DYNAMICS OF RATIONAL MAPS: A CURRENT ON THE BIFURCATION LOCUS

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Abstract. Let $f_\lambda : \mathbf{P}^1 \to \mathbf{P}^1$ be a family of rational maps of degree $d > 1$, parametrized holomorphically by λ in a complex manifold X. We show that there exists a canonical closed, positive (1,1)-current T on X supported exactly on the bifurcation locus $B(f) \subset X$. If X is a Stein manifold, then the stable regime $X - B(f)$ is also Stein. In particular, each stable component in the space Poly_d (or Rat_d) of all polynomials (or rational maps) of degree d is a domain of holomorphy.

1. Introduction

It is well-known that for a rational map $f : \mathbf{P}^1 \to \mathbf{P}^1$ of degree $d > 1$, there is a natural f-invariant measure μ_f supported on the Julia set of f [B],[Ly]. This measure can be described as the weak limit of purely atomic measures,

$$\mu_f = \lim_{n\to\infty} \frac{1}{d^n} \sum_{\{z: f^n(z)=a\}} \delta_z,$$

for any $a \in \mathbf{P}^1$ (with at most two exceptions).

There is also a potential-theoretic description of μ_f, defined in terms of a homogeneous polynomial lift $F : \mathbf{C}^2 \to \mathbf{C}^2$ of f. The potential function on $\mathbf{C}^2$ is given by

$$h(z) = \lim_{m\to\infty} \frac{1}{d^m} \log \|F^m(z)\|, \tag{1}$$

and the (1,1)-current $\partial\bar\partial h$ satisfies

$$\pi^*\mu_f = \frac{i}{\pi}\partial\bar\partial h$$

where π is the canonical projection $\mathbf{C}^2 - \{0\} \to \mathbf{P}^1$ [HP]. In particular, when f is a monic polynomial, this definition reduces to

$$\mu_f = \frac{i}{\pi}\partial\bar\partial G = \frac{1}{2\pi}\Delta G \, dx \wedge dy,$$

where $G : \mathbf{C} \to [0, \infty)$ is the Green's function for the complement of the filled Julia set $K(f) = \{z : f^n(z) \not\to \infty \text{ as } n \to \infty\}$.

In this paper, we construct a (1,1)-current on the **parameter space** of a holomorphic family of rational maps, supported exactly on the bifurcation locus (just as μ_f is supported exactly on the Julia set).

Let X be a complex manifold. A **holomorphic family of rational maps f over X** is a holomorphic map $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$. For each parameter $\lambda \in X$, we obtain a rational map $f_\lambda : \mathbf{P}^1 \to \mathbf{P}^1$ with Julia set $J(f_\lambda)$. The **bifurcation locus** $B(f)$ of the family f over X is the set of all $\lambda_0 \in X$ for which $\lambda \mapsto J(f_\lambda)$ is a discontinuous function (in the Hausdorff topology) in any neighborhood of λ_0 (§2).

Theorem 1.1. *Let $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$ be a holomorphic family of rational maps on $\mathbf{P}^1$ of degree $d > 1$. Then there exists a canonical closed, positive (1,1)-current $T(f)$ on X such that the support of $T(f)$ is $B(f)$, the bifurcation locus of f.*

By general properties of positive currents (Lemma 3.3), we have

Corollary 1.2. *If X is a Stein manifold, then $X - B(f)$ is also Stein.*

Let Rat_d and Poly_d denote the "universal families" of all rational maps and of all monic polynomials of degree exactly $d > 1$. We have $\mathrm{Poly}_d \simeq \mathbf{C}^d$ and $\mathrm{Rat}_d \simeq \mathbf{P}^{2d+1} - V$, where V is a resultant hypersurface. In particular, Rat_d and Poly_d are Stein manifolds.

Corollary 1.3. *Every stable component in Rat_d and Poly_d is a domain of holomorphy (i.e. a Stein open subset).*

Corollary 1.3 answers a question posed by McMullen in [M2], motivated by analogies between rational maps and Teichmüller space. Bers and Ehrenpreis showed that finite-dimensional Teichmüller spaces are domains of holomorphy [BE].

Sketch proof of Theorem 1.1. Consider a holomorphic family of homogeneous polynomial maps $\{F_\lambda\}$ on $\mathbf{C}^2$, locally lifting the holomorphic family f over X. Let $\{h_\lambda\}$ be the corresponding potential functions on $\mathbf{C}^2$ defined by equation (1). The function $h_\lambda(z)$ is plurisubharmonic in both $\lambda \in X$ and $z \in \mathbf{C}^2$, and it is pluriharmonic in z away from $\pi^{-1}(J(f_\lambda))$. Suppose for simplicity that we have holomorphic functions $c_j : X \to \mathbf{P}^1$, $j = 1, \ldots, 2d-2$, parametrizing the critical points of f_λ in $\mathbf{P}^1$. We choose lifts $\tilde{c}_j$ from a neighborhood in X to $\mathbf{C}^2$ so that $c_j = \pi \circ \tilde{c}_j$ and define the plurisubharmonic function

$$H(\lambda) = \sum_j h_\lambda(\tilde{c}_j(\lambda)).$$

The desired (1,1)-current on X is defined by

$$T(f) = \frac{i}{\pi}\partial\bar{\partial}H,$$

independent of the choices of $\{F_\lambda\}$ and $\tilde{c}_j$. It is supported on $B(f)$ since H fails to be pluriharmonic exactly when a critical point $c_j(\lambda)$ passes through the Julia set $J(f_\lambda)$. □

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2. Stability

Let $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$ be a holomorphic family of rational maps of degree $d > 1$. The Julia sets of such a family are said to **move holomorphically** at a point $\lambda_0 \in X$ if there is a family of injections $\phi_\lambda : J_{\lambda_0} \to \mathbf{P}^1$, holomorphic in λ near λ_0 with $\phi_{\lambda_0} =$ id, such that $\phi_\lambda(J_{\lambda_0}) = J_\lambda$ and $\phi_\lambda \circ f_{\lambda_0}(z) = f_\lambda \circ \phi_\lambda(z)$. In other words, ϕ_λ provides a conjugacy between f_{λ_0} and f_λ on their Julia sets. The family of rational maps f over X is **stable** at $\lambda_0 \in X$ if any of the following equivalent conditions are satisfied [M1, Theorem 4.2]:

(1) The number of attracting cycles of f_λ is locally constant at λ_0.
(2) The maximum period of an attracting cycle of f_λ is locally bounded at λ_0.
(3) The Julia set moves holomorphically at λ_0.
(4) For all λ sufficiently close to λ_0, every periodic point of f_λ is attracting, repelling, or persistently indifferent.
(5) The Julia set J_λ depends continuously on λ (in the Hausdorff topology) in a neighborhood of λ_0.

Suppose also that each of the $2d - 2$ critical points of f_λ are parametrized by holomorphic functions $c_j : X \to \mathbf{P}^1$. Then the following conditions are equivalent to those above:

(6) For each j, the family of functions $\{\lambda \mapsto f^n_\lambda(c_j(\lambda)\}_{n \geq 0}$ is normal in some neighborhood of λ_0.
(7) For all nearby λ, $c_j(\lambda) \in J_\lambda$ if and only if $c_j(\lambda_0) \in J_{\lambda_0}$.

We let $S(f) \subset X$ denote the set of stable parameters and define the **bifurcation locus** $B(f)$ to be the complement $X - S(f)$. Mañé, Sad, and Sullivan showed that $S(f)$ is open and dense in X [MSS, Theorem A].

Example. In the family $f_c(z) = z^2 + c$, the bifurcation locus is $B(f) = \partial M$, where $M = \{c \in \mathbf{C} : f^n_c(0) \not\to \infty \text{ as } n \to \infty\}$ is the Mandelbrot set [M1, Theorem 4.6].

Lemma 2.1. *If $B(f)$ is contained in a complex hypersurface $D \subset X$, then $B(f)$ is empty.*

Proof. Suppose there exists $\lambda_0 \in B(f)$. By characterization (4) of stability, any neighborhood U of λ_0 must contain a point λ_1 at which the multiplier $m(\lambda)$ of a periodic cycle for f_λ is passing through the unit circle. In other words, the holomorphic function $m(\lambda)$ defined in a neighborhood N of λ_1 is non-constant with $|m(\lambda_1)| = 1$. The set $\{\lambda \in N : |m(\lambda)| = 1\}$ lies in the bifurcation locus and cannot be completely contained in a hypersurface. □

3. Stein manifolds and positive currents

Let X be a paracompact complex manifold and $\mathcal{O}(X)$ its ring of holomorphic functions. Then X is a **Stein manifold** if the following three conditions are satisfied:

- for any $x \in X$ there exists a neighborhood U of x and $f_1, \ldots, f_n \in \mathcal{O}(X)$ defining local coordinates on U;
- for any $x \neq y \in X$, there exists an $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$; and
- for any compact set K in X, the holomorphic hull
$$\hat{K} = \{x \in X : |f(x)| \leq \sup_K |f| \text{ for all } f \in \mathcal{O}(X)\}$$
is also compact in X.

An open domain Ω in X is **locally Stein** if every boundary point $p \in \partial\Omega$ has a neighborhood U such that $U \cap \Omega$ is Stein.

Properties of Stein manifolds. The Stein manifolds are exactly those which can be embedded as closed complex submanifolds of $\mathbf{C}^N$. If Ω is an open domain in $\mathbf{C}^n$ then Ω is Stein if and only if Ω is pseudoconvex if and only if Ω is a domain of holomorphy. An open domain in a Stein manifold is Stein if and only if it is locally Stein. Also, an open domain in complex projective space $\mathbf{P}^n$ is Stein if and only if it is locally Stein and not all of $\mathbf{P}^n$. See, for example, [H] and the survey article by Siu [S].

Examples. (1) $\mathbf{C}^N$ is Stein. (2) The space of all monic polynomials of degree d, $\mathrm{Poly}_d \simeq \mathbf{C}^d$, is Stein. (3) $\mathbf{P}^n - V$ for a hypersurface V is Stein. If V is the zero locus of degree d homogeneous polynomial F and $\{g_j\}$ a basis for the vector space of homogeneous polynomials of degree d, then the map $(g_1/F, \ldots, g_N/F)$ embeds $\mathbf{P}^n - V$ as a closed complex submanifold of $\mathbf{C}^N$. (4) The space Rat_d of all rational maps $f(z) = P(z)/Q(z)$ on $\mathbf{P}^1$ of degree exactly d is Stein. Indeed, parameterizing f by the coefficients of P and Q defines an isomorphism $\mathrm{Rat}_d \simeq \mathbf{P}^{2d+1} - V$, where V is the resultant hypersurface given by the condition $\gcd(P, Q) \neq 1$.

A (p, q)**-current** T on a complex manifold of dimension n is an element of the dual space to smooth $(n-p, n-q)$-forms with compact support. See [HP], [Le], and [GH] for details. The wedge product of a (p, q)-current T with any smooth $(n-p, n-q)$-form α defines a distribution by $(T \wedge \alpha)(f) = T(f\alpha)$ for $f \in C^\infty_c(X)$. Recall that a distribution δ is positive if $\delta(f) \geq 0$ for functions $f \geq 0$. A (p, p)-current is **positive** if for any system of $n - p$ smooth (1,0)-forms with compact support, $\{\alpha_1, \ldots, \alpha_{n-p}\}$, the product
$$T \wedge (i\alpha_1 \wedge \bar{\alpha}_1) \wedge \cdots \wedge (i\alpha_{n-p} \wedge \bar{\alpha}_{n-p})$$
is a positive distribution.

An upper-semicontinuous function h on a complex manifold X is **plurisubharmonic** if $h|\mathbf{D}$ is subharmonic for any complex analytic disk $\mathbf{D}^1$ in X. The current $T = i\partial\bar{\partial}h$ is positive for any plurisubharmonic h, and $T \equiv 0$

if and only if h is pluriharmonic. The "$\partial\bar{\partial}$-Poincaré Lemma" says that any closed, positive (1,1)-current T on a complex manifold is locally of the form $i\partial\bar{\partial}h$ for some plurisubharmonic function h [GH].

The next three Lemmas show that the "region of pluriharmonicity" of a plurisubharmonic function is locally Stein. See [C, Theorem 6.2], [U, Lemma 2.4], [FS, Lemma 5.3], and [R, Theorem II.2.3] for similar statements.

Lemma 3.1. *Suppose h is plurisubharmonic on the open unit polydisk $\mathbf{D}^2$ in $\mathbf{C}^2$ and h is pluriharmonic on the "Hartogs domain"*

$$\Omega_\delta = \{(z,w) : |z| < 1, |w| < \delta\} \cup \{(z,w) : 1-\delta < |z| < 1, |w| < 1\}.$$

Then h is pluriharmonic on $\mathbf{D}^2$.

Proof. Let H be a holomorphic function on Ω_δ such that $h = \operatorname{Re} H$. Any holomorphic function on Ω_δ extends to $\mathbf{D}^2$, and extending H we have $h \leq \operatorname{Re} H$ on $\mathbf{D}^2$ since h is plurisubharmonic. The set $A = \{z \in \mathbf{D}^2 : h = \operatorname{Re} H\}$ is closed by upper-semi-continuity of h. If A has a boundary point $w \in \mathbf{D}^2$, then for any ball $B(w)$ about w, we have

$$\begin{aligned} h(w) &= \operatorname{Re} H(w) \\ &= \frac{1}{|B(w)|}\int_{B(w)} \operatorname{Re} H \\ &> \frac{1}{|B(w)|}\int_{B(w)} h \end{aligned}$$

since $\operatorname{Re} H > h$ on a set of positive measure in $B(w)$. This inequality, however, contradicts the sub-mean-value property of the subharmonic function h. Therefore $A = \mathbf{D}^2$ and h is pluriharmonic on the polydisk. □

Lemma 3.2. *Let X be a complex manifold. If an open subset $\Omega \subset X$ is not locally Stein, there is a $\delta > 0$ and an embedding*

$$e : \mathbf{D}^2 \to X$$

so that $e(\Omega_\delta) \subset \Omega$ but $e(\mathbf{D}^2) \not\subset \Omega$.

Proof. Suppose Ω is not locally Stein at $x \in \partial\Omega$. By choosing local coordinates in a Stein neighborhood U of x in X, we may assume that U is a pseudoconvex domain in $\mathbf{C}^n$. Then $\Omega_0 = U \cap \Omega$ is not pseudoconvex and the function $\phi(z) = -\log d_0(z)$ is not plurisubharmonic near $x \in \partial\Omega$. Here, d_0 is the Euclidean distance function to the boundary of Ω_0.

If ϕ is not plurisubharmonic at the point $z_0 \in U \cap \Omega$, then there is a one-dimensional disk $\alpha : \mathbf{D}^1 \to \Omega$ centered at z_0 such that $\int_{\partial\mathbf{D}^1} \phi < \phi(z_0)$ (identifying the disk with its image $\alpha(\mathbf{D}^1)$). Let ψ be a harmonic function on $\mathbf{D}^1$ so that $\psi = \phi$ on $\partial\mathbf{D}^1$. Then $\psi(z_0) < \phi(z_0)$. Let Ψ be a holomorphic function on $\mathbf{D}^1$ with $\psi = \operatorname{Re}\Psi$.

Now, let $p \in \partial\Omega$ be such that $d_0(z_0) = |z_0 - p|$. Let $e : \mathbf{D}^2 \to U$ be given by

$$e(z_1, z_2) = \alpha(z_1) + z_2(1-\varepsilon)e^{-\Psi(z_1)}(p - z_0).$$

That is, the two-dimensional polydisk is embedded so that at each point $z_1 \in \mathbf{D}^1$ there is a disk of radius $|(1-\varepsilon)\exp(-\Psi(z_1))|$ in the direction of $p - z_0$. If ε is small enough we have a Hartogs-type subset of the polydisk contained in Ω but the polydisk is not contained in Ω since $d_0(z_0, \partial\Omega) = \exp(-\phi(z_0)) < \exp(-\psi(z_0))$. ☐

Lemma 3.3. *Let T be a closed, positive (1,1)-current on a complex manifold X. Then $\Omega = X - \operatorname{supp}(T)$ is locally Stein.*

Proof. Let p be a boundary point of Ω. Choose a Stein neighborhood U of p in X so that $T = i\partial\bar{\partial}h$ for some plurisubharmonic function h on U. By definition of Ω, h is pluriharmonic on $U \cap \Omega$.

If Ω is not locally Stein at p, then by Lemma 3.2, we can embed a two-dimensional polydisk into U so that a Hartogs-type domain Ω_δ lies in Ω, but the polydisk is not contained in Ω. By Lemma 3.1, h must be pluriharmonic on the whole polydisk, contradicting the definition of Ω. ☐

Corollary 3.4. *If X is Stein, then so is $X - \operatorname{supp} T$.*

Example. If X is a Stein manifold and V a hypersurface, then $V = \operatorname{supp} T$ for a positive (1,1)-current T given locally by $T = \frac{i}{\pi}\partial\bar{\partial}\log|f|$, where V is the zero set of f. Lemma 3.3 shows that $X - V$ is locally Stein, and thus Stein. Similarly, $\mathbf{P}^n - V$ is Stein for any hypersurface V.

4. The potential function of a rational map

Let $f : \mathbf{P}^n \to \mathbf{P}^n$ be a holomorphic map. Let $F : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ be a lift of f to a homogeneous polynomial, unique up to scalar multiple, so that $\pi \circ F = f \circ \pi$ where π is the projection $\mathbf{C}^{n+1}\backslash 0 \to \mathbf{P}^n$. Let d be the degree of the components of F; then f has topological degree d^n.

Assume that $d > 1$. Following [HP], we define the **potential function** of F by

$$h_F(z) = \lim_{m\to\infty} \frac{1}{d^m} \log \|F^m(z)\|.$$

The limit converges uniformly on compact subsets of $\mathbf{C}^{n+1} - 0$, and $h_F(z)$ is plurisubharmonic on $\mathbf{C}^{n+1}$ since $\log\|\cdot\|$ is plurisubharmonic. Let $\Omega_F \subset \mathbf{C}^{n+1}$ be the basin of attraction of the origin for F; that is,

$$\Omega_F = \{x \in \mathbf{C}^{n+1} : F^m(x) \to 0 \text{ as } m \to \infty\}.$$

Note that Ω_F is open and bounded.

From the definition, we obtain the following properties of the potential function h_F [HP]:

1. $h_F(\alpha z) = h_F(z) + \log|\alpha|$ for $\alpha \in \mathbf{C}^*$;
2. $\Omega_F = \{z : h_F(z) < 0\}$; and
3. h_F is independent of the choice of norm $\|\cdot\|$ on $\mathbf{C}^{n+1}$.

Theorem 4.1. (Hubbard-Papadopol, Ueda, Fornaess-Sibony) *The support of the positive (1,1)-current*

$$\omega_f = \frac{i}{\pi}\partial\bar{\partial}h_F$$

on $\mathbf{C}^{n+1} - 0$ *is equal to the preimage of the Julia set* $\pi^{-1}(J(f))$*. If* $n = 1$*, then the Brolin-Lyubich measure* μ_f *satisfies* $\pi^*\mu_f = \omega_f$*.*

Proof. See [HP, Theorem 4.1] for $n = 1$ and [U, Theorem 2.2], [FS, Theorem 2.12] for $n > 1$. □

From Corollary 3.4, we obtain the following ([U, Theorem 2.3], [FS, Theorem 5.2]):

Corollary 4.2. (Ueda, Fornaess-Sibony) *The Fatou components of* $f : \mathbf{P}^n \to \mathbf{P}^n$ *are Stein.*

5. The bifurcation current

In this section we complete the proof of Theorem 1.1. Let $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$ be a holomorphic family of rational maps on $\mathbf{P}^1$ of degree $d > 1$. Let $\{F_\lambda\}$ be a holomorphic family of homogeneous polynomials on $\mathbf{C}^2$, locally lifting the family f, and let h_λ denote the potential function of F_λ (§4). The potential function $h_\lambda(z)$ is plurisubharmonic as a function of the pair (λ, z).

Fix $\lambda_0 \in X$. In a neighborhood U of λ_0, we can choose coordinates on $\mathbf{P}^1$ so that ∞ is not a critical point of f_λ, $\lambda \in U$. For $z \in \mathbf{P}^1 - \{\infty\}$, let $\tilde{z} = (z, 1) \in \mathbf{C}^2$. Define a function H on U by

$$H(\lambda) = \sum_{\{c: f'_\lambda(c)=0\}} h_\lambda(\tilde{c}),$$

where the critical points are counted with multiplicity. Now, let $N(\lambda)$ be the number of critical points of the rational map f_λ (counted without multiplicity). Let

$$D(f) = \{\lambda_0 \in X : N(\lambda) \text{ does not have a local maximum at } \lambda = \lambda_0\}.$$

Then $D(f)$ is a complex hypersurface in X, since it is defined by the vanishing of a discriminant. If $\lambda_0 \notin D(f)$, there exists a neighborhood U of λ_0 and holomorphic functions $c_j : U \to \mathbf{P}^1$, $j = 1, \ldots, 2d-2$, parametrizing the critical points of f_λ, such that $\infty \notin c_j(U)$ for all j. In this case, we can express H as the sum

$$H(\lambda) = \sum_j H_j(\lambda)$$

of the plurisubharmonic functions

$$\begin{aligned} H_j(\lambda) &= h_\lambda \circ \tilde{c}_j(\lambda) \\ &= \lim_{m\to\infty} \frac{1}{d^m} \log \|F^m_\lambda(\tilde{c}_j(\lambda))\|. \end{aligned}$$

For any $\lambda_0 \in X$, then, H is defined and continuous in a neighborhood U of λ_0 and plurisubharmonic on $U - D(f)$; therefore H is plurisubharmonic on U.

The **bifurcation current** T is the positive (1,1)-current on parameter space X given locally by

$$T = \frac{i}{\pi}\partial\bar{\partial}H.$$

The next Lemma shows that T is globally well-defined on X.

Lemma 5.1. *The current $T = \frac{i}{\pi}\partial\bar{\partial}H$ is independent of (a) the choice of lifts $\tilde{c}_j$ of c_j and (b) the choice of lifts F_λ of f_λ.*

Proof. Suppose we define a new lift $\hat{c}_j(\lambda) = t(\lambda) \cdot \tilde{c}(\lambda)$ for some holomorphic function t taking values in $\mathbf{C}^*$. Property (1) of the potential function h_λ (§4) implies that $h_\lambda(\hat{c}(\lambda)) = h_\lambda(\tilde{c}(\lambda)) + \log|t(\lambda)|$ and $i\partial\bar{\partial}H$ is unchanged since $\log|t(\lambda)|$ is pluriharmonic, proving (a). If the lifted family $\{F_\lambda\}$ is similarly replaced by $\{t(\lambda) \cdot F_\lambda\}$, a computation shows that h_λ is changed only by the addition of the pluriharmonic term $\frac{1}{d-1}\log|t(\lambda)|$ where d is the degree of the f_λ. This proves (b). □

Lemma 5.2. *A parameter λ_0 lies in the stable regime $S(f) \subset X$ if and only if the function H is pluriharmonic in a neighborhood of λ_0.*

Proof. Let us first suppose that $\lambda_0 \in S(f)$ is not in $D(f)$ (in the notation above). By characterization (6) of stability (§2), for each j, the family of functions $\{\lambda \mapsto f_\lambda^m(c_j(\lambda))\}$ is normal in a neighborhood V of λ_0; hence, there exists a subsequence converging uniformly on compact subsets to a holomorphic function $g_j(\lambda)$. As in [HP, Prop 5.4], we can shrink our neighborhood V if necessary to find a norm $\|\cdot\|$ on $\mathbf{C}^2$ so that $\log\|\cdot\|$ is pluriharmonic on $\pi^{-1}(g_j(V))$; e.g., if $g_j(V)$ is disjoint from $\{|x| = |y|\}$, we can choose norm $\|(x, y)\| = \max\{|x|, |y|\}$. Then, on any compact set in V, the functions

$$\lambda \mapsto \frac{1}{d^{m_k}} \log\|F_\lambda^{m_k}(\tilde{c}_j(\lambda))\|$$

are pluriharmonic if k is large enough. By property (3) of the potential function h_λ (§4), this subsequence converges uniformly to H_j. Therefore, H is pluriharmonic on V.

If λ_0 lies in $D(f) \cap S(f)$, then H is defined and continuous on a neighborhood V of λ_0 and pluriharmonic on $V - D(f)$. As $D(f)$ has codimension 1, H must be pluriharmonic on all of V.

For the converse, let us suppose again that $\lambda_0 \notin D(f)$ and that H is pluriharmonic in a neighborhood of λ_0. Each H_j is pluriharmonic and so we may write $H_j = \operatorname{Re} G_j$ in a neighborhood V of λ_0. In analogy with [U, Prop. 2.1], we define new lifts $\hat{c}_j(\lambda) = e^{-G_j(\lambda)} \cdot \tilde{c}(\lambda)$ of the c_j and compute

$$\begin{aligned} h_\lambda(\hat{c}_j(\lambda)) &= h_\lambda(\tilde{c}(\lambda)) + \log|e^{-G_j(\lambda)}| \\ &= h_\lambda(\tilde{c}(\lambda)) - \operatorname{Re} G_j \\ &= H_j - H_j \\ &= 0. \end{aligned}$$

By property (2) of h_λ, this implies that $\hat{c}_j(\lambda)$ lies in $\partial\Omega_\lambda$ for all $\lambda \in V$. If V is small enough, the set $\cup_{\lambda\in V}(\{\lambda\} \times \partial\Omega_\lambda)$ has compact closure in $X \times \mathbf{C}^2$. As the functions F_λ preserve $\partial\Omega_\lambda$, the family $\{\lambda \mapsto F_\lambda^n(\hat{c}_j(\lambda))\}$ is uniformly bounded and thus normal. Of course, $f_\lambda^n \circ c_j = \pi \circ F_\lambda^n \circ \hat{c}_j$ demonstrating that λ_0 is a stable parameter by (6) of Section 2.

Finally suppose that H is pluriharmonic in a neighborhood U of parameter $\lambda_0 \in D(f)$. Then $U - D(f)$ lies in the stable regime and Lemma 2.1 shows that all of U must belong to $S(f)$. □

Proof of Theorem 1.1. Let T be the bifurcation current defined above for the family of rational maps f over X. By Lemma 5.2, the support of T is the bifurcation locus $B(f)$. □

Corollaries 1.2 and 1.3 now follow immediately from Corollary 3.4.

6. Examples

Example 6.1. In the family $\{f_\lambda(z) = z^d + \lambda\}$, $\lambda \in \mathbf{C}$, the bifurcation current T takes the form

$$T = \frac{d-1}{d}\left(\frac{i}{\pi}\partial\bar{\partial}G\right)$$

where G is the Green's function for the complement of the "degree d Mandelbrot set" $M_d = \{\lambda : f_\lambda^n(0) \not\to \infty \text{ as } n \to \infty\}$. That is, T is a multiple of harmonic measure supported on ∂M_d. The T-mass of ∂M_d is $(d-1)/d$.

Proof. If G_λ denotes the Green's function for the complement of the filled Julia set $K(f_\lambda) = \{z : f_\lambda^n(z) \not\to \infty \text{ as } n \to \infty\}$, then $G(\lambda) = G_\lambda(\lambda)$ (see e.g. [CG, VIII.4]). By [HP, Prop 8.1], we have

$$h_\lambda(x, y) = G_\lambda(x/y) + \log|y|$$

where (x, y), $y \neq 0$, is a point of $\mathbf{C}^2$. Note that $d-1$ of the critical points of f_λ are at $z = 0$ and the other $d - 1$ are at $z = \infty$. Computing, we find

$$\begin{aligned} T &= \frac{i}{\pi}\sum_j \partial\bar{\partial} h_\lambda(\tilde{c}_j(\lambda)) \\ &= (d-1)\frac{i}{\pi}\partial\bar{\partial} h_\lambda(0,1) \\ &= (d-1)\frac{i}{\pi}\partial\bar{\partial} G_\lambda(0) \\ &= \frac{d-1}{d}\frac{i}{\pi}\partial\bar{\partial} G_\lambda(\lambda). \end{aligned}$$

□

Example 6.2. Let f be a polynomial of degree d, μ_f the Brolin-Lyubich measure on the Julia set, and G_f the Green's function for the complement of the filled Julia set (defined in the previous example). The Lyapunov exponent of f satisfies ([Prz],[Mn])

$$L(f) = \log d + \sum_{\{c\in\mathbf{C}: f'(c)=0\}} G_f(c).$$

If $\{f_\lambda\}$ is any holomorphic family of polynomials, the Lyapunov exponent as a function of the parameter is a potential function for the bifurcation current; that is,

$$T = \frac{i}{\pi}\partial\bar{\partial} L.$$

In the sequel, we examine further the connection between the bifurcation current and the Lyapunov exponent.

References

[BE] Bers and Ehrenpreis. Holomorphic Convexity of Teichmuller Space. *Bull. AMS* **70**(1964), 761–764.

[B] H. Brolin. Invariant sets under iteration of rational functions. *Ark. Mat.* **6**(1965), 103–144.

[CG] L. Carleson and T. W. Gamelin. *Complex dynamics.* Springer-Verlag, New York, 1993.

[C] U. Cegrell. Plurisubharmonic functions. *Proc. London Math. Soc.* **36**(1978), 310–336.

[FS] J. E. Fornaess and N. Sibony. Complex dynamics in higher dimension. II. In *Modern methods in complex analysis (Princeton, NJ, 1992)*, pages 135–182. Princeton Univ. Press, Princeton, NJ, 1995.

[GH] P. Griffiths and J. Harris. *Principles of algebraic geometry.* John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.

[H] L. Hörmander. *An introduction to complex analysis in several variables.* North-Holland Publishing Co., Amsterdam, 1990.

[HP] J. Hubbard and P. Papadopol. Superattractive fixed points in $\mathbf{C}^n$. *Indiana Univ. Math. J.* **43**(1994), 321–365.

[Le] P. Lelong. *Fonctions plurisousharmoniques et formes différentielles positives.* Gordon & Breach, Paris, 1968.

[Ly] M. Lyubich. Entropy properties of rational endomorphisms of the Riemann sphere. *Ergodic Theory Dynamical Systems* **3**(1983), 351–385.

[MSS] R. Mañé, P. Sad, and D. Sullivan. On the dynamics of rational maps. *Ann. Sci. Ec. Norm. Sup.* **16**(1983), 193–217.

[Mn] Ricardo Mañé. The Hausdorff dimension of invariant probabilities of rational maps. In *Dynamical systems, Valparaiso 1986*, pages 86–117. Springer, Berlin, 1988.

[M1] C. McMullen. *Complex dynamics and renormalization.* Princeton University Press, Princeton, NJ, 1994.

[M2] C. McMullen. Rational maps and Teichmüller space. In *Lecture Notes in Mathematics*, volume 1574, pages 430–433. Springer-Verlag, 1994.

[Prz] Feliks Przytycki. Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map. *Invent. Math.* **80**(1985), 161–179.

[R] R.M. Range. *Holomorphic functions and integral representations in several complex variables.* Springer-Verlag, New York, 1986.

[S] Y.-T. Siu. Pseudoconvexity and the Problem of Levi. *Bulletin of the AMS* **84**(1978), 481–512.

[U] T. Ueda. Fatou sets in complex dynamics on projective spaces. *J. Math. Soc. Japan* **46**(1994), 545–555.