the parameter. The special case we consider in Theorem 1.1 allows us a characterization by the simple condition $a^d \neq b^d$. For more general one-parameter families $\{f_t\}$ and pairs $a(t), b(t)$, a precise condition characterizing finiteness is not as clear to formulate, and the complex analysis becomes more delicate. However, most of the number-theoretic parts of our argument can easily be adapted to a more general setting. It would also be interesting to study analogs of Theorem 1.1 for, say, an n -dimensional family of rational maps in which $n + 1$ points are required to be simultaneously preperiodic.

1.2. Motivation and historical background. The motivation for Theorem 1.1 came from a topic of discussion at the AIM workshop “The uniform boundedness conjecture in arithmetic dynamics” in Palo Alto in January 2008. By analogy with questions due to David Masser concerning simultaneous torsion sections on families of elliptic curves, Umberto Zannier asked:

Question 1.4. *Is the set of complex numbers $c \in \mathbb{C}$ such that 0 and 1 are both preperiodic for $z^2 + c$ finite?*

Theorem 1.1 provides an affirmative answer to Question 1.4, in analogy with the following recent theorem of Masser and Zannier:

Theorem 1.5. [MZ08, MZ09] *The set of complex numbers $\lambda \neq 0, 1$ such that both $P_\lambda = (2, \sqrt{2(2-\lambda)})$ and $Q_\lambda = (3, \sqrt{6(3-\lambda)})$ have finite order on the Legendre elliptic curve E_λ defined by $Y^2 = X(X-1)(X-\lambda)$ is finite.*

As in Theorem 1.1, one shows fairly easily that there are infinitely many λ such that either P_λ or Q_λ alone has finite order; however, in each case the set of such λ is rather sparse (for example, it is countable), and imposing both torsion conditions at once makes the set of λ finite. There is nothing special about the numbers 2 and 3; Masser and Zannier have announced that they can extend the main result of [MZ08, MZ09] to much more general linearly independent ‘sections’ P_λ, Q_λ .

As it stands, the proof of Theorem 1.1 is not effective (nor is the proof of Theorem 1.5), as we have no control over the number of parameters c for which two

points might be simultaneously preperiodic. For example, it is easily checked that 0 and 1 are both preperiodic for $z^2 + c$ when $c \in \{0, -1, -2\}$. A straightforward computation with Mathematica shows that there are no other values of c for which $f_c^{(\ell_0)}(0) = f_c^{(m_0)}(0)$ and $f_c^{(\ell_1)}(1) = f_c^{(m_1)}(1)$ with $0 \leq m_i < \ell_i < 15$ for $i = 0, 1$ and $\ell_0 + \ell_1 < 20$. Letting $S_{0,1}$ denote the set of parameters c for which both 0 and 1 are preperiodic, the following conjecture seems plausible:

Conjecture 1.6. $S_{0,1} = \{0, -1, -2\}$.

1.3. Brief overview of the proof of Theorem 1.1. A key role in the proof of Theorem 1.1 is played by certain generalizations of the famous *Mandelbrot set*. For $a \in \mathbb{C}$, let M_a denote the set of all $c \in \mathbb{C}$ such that a stays bounded under iteration of $z^d + c$. When $d = 2$ and $a = 0$, M_a is just the usual Mandelbrot set. See Figure 1. We let μ_a denote the *equilibrium measure* on M_a relative to ∞ , in the sense of complex potential theory; by classical results, μ_a is a probability measure whose support is equal to ∂M_a .

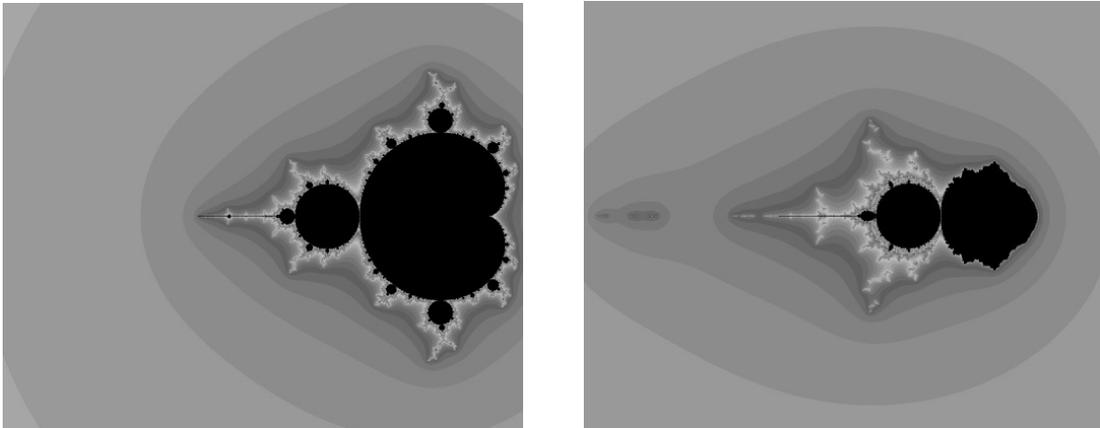


FIGURE 1. At left, the Mandelbrot set M_0 , and at right, the set M_1 for the quadratic family $z^2 + c$. The images are centered at the same point; the shading indicates level sets of the Green's function.

With this terminology in mind, an overview of the proof of Theorem 1.1 is as follows. Assume that there is an infinite sequence c_1, c_2, \dots of complex numbers such that a and b are both preperiodic for $z^d + c_n$ for all n .

Case 1: a, b are *algebraic* numbers. In this case, all c_n 's must also be algebraic. Let δ_n be the discrete probability measure on \mathbb{C} supported equally on the Galois conjugates of c_n . Using the fact that a is preperiodic for each c_n , an arithmetic equidistribution theorem based on the product formula for number fields shows that the measures δ_n converge weakly to the equilibrium measure μ_a for M_a on $\mathbb{P}^1(\mathbb{C})$. By symmetry, the measures δ_n also converge weakly to μ_b . Thus $\mu_a = \mu_b$, which implies that $M_a = M_b$.

A complex-analytic argument using Green's functions and univalent function theory then shows that $a^d = b^d$. (In the special case $a = 0$ and $b = 1$ corresponding to Question 1.4, one can show directly that $M_0 \neq M_1$; for example, $i \in M_0$ but $i \notin M_1$. See Figure 1.)

Case 2: a is transcendental. In this case, one can show that b is also transcendental, and that a, b , and all the c_n 's are defined over the algebraic closure \bar{k} of $k = \bar{\mathbb{Q}}(a)$ in \mathbb{C} . The field k is isomorphic to the field $\bar{\mathbb{Q}}(T)$ of rational functions over the constant field $\bar{\mathbb{Q}}$, and in particular k has a (non-archimedean) product formula structure on it. An arithmetic equidistribution theorem based on the product formula for function fields, together with the assumption that a is preperiodic for each c_n , shows that for every place v of k , the v -adic analogue of the measures δ_n above converge weakly on the Berkovich projective line $\mathbb{P}_{\text{Berk},v}^1$ over \mathbb{C}_v to a probability measure whose support is the v -adic analogue $M_{a,v} \subseteq \mathbb{P}_{\text{Berk},v}^1$ of M_a . (Here \mathbb{C}_v denotes the completion of an algebraic closure of the v -adic completion k_v .) By symmetry, it follows that $M_{a,v} = M_{b,v}$ for all places v of k . A theorem of Benedetto implies that for every $c \in \bar{k}$, and hence for every complex number c , a is preperiodic for $z^d + c$ if and only if b is. We deduce using Montel's theorem that $M_a = M_b$, and finish the argument as in Case 1.

2. POTENTIAL THEORY BACKGROUND

In this section we discuss some results from potential theory which are used in the rest of the paper.

2.1. Complex potential theory. Let E be a compact subset of \mathbb{C} . The *logarithmic capacity* $\gamma(E)$ of E relative to ∞ is $e^{-V(E)}$, where

$$(2.1) \quad -\log \gamma(E) = V(E) = \inf_{\nu} \iint_{E \times E} -\log |x - y| d\nu(x) d\nu(y).$$

The infimum in (2.1) is over all probability measures ν supported on E . If $\gamma(E) > 0$ (equivalently, $V(E) < \infty$), then there is a unique probability measure μ_E which achieves the infimum in (2.1), called the *equilibrium measure* for E . The support of μ_E is contained in the “outer boundary” of E , i.e., in the boundary of the unbounded component U_E of $\mathbb{C} \setminus E$.

If $\gamma(E) > 0$, the *Green's function* G_E is defined by

$$G_E(z) = V(E) + \int_E \log |z - w| d\mu_E(w);$$

it is a nonnegative real-valued subharmonic function on \mathbb{C} . The following facts are well known; we include some proofs for lack of a convenient reference.

Lemma 2.2. *Let E be a compact subset of \mathbb{C} for which $\gamma(E) = e^{-V(E)} > 0$, and let U be the unbounded component of $\mathbb{C} \setminus E$. Then:*

- (1) $G_E(z) = V(E) + \log |z| + o(1)$ for $|z|$ sufficiently large.
- (2) If $G : \mathbb{C} \rightarrow \mathbb{R}$ is a continuous subharmonic function which is harmonic on U , identically zero on E , and such that $G(z) - \log^+ |z|$ is bounded, then $G = G_E$.
- (3) If $G_E(z) = 0$ for all $z \in E$, then G_E is continuous on \mathbb{C} , $\text{Supp } \mu_E = \partial U$, and $G_E(z) > 0$ if and only if $z \in U$.

Proof. Assertion (1) is [Ran95, Theorem 5.2.1].

For (2), first note that G_E is continuous at every point $q \in E$ where $G_E(q) = 0$. Indeed, G_E is upper semicontinuous and bounded below by zero, so

$$(2.3) \quad 0 \leq \liminf_{z \rightarrow q} G_E(z) \leq \limsup_{z \rightarrow q} G_E(z) \leq G_E(q) = 0.$$

By Frostman's Theorem ([Ran95, Theorem 3.3.4]), G_E is identically zero on $\mathbb{C} \setminus U$ outside a set $e \subset \partial U$ of capacity 0, and hence the same is true for $f := G_E - G$. Since G_E is continuous on $\mathbb{C} \setminus e$ and G is continuous everywhere, f is continuous outside e . And by assumption, f is harmonic and bounded on U . By the Extended Maximum Principle [Ran95, Proposition 3.6.9], we conclude that $f \equiv 0$ on U , and hence on $\mathbb{C} \setminus e$. Thus $G_E(z) = G(z)$ for all $z \in \mathbb{C} \setminus e$. Since e has measure zero by [Ran95, Corollary 3.2.4], the generalized Laplacians $\Delta(G_E)$ and $\Delta(G)$ coincide. Since G_E and G are both subharmonic on \mathbb{C} , it follows from Weyl's Lemma [Ran95, Lemma 3.7.10] that f is harmonic on all of \mathbb{C} . Since f is also bounded, Liouville's Theorem [Ran95, Corollary 2.3.4] implies that f is identically zero. This proves (2).

The continuity assertion in (3) follows from (2.3), and the rest of (3) follows easily from the Maximum Principle. \square

2.2. Non-archimedean potential theory. In [BR10] (see also [FRL06, Thu05]), one finds non-archimedean Berkovich space analogs of various classical results from complex potential theory, including a theory of Laplacians, harmonic functions, subharmonic functions, Green's functions, and capacities. These results closely parallel the classical theory over \mathbb{C} . For the reader's convenience, we give a quick summary in this section of the results from [BR10] which are used in the present paper.² Although this theory is used heavily in the proofs of Lemma 2.5 and Theorem 2.8, the reader who wishes to accept these results as "black boxes" does not need a detailed understanding of non-archimedean potential theory in order to understand the proof of Theorem 1.1 below.

²Amaury Thuillier has independently developed non-archimedean potential theory on $\mathbb{P}_{\text{Berk}, K}^1$ [Thu05], and in fact his results are formulated in the context of arbitrary Berkovich curves, and without assuming that the field K is algebraically closed. Also, Charles Favre and Juan Rivera-Letelier [FRL04, FRL06] have independently developed most of the non-archimedean potential theory needed for the present applications to complex dynamics; their work relies heavily on potential theory for \mathbb{R} -trees as developed in the book by Favre and Jonsson [FJ04].

Let K be an algebraically closed field which is complete with respect to some non-trivial absolute value $|\cdot|$. The *Berkovich affine line* $\mathbb{A}_{\text{Berk}}^1 = \mathbb{A}_{\text{Berk},K}^1$ over K is a locally compact, Hausdorff, path-connected space containing K (with the given metric topology) as a dense subspace. As a topological space, $\mathbb{A}_{\text{Berk},K}^1$ is the set of all multiplicative seminorms $[\cdot]_x : K[T] \rightarrow \mathbb{R}$ on the polynomial ring $K[T]$ which extend the given absolute value on K , endowed with the weakest topology for which $x \mapsto [f]_x$ is continuous for all $f \in K[T]$. The *Berkovich projective line* $\mathbb{P}_{\text{Berk},K}^1$ can be identified with the one-point compactification of $\mathbb{A}_{\text{Berk},K}^1$, with the extra point denoted ∞ . It is a consequence of the Gelfand-Mazur theorem that if $K = \mathbb{C}$, then $\mathbb{A}_{\text{Berk},\mathbb{C}}^1$ is homeomorphic to \mathbb{C} (and $\mathbb{P}_{\text{Berk}}^1$ is homeomorphic to the Riemann sphere $\mathbb{P}^1(\mathbb{C})$). When K is non-archimedean, however, there are infinitely many multiplicative seminorms $x \in \mathbb{A}_{\text{Berk},K}^1$ which do not come from evaluation at a point of K ; for example, the *Gauss point* $\zeta_{\text{Gauss}} \in \mathbb{A}_{\text{Berk},K}^1$ corresponds to the seminorm $[f]_{\zeta_{\text{Gauss}}} := \sup_{z \in K, |z| \leq 1} |f(z)|$.

For the rest of this section, we assume that the absolute value on K is *non-archimedean* and non-trivial. If $z \in \mathbb{A}_{\text{Berk}}^1$, we will sometimes write $|z|$ instead of the more cumbersome $[T]_z$; the function $z \mapsto |z|$ is a natural extension of the absolute value on K to $\mathbb{A}_{\text{Berk}}^1$. Similarly, if $f \in K[T]$ we will sometimes write $|f(z)|$ instead of $[f(T)]_z$.

There is a canonical extension of the fundamental potential kernel $-\log|x - y|$ to $\mathbb{A}_{\text{Berk}}^1$. It can be defined as $-\log\delta(x, y)$, where $\delta(x, y)$ (called the *Hsia kernel* in [BR10]) is defined as

$$\delta(x, y) := \limsup_{\substack{z, w \in K \\ z \rightarrow x, w \rightarrow y}} |z - w|.$$

Let E be a compact subset of $\mathbb{A}_{\text{Berk}}^1$. The *logarithmic capacity* $\gamma(E)$ of E relative to ∞ is $e^{-V(E)}$, where

$$(2.4) \quad -\log\gamma(E) = V(E) = \inf_{\nu} \iint_{E \times E} -\log\delta(x, y) d\nu(x) d\nu(y).$$

The infimum in (2.4) is over all probability measures ν supported on E . If $\gamma(E) > 0$ (equivalently, $V(E) < \infty$), there is again a unique probability measure μ_E which achieves the infimum in (2.4), called the *equilibrium measure* for E relative to ∞ . The support of μ_E is contained in the outer boundary of E (the boundary of the unbounded component of $\mathbb{A}_{\text{Berk}}^1 \setminus E$).

If $\gamma(E) > 0$, the *Green's function of E relative to infinity* is defined by

$$G_E(z) = V(E) + \int_E \log\delta(z, w) d\mu_E(w);$$

it is a nonnegative real-valued subharmonic (in the sense of [BR10, Chapter 8]) function on $\mathbb{A}_{\text{Berk}}^1$. For example, if $E = \mathcal{D}(0, 1)$ is the closed unit disc in $\mathbb{A}_{\text{Berk}}^1$, defined

as

$$\mathcal{D}(0, 1) = \{x \in \mathbb{A}_{\text{Berk}}^1 : |x| \leq 1\},$$

then

$$G_E(z) = \log \max\{|z|, 1\}.$$

The following is the non-archimedean counterpart of Lemma 2.2:

Lemma 2.5. *Let E be a compact subset of $\mathbb{A}_{\text{Berk}}^1$ for which $\gamma(E) = e^{-V(E)} > 0$, and let U be the unbounded component of $\mathbb{A}_{\text{Berk}}^1 \setminus E$. Then:*

- (1) $G_E(z) = V(E) + \log |z|$ for all $z \in \mathbb{A}_{\text{Berk}}^1$ with $|z|$ sufficiently large.
- (2) If $G : \mathbb{A}_{\text{Berk}}^1 \rightarrow \mathbb{R}$ is a continuous subharmonic function which is harmonic on U , identically zero on E , and such that $G(z) - \log^+ |z|$ is bounded, then $G = G_E$.
- (3) If $G_E(z) = 0$ for all $z \in E$, then G_E is continuous on $\mathbb{A}_{\text{Berk}}^1$, $\text{Supp } \mu_E = \partial U$, and $G_E(z) > 0$ if and only if $z \in U$.

Proof. Assertion (1) follows from [BR10, Proposition 7.37(A7)], and (3) is [BR10, Corollary 7.39].

For (2), note that by [BR10, Proposition 7.37(A4)], G_E is identically zero on $\mathbb{A}_{\text{Berk}}^1 \setminus U$ outside a set $e \subset \partial U$ of capacity 0, and hence the same is true for $f := G_E - G$. Since G_E is continuous on $\mathbb{A}_{\text{Berk}}^1 \setminus e$ by [BR10, Proposition 7.37(A5)] and G is continuous everywhere, f is continuous outside e . And by assumption, f is harmonic and bounded on U . By the Strong Maximum Principle [BR10, Proposition 7.17], we conclude that $f \equiv 0$ on U . Thus $G_E(z) = G(z)$ for all $z \in \mathbb{A}_{\text{Berk}}^1 \setminus e$.

Note that G_E is subharmonic on $\mathbb{A}_{\text{Berk}}^1$ by [BR10, Example 8.9] and G is subharmonic on $\mathbb{A}_{\text{Berk}}^1$ by assumption. Since $e \subset \mathbb{P}^1(K)$ by [BR10, Example 6.3], and the Laplacian of a function on $\mathbb{P}_{\text{Berk}}^1$ depends only on its restriction to $\mathbb{P}_{\text{Berk}}^1 \setminus \mathbb{P}^1(K)$ (see [BR10, Remark 5.12]), we have $\Delta_{\mathbb{A}_{\text{Berk}}^1}(G_E) = \Delta_{\mathbb{A}_{\text{Berk}}^1}(G)$. Since G_E and G are both subharmonic on $\mathbb{A}_{\text{Berk}}^1$, have the same Laplacian, and agree on $\mathbb{A}_{\text{Berk}}^1 \setminus K$, it follows from [BR10, Corollary 8.37] that $G = G_E$ on $\mathbb{A}_{\text{Berk}}^1$. \square

2.3. Adelic equidistribution of small points. In this section, we state the arithmetic equidistribution result needed for our proof of Theorem 1.1. In order to state the result (Theorem 2.8 below), we first need some definitions.

Definition. A *product formula field* is a field k , together with the following extra data:

- (1) a set \mathcal{M}_k of non-trivial absolute values on k (which we may assume to be pairwise inequivalent), and
- (2) for each $v \in \mathcal{M}_k$, an integer $N_v \geq 1$

such that

- (3) for each $\alpha \in k^\times$, we have $|\alpha|_v = 1$ for all but finitely many $v \in \mathcal{M}_k$, and

(4) every $\alpha \in k^\times$ satisfies the *product formula*

$$\prod_{v \in \mathcal{M}_k} |\alpha|_v^{N_v} = 1.$$

The most important examples of product formula fields are number fields and function fields of normal projective varieties (see [Lan83, §2.3] or [BG06, §1.4.6]). It is known (see [Art06, Chapter 12, Theorem 3]) that a product formula field for which at least one $v \in \mathcal{M}_k$ is archimedean must be a number field. If all $v \in \mathcal{M}_k$ are non-archimedean, then we define the *constant field* k_0 of k to be the set of all $\alpha \in k$ such that $|\alpha|_v \leq 1$ for all $v \in \mathcal{M}_k$. By the product formula, if $\alpha \in k_0$ is nonzero then in fact $|\alpha|_v = 1$ for all $v \in \mathcal{M}_k$.

Remark 2.6. Any finitely generated extension k of an algebraically closed field k_0 can be endowed with a product formula structure in such a way that the field of constants of k is k_0 (cf. [BG06, Lemma 1.4.10]). Indeed, every such field k can be (non-canonically) identified with the function field of a normal projective variety X/k_0 . Choosing such an X , one endows the function field with a product formula structure in which the places $v \in \mathcal{M}_k$ (which are all non-archimedean) correspond to prime divisors on X . See [BG06, §1.4] for further details.

Let k be a product formula field and let \bar{k} (resp. k^{sep}) denote a fixed algebraic closure (resp. separable closure) of k . For $v \in \mathcal{M}_k$, let k_v be the completion of k at v , let \bar{k}_v be an algebraic closure of k_v , and let \mathbb{C}_v denote the completion of \bar{k}_v . For each $v \in \mathcal{M}_k$, we fix an embedding of \bar{k} in \mathbb{C}_v extending the canonical embedding of k in k_v , and view this embedding as an identification. By the discussion above, if v is archimedean then $\mathbb{C}_v \cong \mathbb{C}$. For each $v \in \mathcal{M}_k$, we let $\mathbb{P}_{\text{Berk},v}^1$ denote the Berkovich projective line over \mathbb{C}_v , which we take to mean $\mathbb{P}^1(\mathbb{C})$ if v is archimedean.

A *compact Berkovich adelic set* (relative to ∞) is a set of the form

$$\mathbb{E} = \prod_v E_v$$

where E_v is a nonempty compact subset of $\mathbb{A}_{\text{Berk},v}^1 = \mathbb{P}_{\text{Berk},v}^1 \setminus \{\infty\}$ for each $v \in \mathcal{M}_k$, and where E_v is the closed unit disc $\mathcal{D}(0, 1)$ in $\mathbb{A}_{\text{Berk},v}^1$ for all but finitely many non-archimedean $v \in \mathcal{M}_k$.

For each $v \in \mathcal{M}_k$, let $\gamma(E_v)$ be the logarithmic capacity of E_v relative to ∞ ; see (2.1) and (2.4). The *logarithmic capacity* (relative to ∞) of a compact Berkovich adelic set \mathbb{E} , denoted $\gamma(\mathbb{E})$, is

$$\gamma(\mathbb{E}) = \prod_v \gamma(E_v)^{N_v}.$$

We will assume throughout the rest of this section that $\gamma(\mathbb{E}) \neq 0$, i.e., that $\gamma(E_v) > 0$ for all $v \in \mathcal{M}_k$.

For each $v \in \mathcal{M}_k$, let $G_v : \mathbb{A}_{\text{Berk},v}^1 \rightarrow \mathbb{R}$ be the Green's function for E_v relative to ∞ , i.e., $G_v(z) = G_{E_v}(z)$. If $S \subset k^{\text{sep}}$ is any finite set invariant under $\text{Gal}(k^{\text{sep}}/k)$, we define the *height of S relative to \mathbb{E}* , denoted $h_{\mathbb{E}}(S)$, by

$$(2.7) \quad h_{\mathbb{E}}(S) = \sum_{v \in \mathcal{M}_k} N_v \left(\frac{1}{|S|} \sum_{z \in S} G_v(z) \right).$$

By Galois-invariance, the sum $\sum_{z \in S} G_v(z)$ does not depend on our choice of an embedding of \bar{k} into \mathbb{C}_v .

If $z \in k^{\text{sep}}$, let $S_k(z) = \{z_1, \dots, z_n\}$ denote the set of $\text{Gal}(k^{\text{sep}}/k)$ -conjugates of z over k , where $n = [k(z) : k]$. We define a function $h_{\mathbb{E}} : k^{\text{sep}} \rightarrow \mathbb{R}_{\geq 0}$ by setting $h_{\mathbb{E}}(z) = h_{\mathbb{E}}(S_k(z))$. If $E_v = \mathcal{D}(0,1)$ for all $v \in \mathcal{M}_k$, then $G_v(z) = \log_v^+ |z|_v$ for all $v \in \mathcal{M}_k$ and all $z \in k^{\text{sep}}$, and $h_{\mathbb{E}}$ coincides with the *standard logarithmic Weil height* h .

Finally, we let μ_v denote the equilibrium measure for E_v relative to ∞ . We can now state the needed equidistribution result [BR10, Theorem 7.52]:

Theorem 2.8. *Let k be a product formula field and let \mathbb{E} be a compact Berkovich adelic set with $\gamma(\mathbb{E}) = 1$. Suppose S_n is a sequence of $\text{Gal}(k^{\text{sep}}/k)$ -invariant finite subsets of k^{sep} with $|S_n| \rightarrow \infty$ and $h_{\mathbb{E}}(S_n) \rightarrow 0$. Fix $v \in \mathcal{M}_k$, and for each n let δ_n be the discrete probability measure on $\mathbb{P}_{\text{Berk},v}^1$ supported equally on the elements of S_n . Then the sequence of measures $\{\delta_n\}$ converges weakly to μ_v on $\mathbb{P}_{\text{Berk},v}^1$.*

Remark 2.9. When k is a number field, a slightly weaker version of Theorem 2.8 is proved in [BR06] and a closely related result (which also generalizes Theorem 5.3 below) is proved in [FRL06].

For concreteness, we explicitly state the special case of Theorem 2.8 in which k is a number field and S_n is the $\text{Gal}(\bar{k}/k)$ -orbit of a point $z_n \in \bar{k}$ (note in this case that $h_{\mathbb{E}}(z_n) \rightarrow 0$ implies $|S_n| \rightarrow \infty$ by Northcott's theorem):

Corollary 2.10. *Let k be a number field, and let \mathbb{E} be a compact Berkovich adelic set with $\gamma(\mathbb{E}) = 1$. Suppose $\{z_n\}$ is a sequence of distinct points of \bar{k} with $h_{\mathbb{E}}(z_n) \rightarrow 0$. Fix a place v of k , and for each n let δ_n be the discrete probability measure on $\mathbb{P}_{\text{Berk},v}^1$ supported equally on the $\text{Gal}(\bar{k}/k)$ -conjugates of z_n . Then the sequence of measures $\{\delta_n\}$ converges weakly to μ_v on $\mathbb{P}_{\text{Berk},v}^1$.*

Remark 2.11. When $k = \mathbb{Q}$ and \mathbb{E} is the *trivial* Berkovich adelic set (i.e., E_v is the v -adic unit disc for all v), Corollary 2.10 is Bilu's equidistribution theorem [Bil97] for v archimedean, and it is Chambert-Loir's generalization of Bilu's theorem [CL06] for v non-archimedean.

Remark 2.12. If k is a number field and $\gamma(\mathbb{E}) < 1$, there are only finitely many $z \in \bar{k}$ with $h_{\mathbb{E}}(z) = 0$; this follows from the adelic version of the Fekete-Szegő theorem

proved in [BR10, Theorem 6.28]. This observation helps explain the role played by the condition $\gamma(\mathbb{E}) = 1$ in Theorem 2.8 and Corollary 2.10.

3. GENERALIZED MANDELBROT SETS

Let K be an algebraically closed field which is complete with respect to a nontrivial (archimedean or nonarchimedean) absolute value. Fix an integer $d \geq 2$, and for $c \in K$ let $f_c(z) = z^d + c$. We denote by $f_c^{(n)}$ the n^{th} iterate of f_c . In this section, we introduce a family of generalized Mandelbrot sets, defined as the set of parameters c for which a given point $z = a$ remains bounded under iteration.

3.1. The archimedean case. If $K = \mathbb{C}$, define the *generalized Mandelbrot set* M_a for $a \in \mathbb{C}$ by

$$(3.1) \quad M_a = \left\{ c \in \mathbb{C} : \sup_n |f_c^{(n)}(a)| < \infty \right\}.$$

When $d = 2$ and $a = 0$, M_a is the usual Mandelbrot set. It is clear that every parameter $c \in K$ for which a is preperiodic for $z^d + c$ is contained in M_a . See Figure 1.

We need some basic potential-theoretic properties of M_a . The proofs follow the same reasoning as for the Mandelbrot set, but we provide some details for the reader's convenience. Recall that for each fixed f_c , the Green's function for the filled Julia set

$$K_c = \left\{ z : \sup_n |f_c^{(n)}(z)| < \infty \right\}$$

of f_c is given by the *escape rate*

$$G_c(z) = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log^+ |f_c^{(n)}(z)|.$$

These escape-rate functions are continuous in both c and z , and $G_c(z) = 0$ if and only if $z \in K_c$. In fact, as a locally uniform limit of plurisubharmonic functions, the function

$$(c, z) \mapsto G_c(z)$$

is plurisubharmonic on $\mathbb{C} \times \mathbb{C}$. For each fixed c there is an analytic homeomorphism ϕ_c , defining the *Böttcher coordinate* $w = \phi_c(z)$ near ∞ , which satisfies $\phi_c(f_c(z)) = (\phi_c(z))^d$ and $G_c(z) = \log |\phi_c(z)|$. The map ϕ_c sends the domain

$$V_c := \{z \in \mathbb{C} : G_c(z) > G_c(0)\}$$

conformally to the punctured disk $\{w \in \mathbb{C} : |w| > e^{G_c(0)}\}$. The Böttcher coordinate is uniquely determined if we require that ϕ_c has derivative 1 at infinity. See for example [DH84], [CG93, §2.4], and [BH88].

Lemma 3.2. *For each fixed $a \in \mathbb{C}$, we have $G_c(a^d + c) > G_c(0)$ for all c sufficiently large. Consequently, the value $f_c(a)$ lies in the domain V_c of the conjugating isomorphism ϕ_c .*

Proof. The proof relies on a standard distortion theorem for univalent functions [Rud87, Theorem 14.14]: for any holomorphic function

$$h(z) = z + \sum_{n \geq 2} a_n z^n$$

which is univalent on a disk of radius $1/r$, we have $|a_2| \leq 2r$. The conjugating isomorphism ϕ_c satisfies $\phi_c(z) = z + O(1/z)$ for z near infinity. Let U_R denote the domain $\{|z| > R\}$ in the complex plane. Setting $R_c = e^{G_c(0)}$, the inverse function $\phi_c^{-1} : U_{R_c} \rightarrow V_c$ is univalent on U_{R_c} , and it also has expansion $\phi_c^{-1}(z) = z + O(1/z)$. For any $w \notin V_c$, consider the univalent function

$$h_w(z) = \frac{1}{\phi_c^{-1}(1/z) - w} = z + wz^2 + \dots$$

We conclude that $|w| \leq 2R_c$, and therefore $V_c \supseteq U_{2R_c}$. This argument appears in [BH88, §3].

In particular, since $G_c(w) = G_c(0)/d$ and thus $w \notin V_c$ for every w with $f_c(w) = 0$, the critical point $z = 0$ and all of its preimages $(-c)^{1/d}$ must lie in the closed disk of radius $2R_c$. Thus $|c| \leq 2^d R_c^d$. This implies that $R_c \rightarrow \infty$ as $c \rightarrow \infty$.

Note that $|\phi_c(c)| = R_c^d$. When c is large enough so that $R_c^d/2 > 2R_c$, we apply the same distortion estimate to conclude that $\phi_c(U_{R_c^d/2}) \supseteq U_{R_c^d}$, so $|c| \geq R_c^d/2$. It follows that for any fixed a , since $R_c \rightarrow \infty$ with c , we have $|a^d + c| \geq R_c^d/2 - |a|^d > 2R_c$ for all sufficiently large c . Thus $f_c(a) = a^d + c$ lies in V_c and has escape rate $G_c(a^d + c) > G_c(0)$. \square

Proposition 3.3. *For each $a \in \mathbb{C}$, the generalized Mandelbrot set M_a satisfies:*

- (1) M_a is a compact and full subset of \mathbb{C} ;
- (2) the function $G_a(c) := G_c(a^d + c)$ defines the Green's function for M_a and satisfies $G_a(c) = 0$ for all $c \in M_a$;
- (3) the function $\Phi_a(c) := \phi_c(a^d + c)$ defines a conformal isomorphism in a neighborhood of infinity, and it is uniquely determined by the conditions $G_a(c) = \log |\Phi_a(c)|$ and $\Phi'_a(\infty) = 1$;
- (4) the logarithmic capacity is $\gamma(M_a) = 1$; and
- (5) the support of the equilibrium measure μ_{M_a} on M_a is equal to the boundary ∂M_a .

Proof. The set M_a is closed because $M_a = \{c : G_c(a) = 0\}$ and $(c, z) \mapsto G_c(z)$ is continuous. It is bounded by Lemma 3.2: for all sufficiently large c , the escape rate of a is positive and therefore $f_c^{(n)}(a) \rightarrow \infty$. The maximum principle applied to the

subharmonic function $c \mapsto G_c(a)$ implies that M_a is full (meaning that its complement is connected), completing the proof of statement (1).

The conjugating isomorphisms ϕ_c satisfy

$$\phi_c(z) = z \prod_{n=0}^{\infty} \left(1 + \frac{c}{(f_c^{(n)}(z))^d} \right)^{1/d^{n+1}}$$

on their domains $\{z : G_c(z) > G_c(0)\}$; see [DH84] or [CG93, §VIII.3]. By Lemma 3.2, the function $\Phi_a(c)/c = \phi_c(a^d + c)/c$ can be expressed by this infinite product for c near infinity. The terms in the infinite product each tend to 1 as $c \rightarrow \infty$, so (setting $\Phi_a(\infty) = \infty$) we conclude $\Phi'_a(\infty) = 1$. In particular, Φ_a defines a conformal isomorphism in a neighborhood of infinity.

The Green's function G_c for the filled Julia set K_c satisfies $G_c(z) = \log |\phi_c(z)|$ where defined. The function $G_a(c) = G_c(a^d + c)$ therefore satisfies $G_a(c) = \log |\Phi_a(c)| = \log |c + O(1)| = \log |c| + o(1)$ for all c large. Furthermore, G_a is harmonic on $\mathbb{C} \setminus M_a$, as a locally uniform limit of the harmonic functions $c \mapsto G_n(c) = d^{-n} \log |f_c^{(n)}(a^d + c)|$, and $G_a(c) = 0$ if and only if $c \in M_a$; we conclude that G_a is the Green's function for M_a . The conditions stated in (3) clearly determine Φ_a uniquely near ∞ . Statement (4) follows because $G_a(c) = \log |c| + o(1)$ near infinity.

Finally, statement (5) follows from Lemma 2.2, because M_a is full. \square

Fix a degree $d \geq 2$. For each $a \in \mathbb{C}$, define

$$\text{Preper}(a) := \{c \in \mathbb{C} : a \text{ is preperiodic for } z^d + c\}.$$

Combining Proposition 3.3 with Montel's theorem, we obtain:

Theorem 3.4. *For each degree $d \geq 2$ and any $a, b \in \mathbb{C}$, the following are equivalent:*

- (1) $M_a = M_b$
- (2) $\mu_{M_a} = \mu_{M_b}$
- (3) $a^d = b^d$
- (4) $\text{Preper}(a) = \text{Preper}(b)$.

Proof. First suppose that $a^d = b^d$. Then for every c , we have $f_c(a) = a^d + c = b^d + c = f_c(b)$, so a is preperiodic for f_c if and only if b is preperiodic for f_c . Thus (3) implies (4).

Now assume (4) and consider the sequence of functions $g_n(c) := f_c^{(n)}(a)$. This sequence forms a normal family except on the boundary ∂M_a . Consider the holomorphic functions $h_1(c) = a$ and $h_2(c) = a^d + c$. First note that $a^d + c = a$ implies that a is a fixed point for f_c , so $c = a - a^d \in \text{Preper}(a) \subset M_a$. Now fix an open set U intersecting ∂M_a which does not contain the parameter $c = a - a^d$, so the images of h_1 and h_2 are distinct throughout U . Then by Montel's theorem, the failure of normality of $\{g_n\}$ implies that the image of g_n must coincide with that of h_1 or h_2 for some n and some $c \in U$. In particular, there is an iterate n so that either $f_c^{(n)}(a) = a$ or $f_c^{(n)}(a) = f_c(a)$;

we conclude that the set U must intersect $\text{Preper}(a)$. Consequently, the boundary ∂M_a is contained in the closure of $\text{Preper}(a)$. As $\text{Preper}(a) \subset M_a$ by definition and M_a is a full set by Proposition 3.3 (1), the assumption $\text{Preper}(a) = \text{Preper}(b)$ implies that $M_a = M_b$, i.e., (4) implies (1).

Assume that $M_a = M_b$. Clearly the equilibrium measures coincide, so $\mu_{M_a} = \mu_{M_b}$, giving (1) \implies (2). Conversely, the support of μ_{M_a} is equal to the boundary ∂M_a , so again using the fact that M_a is a full set we conclude that (2) implies (1).

Finally, if $M_a = M_b$, then by Proposition 3.3 (3) the uniformizing maps Φ_a and Φ_b coincide on a neighborhood of infinity. In other words, for all large c , we have $\phi_c(a^d + c) = \phi_c(b^d + c)$. The conjugating isomorphisms ϕ_c are injective, so we conclude that $a^d + c = b^d + c$. This shows that (1) implies (3), completing the proof. \square

The following simple statement is used for one implication of Theorem 1.1.

Lemma 3.5. *For each $a \in \mathbb{C}$, the set $\text{Preper}(a)$ is infinite.*

Proof. From Proposition 3.3, the set M_a has capacity 1, so its boundary cannot be a finite set. From the proof of Theorem 3.4, the boundary ∂M_a is contained in the closure of $\text{Preper}(a)$, so the set $\text{Preper}(a)$ must contain infinitely many points. \square

Note that if $a \in \bar{\mathbb{Q}}$, then the set $\text{Preper}(a)$ is a subset of $\bar{\mathbb{Q}}$ with bounded Weil height (since $h_{M_a}(c) = 0$ for all $c \in \text{Preper}(a)$ and the difference $h - h_{M_a}$ is bounded). It is thus a rather ‘‘sparse’’ set (compare with the discussion following Theorem 1.5 above).

3.2. The non-archimedean case. If K is a non-archimedean field, one can define $M_a \subset \mathbb{A}_{\text{Berk},K}^1$ similarly and prove basic potential-theoretic statements about M_a .

Let $g_n(T) = f_T^{(n)}(a)$; this is a monic polynomial in T of degree d^{n-1} which depends on $a \in K$. Define

$$(3.6) \quad M_a := \left\{ c \in \mathbb{A}_{\text{Berk},K}^1 : \sup_n |g_n(c)| < \infty \right\},$$

where $[\cdot]_c$ is the multiplicative seminorm on $K[T]$ corresponding to $c \in \mathbb{A}_{\text{Berk},K}^1$ and $|g_n(c)|$ is shorthand for $[g_n(T)]_c$. (Note that for $c \in K$, we have $[g_n(T)]_c = |g_n(c)| = |f_c^{(n)}(a)|$.)

Proposition 3.7. *For each $a \in K$,*

- (1) *the boundary of M_a coincides with the outer boundary in $\mathbb{A}_{\text{Berk},K}^1$, and it is equal to the support of μ_{M_a} ;*
- (2) *the logarithmic capacity $\gamma(M_a)$ is equal to 1; and*
- (3) *the Green’s function for M_a relative to ∞ is 0 at all points of M_a .*

Proof. Fix $a \in K$, and for $c \in \mathbb{A}_{\text{Berk}}^1$ define

$$(3.8) \quad G_a(c) := \lim_{n \rightarrow \infty} \frac{1}{d^n} \log^+ |g_{n+1}(c)|.$$

Note that for $c \in K$, we have

$$G_a(c) = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log^+ |f_c^{(n+1)}(a)| = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log^+ |f_c^{(n)}(a^d + c)|,$$

which is the same formula we used to define G_a over \mathbb{C} .

The limit in (3.8) exists for all $c \in \mathbb{A}_{\text{Berk}}^1$: if $c \in M_a$, then the limit is zero, while if $c \notin M_a$, then the sequence $\frac{1}{d^n} \log^+ |g_{(n+1)}(c)|$ is eventually constant. Indeed, if $c \notin M_a$ then the sequence $|g_n(c)|$ is unbounded so there exists $N = N(c)$ such that $|g_n(c)| > \max\{1, |c|^{\frac{1}{a}}\}$ for $n \geq N$. By the ultrametric inequality, for $c \in K$ and $n \geq N$ we have

$$|g_{n+1}(c)| = |g_n(c)^d + c| = |g_n(c)|^d > 1,$$

and since K is dense in $\mathbb{A}_{\text{Berk}}^1$ the equality $|g_{n+1}(c)| = |g_n(c)|^d$ holds more generally for all $c \in \mathbb{A}_{\text{Berk}}^1 \setminus M_a$. Thus the sequence $\frac{1}{d^n} \log^+ |g_{(n+1)}(c)|$ is constant for $n \geq N$.

Claim 1: $\frac{1}{d^n} \log^+ |g_{n+1}(c)|$ converges to $G_a(c)$ uniformly on compact subsets of $\mathbb{A}_{\text{Berk}}^1$ (as functions of c).

To see this, one can employ essentially the same argument as in the archimedean case ([BH88, Proposition 1.2]; compare with [BR10, §10.1]). Briefly, fix a compact set $E \subset \mathbb{A}_{\text{Berk}}^1$. Then there is a constant $C > 0$, depending only on E , such that $[T + z^d]_c = |z|^d$ for $c \in E$ and $|z| \geq C$. Thus there are constants $C_1, C_2 > 0$ (depending only on E) such that for $z \in K$ and $c \in E$,

$$C_1 \max(1, |z|^d) \leq \max(1, [T + z^d]_c) \leq C_2 \max(1, |z|^d).$$

Taking logarithms, iterating, and multiplying by d shows that there is a constant $C' > 0$ (depending only on E) such that for each fixed $a \in K$,

$$\left| \frac{1}{d^n} \log^+ |g_{n+1}(c)| - \frac{1}{d^{n-1}} \log^+ |g_n(c)| \right| \leq \frac{C'}{d^n}.$$

A telescoping series argument now gives the desired uniform convergence on E :

$$\begin{aligned} \left| G_a(c) - \frac{1}{d^{n-1}} \log^+ |g_n(c)| \right| &= \left| \sum_{m \geq n} \left(\frac{1}{d^m} \log^+ |g_{m+1}(c)| - \frac{1}{d^{m-1}} \log^+ |g_m(c)| \right) \right| \\ &\leq \sum_{m \geq n} \frac{C'}{d^m} = \frac{C'}{d^n - d^{n-1}}, \end{aligned}$$

proving Claim 1.

Claim 2: G_a is the Green's function for M_a relative to ∞ .

Indeed, G_a is harmonic on $U_a := \mathbb{A}_{\text{Berk}}^1 \setminus M_a$ by [BR10, Example 7.5] and [BR10, Proposition 7.31], since on U_a the function G_a is the limit of the harmonic functions $\frac{1}{d^n} \log |g_{n+1}(c)|$. Moreover, since the sequence of continuous subharmonic functions $\frac{1}{d^n} \log^+ |g_{n+1}(c)|$ converges uniformly to G_a on compact subsets of $\mathbb{A}_{\text{Berk}}^1$, it follows from [BR10, Proposition 8.26(C)] that G_a is continuous and subharmonic on $\mathbb{A}_{\text{Berk}}^1$.

In addition, G_a is zero on M_a , and for $|c| > \max\{1, |a|^d\}$ we have $G_a(c) = \log^+ |c|$. Claim 2 therefore follows from part (2) of Lemma 2.5.

Assertion (3) is now immediate, and assertions (1) and (2) follow from parts (3) and (1) of Lemma 2.5, respectively. \square

3.3. Global generalized Mandelbrot sets. Let k be a product formula field, and fix $a \in k$. For each $v \in \mathcal{M}_k$, define $M_{a,v} \subseteq \mathbb{A}_{\text{Berk}, \mathbb{C}_v}^1$ following the local recipes above. Recall that $\mathbb{A}_{\text{Berk}, \mathbb{C}_v}^1 = \mathbb{C}$ if v is archimedean. Define a compact Berkovich adelic set \mathbf{M}_a by

$$\mathbf{M}_a := \{M_{a,v}\},$$

observing that $M_{a,v} = \mathcal{D}(0, 1)$ whenever $|a|_v \leq 1$. Propositions 3.3 (4) and 3.7 (2) imply that the global capacity $\gamma(\mathbf{M}_a)$ is equal to 1. Moreover, for each $v \in \mathcal{M}_k$ the local Green's function $G_{M_{a,v}} : \mathbb{A}_{\text{Berk}, \mathbb{C}_v}^1 \rightarrow \mathbb{R}_{\geq 0}$ is continuous, with $G_{M_{a,v}}(z) = 0$ if and only if $z \in M_{a,v}$.

If $S \subset \bar{k}$ is any finite set invariant under $\text{Gal}(\bar{k}/k)$, then following (2.7) the height of S relative to \mathbf{M}_a is given by

$$(3.9) \quad h_{\mathbf{M}_a}(S) = \sum_{v \in \mathcal{M}_k} N_v \left(\frac{1}{|S|} \sum_{z \in S} G_{M_{a,v}}(z) \right).$$

Remark 3.10. The adelic height function attached to the usual Mandelbrot set appeared previously in [BH05] and [FRL06].

3.4. The function field setting. For later use, we recall a result of Benedetto and note its relevant consequences. Let k be an *abstract function field*, by which we mean a product formula field for which all $v \in \mathcal{M}_k$ are non-archimedean. A polynomial $\varphi \in k[T]$ is called *trivial over k* if it is conjugate (by an invertible linear map $T \mapsto \alpha T + \beta$ defined over k) to a polynomial defined over the constant field of k .

Theorem 3.11. [Ben05] *Let k be an abstract function field. If $\varphi \in k[T]$ is not trivial over k , then $a \in \bar{k}$ is preperiodic for φ if and only if a belongs to the v -adic filled Julia set of φ (i.e., a stays v -adically bounded under iteration of φ) for all $v \in \mathcal{M}_k$.*

Remark 3.12. If k is a number field, then it is well known and follows easy from Northcott's theorem that $a \in \bar{k}$ is preperiodic for φ if and only if a belongs to the v -adic filled Julia set of φ for all $v \in \mathcal{M}_k$. But if k is an abstract function field and $\varphi \in k[T]$ is trivial over k , then it is easy to see that the conclusion of Theorem 3.11 fails, since every element of the constant field k_0 of k stays v -adically bounded for all v but not every element of k_0 is preperiodic.

Corollary 3.13. *Let k be an abstract function field of characteristic zero such that every field k_v for $v \in \mathcal{M}_k$ also has residue characteristic zero, and fix $a, c \in k$ with c not in the constant field k_0 of k . Then the following are equivalent:*

- (1) a is preperiodic for the iteration of $f_c(z) = z^d + c$.
- (2) c is contained in $M_{a,v}$ for all $v \in \mathcal{M}_k$.
- (3) $h_{\mathbf{M}_a}(c) = 0$.

Proof. By the discussion at the beginning of §3.3, we have $h_{\mathbf{M}_a}(c) = 0$ if and only if c is contained in $M_{a,v}$ for all $v \in \mathcal{M}_k$. We claim that $z^d + c$ is not trivial over k . Assuming the claim, Benedetto's theorem implies that a is preperiodic for f_c if and only if a belongs to the v -adic filled Julia set of f_c for all $v \in \mathcal{M}_k$. The desired result follows, since by definition, a belongs to the v -adic filled Julia set of f_c if and only if $c \in M_{a,v}$.

To prove the claim, suppose for the sake of contradiction that $c \notin k_0$ and $\alpha z + \beta$ conjugates $z^d + c$ into a polynomial defined over k_0 . Then that conjugate is

$$(3.14) \quad \frac{1}{\alpha}(\alpha z + \beta)^d + \frac{1}{\alpha}(c - \beta) = \alpha^{d-1}z^d + \cdots + d\beta^{d-1}z + \frac{c + \beta^d - \beta}{\alpha} \in k_0[z].$$

Since $c \notin k_0$, there exists $v \in \mathcal{M}_k$ such that $|c|_v > 1$. By (3.14) and our assumptions on k , we have $|\alpha|_v = 1$ and $|d\beta^{d-1}|_v = |\beta|_v^{d-1} \leq 1$, hence $|\beta|_v \leq 1$. Thus $|c + \beta^d - \beta|_v > 1$ by the ultrametric inequality, contradicting (3.14). \square

4. PROOF OF THEOREM 1.1

We are now ready to give the proof of our main theorem. We also extract a stronger statement (Theorem 4.1) in the case where the points a and b are algebraic.

4.1. The main theorem. Fix two points a and b in the complex plane and a degree $d \geq 2$. We aim to prove that the set of parameters $c \in \mathbb{C}$ for which both a and b are preperiodic for $f_c(z) = z^d + c$ is infinite if and only if $a^d = b^d$.

Proof of Theorem 1.1. First suppose that $a^d = b^d$. Then a is preperiodic for f_c if and only if b is preperiodic for f_c . From Lemma 3.5, the set of parameters c for which a is preperiodic is infinite.

Now fix a and b in \mathbb{C} , and assume that there is an infinite sequence c_1, c_2, \dots of distinct complex numbers such that a and b are both preperiodic for $z^d + c_n$ for all n .

Case 1: $a, b \in \bar{\mathbb{Q}}$.

In this case, c_n must be algebraic for all n . Indeed, let $g_m(c) = f_c^{(m)}(a)$; this is a monic polynomial in c of degree d^{m-1} with coefficients in the number field $k := \mathbb{Q}(a) \subset \bar{\mathbb{Q}}$. Since a is preperiodic for $f_{c_n}(z)$ ($n = 1, 2, \dots$), there exist integers $\ell > m \geq 1$ (depending on n) such that $g_\ell(c_n) = g_m(c_n)$. Thus c_n is a root of the nonzero polynomial $g_\ell(z) - g_m(z) \in \bar{\mathbb{Q}}[z]$, and hence $c_n \in \bar{k}$ for all n . Further, we

see that a is also preperiodic for all $\text{Gal}(\bar{k}/k)$ -conjugates of c_n , and we deduce that $h_{\mathbf{M}_a}(c_n) = 0$ for all n .

Let δ_n be the discrete probability measure on \mathbb{C} supported equally on the $\text{Gal}(\bar{k}/k)$ -conjugates of c_n . By Corollary 2.10 the measures δ_n converge weakly to the probability measure μ_{M_a} (the equilibrium measure relative to ∞ for the set M_a) on $\mathbb{P}^1(\mathbb{C})$. By symmetry, the measures δ_n also converge weakly to μ_{M_b} . Thus $\mu_{M_a} = \mu_{M_b}$. By Theorem 3.4, we conclude that $a^d = b^d$.

Case 2: a is transcendental.

In this case, b is also transcendental, as otherwise each c_n would be algebraic, contradicting the transcendence of a . In fact, the values a , b , and c_n for all n are defined over the algebraic closure \bar{k} of $k = \bar{\mathbb{Q}}(a)$ in \mathbb{C} . Indeed, for each n there exist integers $\ell > m \geq 1$ such that c_n is a root of the nonzero polynomial $g_\ell(z) - g_m(z) \in k[z]$, and hence $c_n \in \bar{k}$ for all n . Moreover (setting $c = c_n$ for any n), there exist $\ell > m \geq 1$ such that b is a root of the nonzero polynomial $f_c^{(\ell)}(z) - f_c^{(m)}(z) \in \bar{k}[z]$, and hence $b \in \bar{k}$ as well.

Since a is transcendental, the field $k = \bar{\mathbb{Q}}(a)$ is isomorphic to the field $\bar{\mathbb{Q}}(T)$ of rational functions over $\bar{\mathbb{Q}}$, and in particular k can be viewed as a product formula field with $\bar{\mathbb{Q}}$ as its field of constants (cf. Remark 2.6). Since a is preperiodic for $f_{c_n}(z)$, we have $h_{\mathbf{M}_a}(c_n) = 0$ for all n . Fix a place $v \in \mathcal{M}_k$, let \mathbb{C}_v be the completion of an algebraic closure of the v -adic completion k_v , and identify \bar{k} with a subfield of \mathbb{C}_v . Let T_m be the set of $\text{Gal}(\bar{k}/k)$ -conjugates of $c_m \in \bar{k}$, and define

$$S_n = \bigcup_{m=1}^n T_m.$$

Then S_n is a $\text{Gal}(\bar{k}/k)$ -stable subset of \bar{k} , $h_{\mathbf{M}_a}(c) = 0$ for every $c \in S_n$, and $|S_n| \rightarrow \infty$ as $n \rightarrow \infty$. Let δ_n be the discrete probability measure on the Berkovich projective line $\mathbb{P}_{\text{Berk},v}^1$ over \mathbb{C}_v supported equally on the elements of S_n . Let $M_{a,v} \subset \mathbb{P}_{\text{Berk},v}^1$ be the v -adic generalized Mandelbrot set corresponding to a (cf. (3.6)). By Theorem 2.8, the sequence δ_n converges weakly on $\mathbb{P}_{\text{Berk},v}^1$ to the equilibrium measure $\mu_{M_{a,v}}$ for $M_{a,v}$ relative to ∞ . Moreover, by Proposition 3.7, the support of $\mu_{M_{a,v}}$ is equal to $\partial M_{a,v}$.

Applying the same reasoning to b , it follows by symmetry that $M_{a,v} = M_{b,v}$ for all places v of k . Hence, by Corollary 3.13, for each fixed $c \in \bar{k}$, a is preperiodic for $z^d + c$ if and only if b is preperiodic. Recall from the discussion at the beginning of Case 2 that if $c \in \mathbb{C}$ and a is preperiodic for $z^d + c$, then $c \in \bar{k}$. It follows that for every complex number c , a is preperiodic for $z^d + c$ if and only if b is. Theorem 3.4 then implies that $a^d = b^d$, completing the proof of the theorem. \square

4.2. Height bounds. In the case where $a, b \in \bar{\mathbb{Q}}$, the proof of Theorem 1.1 yields the following stronger result:

Theorem 4.1. *Let $a, b \in \bar{\mathbb{Q}}$ with $a^d \neq b^d$. Then there is a real number $\varepsilon > 0$ such that $h_{\mathbf{M}_a}(c) + h_{\mathbf{M}_b}(c) \geq \varepsilon$ for all but finitely many $c \in \bar{\mathbb{Q}}$.*

Proof. If we assume that the result is false, then there exists a sequence c_n of algebraic numbers such $\lim_{n \rightarrow \infty} h_{\mathbf{M}_a}(c_n) = \lim_{n \rightarrow \infty} h_{\mathbf{M}_b}(c_n) = 0$. The proof now follows from the exact same argument as the proof of Case 1 of Theorem 1.1. \square

4.3. Remark: non-arithmetic equidistribution. Fix a point $a \in \mathbb{C}$ and let $f_c(z) = z^d + c$. It follows from the arguments in [DF08] (specifically the proof of their Theorem 1) that the solutions $c \in \mathbb{C}$ to equations

$$f_c^n(a) = f_c^{k(n)}(a),$$

weighted by multiplicity, will equidistribute to the equilibrium measure μ_{M_a} on ∂M_a as $n \rightarrow \infty$ for any sequence $0 \leq k(n) < n$. Thus, a sequence of *complete* sets of solutions to these equations will determine the measure μ_{M_a} . This would give an alternate argument in the proof of Theorem 1.1 under a much stronger hypothesis, treating the algebraic and transcendental cases simultaneously.

However, it should be noted that under this stronger hypothesis, we are able to determine the measure μ_{M_a} and the value of a^d by more classical means, without need for equidistribution. By a straightforward argument using Montel's theorem (as in the proof of Theorem 3.4), we find that the parameters $\{c : f_c^n(a) = f_c^{k(n)}(a)\}$ must accumulate everywhere on ∂M_a as $n \rightarrow \infty$. These parameters determine M_a as a set, from which the measure μ_a is determined by classical potential theory. The value of a^d is then recovered from Theorem 3.4.

5. A VARIANT OF THEOREM 1.1

For a rational function $\varphi \in \mathbb{C}(z)$, let $\text{Preper}(\varphi)$ denote its set of preperiodic points in the Riemann sphere $\hat{\mathbb{C}}$. Our goal in this section is to prove Theorem 1.2, whose statement we recall:

Theorem. *Let $\varphi, \psi \in \mathbb{C}(z)$ be rational functions of degrees at least 2. Then $\text{Preper}(\varphi) \cap \text{Preper}(\psi)$ is infinite if and only if $\text{Preper}(\varphi) = \text{Preper}(\psi)$.*

We conclude this section with the proof of Corollary 1.3, and we state two further consequences of the proof of Theorem 1.2 (Theorems 5.7 and 5.8).

5.1. Equidistribution. The main references for this section are [BR06] and [BR10, Chapter 10]; see also [FRL06]. Let k be a product formula field, and let $\varphi \in k(T)$ be a rational function of degree $d \geq 2$. Associated to φ is the *Call-Silverman canonical height function* $\hat{h}_\varphi : \mathbb{P}^1(\bar{k}) \rightarrow \mathbb{R}_{\geq 0}$. If k is a number field, then a point $P \in \mathbb{P}^1(\bar{k})$ is preperiodic for φ if and only if $\hat{h}_\varphi(P) = 0$. In general, things are a little more subtle; see Theorem 5.5 below.

For each $v \in \mathcal{M}_k$, the v -adic *Arakelov-Green function* of φ is a function

$$g_{\varphi,v} : \mathbb{P}_{\text{Berk},v}^1 \times \mathbb{P}_{\text{Berk},v}^1 \rightarrow \mathbb{R} \cup \{+\infty\}$$

that takes the value $+\infty$ on the intersection of the diagonal with $\mathbb{P}^1(\mathbb{C}_v)$ and is finite-valued elsewhere. There is also a canonical probability measure $\mu_{\varphi,v}$ on $\mathbb{P}_{\text{Berk},v}^1$; when v is archimedean $\mu_{\varphi,v}$ is the measure of maximal entropy for φ on $\mathbb{P}^1(\mathbb{C})$ studied in [Lyu83], [FLM83]. Each of $\mu_{\varphi,v}$ and $g_{\varphi,v}(x, y)$ determines the other by the equation

$$(5.1) \quad \Delta_x g_{\varphi,v}(x, y) = \mu_{\varphi,v} - \delta_y$$

for every fixed $y \in \mathbb{P}_{\text{Berk},v}^1$, where g is normalized so that

$$\iint_{\mathbb{P}_{\text{Berk},v}^1 \times \mathbb{P}_{\text{Berk},v}^1} g_{\varphi,v}(x, y) d\mu_{\varphi,v}(x) d\mu_{\varphi,v}(y) = 0.$$

(The Laplacian in (5.1) is the negative of the one studied in [BR10].) Moreover, for all $x, y \in \mathbb{P}^1(\bar{k})$ with $x \neq y$ we have

$$(5.2) \quad \hat{h}_{\varphi}(x) + \hat{h}_{\varphi}(y) = \sum_{v \in \mathcal{M}_k} N_v g_{\varphi,v}(x, y).$$

If v is archimedean, it is well known that $\text{Supp}(\mu_{\varphi,v})$ is equal to the complex Julia set of φ . For v non-archimedean, the Berkovich Julia set of φ is *defined* in [BR10, Chapter 10] to be the support of $\mu_{\varphi,v}$. This turns out to be equivalent to several other, more topological, characterizations of the Berkovich Julia set.

If S is a finite subset of $\mathbb{P}^1(k^{\text{sep}})$ which is stable under $\text{Gal}(k^{\text{sep}}/k)$, we define

$$\hat{h}_{\varphi}(S) = \frac{1}{|S|} \sum_{P \in S} \hat{h}_{\varphi}(P).$$

The following equidistribution theorem was proved independently by Baker–Rumely [BR06], Chambert-Loir [CL06], and Favre–Rivera-Letelier [FRL06] in the number field case. For the present formulation in terms of an arbitrary product formula field, see [BR10, Theorem 10.24].

Theorem 5.3. *Let k be a product formula field, and let $\varphi \in k(T)$ be a rational function of degree $d \geq 2$. Suppose S_n is a sequence of $\text{Gal}(k^{\text{sep}}/k)$ -invariant finite subsets of k^{sep} with $|S_n| \rightarrow \infty$ and $\hat{h}_{\varphi}(S_n) \rightarrow 0$. Fix $v \in \mathcal{M}_k$, and for each n let δ_n be the discrete probability measure on $\mathbb{P}_{\text{Berk},v}^1$ supported equally on the elements of S_n . Then the sequence of measures $\{\delta_n\}$ converges weakly to $\mu_{\varphi,v}$ on $\mathbb{P}_{\text{Berk},v}^1$.*

5.2. Isotriviality. Let k be an abstract function field; see §3.4 for the definition. A map $\varphi \in k(T)$ of degree d is said to have *good reduction* at $v \in \mathcal{M}_k$ if $\varphi = f/g$ for some $f, g \in \mathcal{O}_v[T]$ whose reductions \bar{f}, \bar{g} are polynomials in $\tilde{k}_v[T]$ with no common roots in any algebraic closure of \tilde{k}_v and such that at least one of \bar{f}, \bar{g} has degree d . (Here \mathcal{O}_v denotes the valuation ring of k_v , and \tilde{k}_v its residue field.)

The following result is proved in [Bak09]; the equivalence of (2) and (3) is due originally to Rivera-Letelier and another proof of the equivalence of (1) and (2) can be found in [PST09].

Proposition 5.4. *Let k be an abstract function field, and let $\varphi \in k(T)$ be a rational map of degree at least 2. Then the following are equivalent:*

- (1) φ is defined over the constant field of k
- (2) φ has good reduction at every $v \in \mathcal{M}_k$
- (3) The canonical measure $\mu_{\varphi, v}$ is a point mass supported at the Gauss point of $\mathbb{P}_{\text{Berk}, v}^1$ for all $v \in \mathcal{M}_k$.

Assume now that the constant field of k is algebraically closed. A rational map $\varphi \in k(T)$ is called *isotrivial over k* if it is conjugate, by a linear fractional transformation defined over \bar{k} , to a rational map defined over the constant field of k . The following result is also proved in [Bak09]:

Theorem 5.5. *Let k be an abstract function field with an algebraically closed field of constants. If $\varphi \in k(T)$ is a rational map of degree $d \geq 2$ which is not isotrivial, then a point $P \in \mathbb{P}^1(\bar{k})$ satisfies $\hat{h}_\varphi(P) = 0$ if and only if P is preperiodic for φ .*

Modulo the assumption that the constant field of k is algebraically closed, Theorem 5.5 is a generalization of Theorem 3.11. See [CH08a, CH08b] for a higher-dimensional generalization of Theorem 5.5 (which also does not require the assumption that the constant field of k is algebraically closed).

5.3. Preperiodic points for rational functions. We can now give the proof of Theorem 1.2.

Proof of Theorem 1.2. First note that the set of preperiodic points of a rational function of degree ≥ 2 is always infinite; see [Bea91, §6.2]. Therefore, the equality $\text{Preper}(\varphi) = \text{Preper}(\psi)$ clearly implies that the intersection is infinite. It remains to prove the reverse implication.

Suppose there is an infinite sequence a_n of complex numbers which are preperiodic for both φ and ψ .

Case 1: $\varphi, \psi \in \bar{\mathbb{Q}}(T)$.

In this case, a_n is algebraic for all n . Let k be a number field over which both φ and ψ are defined. Fix a place v of k and let δ_n be the discrete probability measure on $\mathbb{P}^1(k_v)$ supported equally on the $\text{Gal}(\bar{k}/k)$ -conjugates of a_n . By Theorem 5.3, the

sequence δ_n converges weakly on $\mathbb{P}_{\text{Berk},v}^1$ to both $\mu_{\varphi,v}$ and $\mu_{\psi,v}$, hence $\mu_{\varphi,v} = \mu_{\psi,v}$. By (5.1), we therefore have $g_{\mu_{\varphi,v}}(x, y) = g_{\mu_{\psi,v}}(x, y)$ for all $v \in \mathcal{M}_k$ and all $x, y \in \mathbb{P}_{\text{Berk}}^1$. From the identity (5.2), it follows (letting $y = a_n$ for some n , so that $\hat{h}_{\varphi}(y) = \hat{h}_{\psi}(y) = 0$) that $\hat{h}_{\varphi}(x) = \hat{h}_{\psi}(x)$ for all $x \in \mathbb{P}^1(\bar{k})$. As all preperiodic points lie in \bar{k} , we may conclude that $\text{Preper}(\varphi) = \text{Preper}(\psi)$.

Case 2: Neither φ nor ψ is conjugate to a rational map defined over $\bar{\mathbb{Q}}$.

In this case, all a_n 's are defined over \bar{k} , where k is the finitely generated field extension of $\bar{\mathbb{Q}}$ generated by the coefficients of φ and ψ . Equip k with a product formula structure as in Remark 2.6. Let T_m be the set of $\text{Gal}(\bar{k}/k)$ -conjugates of a_m , and set

$$(5.6) \quad S_n = \bigcup_{m=1}^n T_m.$$

For each $v \in \mathcal{M}_k$, let δ_n be the discrete probability measure on $\mathbb{P}_{\text{Berk},v}^1$ supported equally on the elements of S_n . By Theorem 5.3, the sequence δ_n converges weakly on $\mathbb{P}_{\text{Berk},v}^1$ to both $\mu_{\varphi,v}$ and $\mu_{\psi,v}$, hence $\mu_{\varphi,v} = \mu_{\psi,v}$ for all $v \in \mathcal{M}_k$. From equations (5.1) and (5.2), we deduce that the height functions \hat{h}_{φ} and \hat{h}_{ψ} must coincide on $\mathbb{P}^1(\bar{k})$.

By assumption, neither φ nor ψ is conjugate to a map defined over $\bar{\mathbb{Q}}$. By Theorem 5.5, a point $x \in \mathbb{P}^1(\bar{k})$ is preperiodic for φ (resp. ψ) if and only if $\hat{h}_{\varphi}(x) = 0$ (resp. $\hat{h}_{\psi}(x) = 0$). We conclude that φ and ψ have the same set of preperiodic points in \bar{k} , and therefore the same set of preperiodic points in \mathbb{C} .

Case 3: φ is conjugate to a rational map defined over $\bar{\mathbb{Q}}$.

Replacing φ by a conjugate, we may assume without loss of generality that φ is defined over $\bar{\mathbb{Q}}$. We claim that (in these new coordinates) ψ is defined over $\bar{\mathbb{Q}}$ as well, so that we are back in Case 1.

As in Case 2, all a_n 's are defined over \bar{k} , where k is the finitely generated field extension of $\bar{\mathbb{Q}}$ generated by the coefficients of φ and ψ . We may assume that $k/\bar{\mathbb{Q}}$ is transcendental, since otherwise we're done. So as in Case 2, k can be endowed with a product formula structure with respect to which all places $v \in \mathcal{M}_k$ are non-archimedean and the constant field of k is $\bar{\mathbb{Q}}$. For each $v \in \mathcal{M}_k$, let δ_n be the discrete probability measure on $\mathbb{P}_{\text{Berk},v}^1$ supported equally on the elements of S_n , where S_n is defined as in (5.6). By Theorem 5.3, the sequence δ_n converges weakly on $\mathbb{P}_{\text{Berk},v}^1$ to both $\mu_{\varphi,v}$ and $\mu_{\psi,v}$, hence $\mu_{\varphi,v} = \mu_{\psi,v}$ for all $v \in \mathcal{M}_k$.

Since φ is defined over the constant field $\bar{\mathbb{Q}}$ of k , φ has good reduction at every $v \in \mathcal{M}_k$. Equivalently, $\mu_{\varphi,v}$ is a point mass supported at the Gauss point of $\mathbb{P}_{\text{Berk},v}^1$ for all $v \in \mathcal{M}_k$. As $\mu_{\varphi,v} = \mu_{\psi,v}$ for all $v \in \mathcal{M}_k$, we deduce from Proposition 5.4 that ψ has good reduction at every $v \in \mathcal{M}_k$ and hence is defined over the constant field $\bar{\mathbb{Q}}$ of k , as claimed. \square

5.4. Shared Julia sets and further consequences. Corollary 1.3 states that if rational functions φ and ψ have infinitely many preperiodic points in common, then they must have the same Julia set.

Proof of Corollary 1.3. Assume that φ and ψ each have degree > 1 and that they have infinitely many preperiodic points in common. From Theorem 1.2, they share all of their preperiodic points.

It is well known that repelling periodic points are dense in the Julia set $J(\varphi)$ (see [Mil99, Theorem 14.1]), so clearly $J(\varphi) \subset \overline{\text{Preper}(\varphi)}$. Furthermore, the Julia set has no isolated points by [Mil99, Corollary 4.11], so every point of $J(\varphi)$ is an accumulation point of $\text{Preper}(\varphi)$. On the other hand, preperiodic points form a discrete subset of the Fatou set (see [Pet08, Proposition 2] for a proof). We conclude that the Julia set of a rational map φ of degree at least 2 is equal to the set of accumulation points of $\text{Preper}(\varphi)$. From the equality $\text{Preper}(\varphi) = \text{Preper}(\psi)$, we may therefore deduce that $J(\varphi) = J(\psi)$. \square

Much is known about rational functions in one complex variable which share the same Julia set; see for example [Bea90] (for the polynomial case) and [LP97]. For example, if $\varphi(z) = z^2 + c$ and $\psi(z) = z^2 + c'$ with $c \neq c'$, then φ and ψ have distinct Julia sets; see [Bea90, §4]. For polynomials defined over a number field, we use [KS07, Corollary 25] (which is itself based on the classification in [Bea90]) to strengthen the conclusion of Theorem 1.2:

Theorem 5.7. *Let $\varphi, \psi \in \bar{\mathbb{Q}}[z]$ be polynomials of degrees at least 2 with algebraic coefficients, and fix an embedding of \mathbb{Q} into \mathbb{C} . Then the set of points which are preperiodic for both φ and ψ is infinite if and only if there are positive integers m, n and a linear fractional transformation λ defined over $\bar{\mathbb{Q}}$ such that $\tilde{\varphi} := \lambda^{-1}\varphi\lambda$ and $\tilde{\psi} := \lambda^{-1}\psi\lambda$ satisfy one of the following:*

- (1) $\tilde{\varphi}(z) = z^n$ and $\tilde{\psi}(z) = \eta z^m$ for some root of unity η . In this case $J(\tilde{\varphi}) = J(\tilde{\psi})$ is the complex unit circle.
- (2) $\tilde{\varphi}(z) = \pm T_n(z)$ and $\tilde{\psi}(z) = \pm T_m(z)$, where $T_n(z)$ is the n^{th} Tchebycheff polynomial. In this case $J(\tilde{\varphi}) = J(\tilde{\psi}) = [-2, 2]$.
- (3) $J(\tilde{\varphi}) = J(\tilde{\psi})$ has a Euclidean symmetry group which is cyclic of order k (consisting of rotations about the origin in \mathbb{C}) and there are k^{th} roots of unity η_1 and η_2 and a polynomial $\gamma \in \bar{\mathbb{Q}}[z]$ such that $\eta_1\tilde{\varphi}(z)$ and $\eta_2\tilde{\psi}(z)$ are both iterates of γ .

Proof. If φ, ψ are defined over a number field k and φ, ψ have infinitely many preperiodic points in common, then the proof of Theorem 1.2 shows that $\mu_{\varphi, v} = \mu_{\psi, v}$ for all places v of k and hence that $\hat{h}_{\varphi} = \hat{h}_{\psi}$. Combined with [KS07, Corollary 25], this proves Theorem 5.7. \square

If we replace the sequence a_n of preperiodic points in the proof of Theorem 1.2 with a sequence of points such that $\lim_{n \rightarrow \infty} \hat{h}_\varphi(a_n) = \lim_{n \rightarrow \infty} \hat{h}_\psi(a_n) = 0$, these observations also prove the following result:

Theorem 5.8. *Let $\varphi, \psi \in \bar{\mathbb{Q}}(T)$ be rational functions of degree at least 2, and assume that the canonical height functions \hat{h}_φ and \hat{h}_ψ on $\mathbb{P}^1(\bar{\mathbb{Q}})$ are distinct. (This will be true, in particular, if the complex Julia sets of φ and ψ are distinct for some embedding of $\bar{\mathbb{Q}}$ into \mathbb{C} .) Then there is a real number $\varepsilon > 0$ such that $\hat{h}_\varphi(a) + \hat{h}_\psi(a) \geq \varepsilon$ for all but finitely many $a \in \bar{\mathbb{Q}}$.*

REFERENCES

- [Art06] E. Artin. *Algebraic numbers and algebraic functions*. AMS Chelsea Publishing, Providence, RI, 2006. Reprint of the 1967 original.
- [Bak08] M. Baker. An introduction to Berkovich analytic spaces and non-Archimedean potential theory on curves. In *p-adic geometry*, volume 45 of *Univ. Lecture Ser.*, pages 123–174. Amer. Math. Soc., Providence, RI, 2008.
- [Bak09] M. Baker. A finiteness theorem for canonical heights attached to rational maps over function fields. *J. Reine Angew. Math.*, 626:205–233, 2009.
- [Bea90] A. F. Beardon. Symmetries of Julia sets. *Bull. London Math. Soc.*, 22(6):576–582, 1990.
- [Bea91] A. F. Beardon. *Iteration of rational functions*, volume 132 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.
- [Ben05] R. L. Benedetto. Heights and preperiodic points of polynomials over function fields. *Int. Math. Res. Not.*, 62:3855–3866, 2005.
- [Ben10] R. L. Benedetto. Nonarchimedean dynamics in dimension one: Lecture notes. preprint. Available at <http://math.arizona.edu/~swc/aws/>, 87 pages, 2010.
- [BG06] E. Bombieri and W. Gubler. *Heights in Diophantine Geometry*, volume 3 of *New Mathematical Monographs*. Cambridge Univ. Press, Cambridge, 2006.
- [BH88] B. Branner and J. H. Hubbard. The iteration of cubic polynomials. I. The global topology of parameter space. *Acta Math.*, 160(3-4):143–206, 1988.
- [BH05] M. Baker and L. C. Hsia. Canonical heights, transfinite diameters, and polynomial dynamics. *J. Reine Angew. Math.*, 585:61–92, 2005.
- [Bil97] Y. Bilu. Limit distribution of small points on algebraic tori. *Duke Math.*, 89:465–476, 1997.
- [BR06] M. Baker and R. Rumely. Equidistribution of small points, rational dynamics, and potential theory. *Ann. Inst. Fourier (Grenoble)*, 56(3):625–688, 2006.
- [BR10] M. Baker and R. Rumely. *Potential theory and dynamics on the Berkovich projective line*, volume 159 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.
- [CCL10] S. Cantat and A. Chambert-Loir. Dynamique p-adique (d’après des exposés de Jean-Christophe Yoccoz). preprint. Available at [arXiv:0912.5413](https://arxiv.org/abs/0912.5413), 47 pages, 2010.
- [CG93] L. Carleson and T. W. Gamelin. *Complex Dynamics*. Springer-Verlag, New York, 1993.
- [CH08a] Z. Chatzidakis and E. Hrushovski. Difference fields and descent in algebraic dynamics. I. *J. Inst. Math. Jussieu*, 7(4):653–686, 2008.
- [CH08b] Z. Chatzidakis and E. Hrushovski. Difference fields and descent in algebraic dynamics. II. *J. Inst. Math. Jussieu*, 7(4):687–704, 2008.
- [CL06] A. Chambert-Loir. Mesures et équidistribution sur les espaces de Berkovich. *J. Reine Angew. Math.*, 595:215–235, 2006.

- [DF08] R. Dujardin and C. Favre. Distribution of rational maps with a preperiodic critical point. *Amer. J. Math.*, 130(4):979–1032, 2008.
- [DH84] A. Douady and J. H. Hubbard. *Étude dynamique des polynômes complexes.*, volume 84 of *Publications Mathématiques d’Orsay*. Université de Paris-Sud, Département de Mathématiques, Orsay, 1984.
- [FJ04] C. Favre and M. Jonsson. *The valuative tree*, volume 1853 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2004.
- [FLM83] A. Freire, A. Lopes, and R. Mañé. An invariant measure for rational maps. *Bol. Soc. Brasil. Mat.*, 14(1):45–62, 1983.
- [FRL04] C. Favre and J. Rivera-Letelier. Théorème d’équidistribution de Brolin en dynamique p -adique. *C. R. Math. Acad. Sci. Paris*, 339(4):271–276, 2004.
- [FRL06] C. Favre and J. Rivera-Letelier. Équidistribution quantitative des points de petite hauteur sur la droite projective. *Math. Ann.*, 335(2):311–361, 2006.
- [KS07] S. Kawaguchi and J. H. Silverman. Dynamics of projective morphisms having identical canonical heights. *Proc. Lond. Math. Soc. (3)*, 95(2):519–544, 2007.
- [Lan83] S. Lang. *Fundamentals of Diophantine geometry*. Springer-Verlag, New York, 1983.
- [LP97] G. Levin and F. Przytycki. When do two rational functions have the same Julia set? *Proc. Amer. Math. Soc.*, 125(7):2179–2190, 1997.
- [Lyu83] M. Lyubich. Entropy properties of rational endomorphisms of the Riemann sphere. *Ergodic Theory Dynamical Systems*, 3(3):351–385, 1983.
- [Mil99] J. Milnor. *Dynamics in one complex variable*. Friedr. Vieweg & Sohn, Braunschweig, 1999. Introductory lectures.
- [Mim97] A. Mimar. *On the preperiodic points of an endomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ which lie on a curve*. PhD thesis, Columbia University, 1997.
- [MZ08] D. Masser and U. Zannier. Torsion anomalous points and families of elliptic curves. *C. R. Math. Acad. Sci. Paris*, 346(9-10):491–494, 2008.
- [MZ09] D. Masser and U. Zannier. Torsion anomalous points and families of elliptic curves. To appear in *Amer. J. Math.*, 2009.
- [Pet08] C. Petsche. S -integral preperiodic points by dynamical systems over number fields. *Bull. Lond. Math. Soc.*, 40(5):749–758, 2008.
- [PST09] C. Petsche, L. Szpiro, and M. Tepper. Isotriviality is equivalent to potential good reduction for endomorphisms of \mathbb{P}^N over function fields. *J. Algebra*, 322(9):3345–3365, 2009.
- [Ran95] T. Ransford. *Potential theory in the complex plane*, volume 28 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1995.
- [RL03] J. Rivera-Letelier. Dynamique des fonctions rationnelles sur des corps locaux. *Astérisque*, (287):xv, 147–230, 2003. Geometric methods in dynamics. II.
- [Rud87] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.
- [Thu05] A. Thuillier. *Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Applications à la théorie d’Arakelov*. PhD thesis, University of Rennes, 2005. Preprint. Available at <http://tel.ccsd.cnrs.fr/documents/archives0/00/01/09/90/index.html>.
- [YZ09] X. Yuan and S. Zhang. Calabi theorem and algebraic dynamics. Preprint, November 2009.

SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA GA 30332-0160,
USA

E-mail address: `demarco@math.uic.edu`

DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE, UNIVERSITY OF ILLI-
NOIS AT CHICAGO, CHICAGO, IL 60607-7045