1. Classification of 1-manifolds

**Theorem 1.1.** Let $M$ be a connected 1–manifold. Then $M$ is diffeomorphic either to $[0, 1]$, $[0, 1)$, $(0, 1)$, or $S^1$.

We know that none of these four manifolds are not diffeomorphic to each other, since the number of points in $\partial M$ is a diffeomorphism invariant, and compactness distinguishes $S^1$ from $(0, 1)$. The intuition behind the proof is this. Start at a point in $M$, you know its neighborhood is diffeomorphic to $(-\varepsilon, \varepsilon)$. So go to the right as far as you can. Only three things can happen: either you hit a closed endpoint, you go forever, which looks like an open endpoint, or you end up coming back to where you started from the left side. In this latter case you must be diffeomorphic to $S^1$. Otherwise, go to the left and far as you can, and you will hit either a closed or an open endpoint in that direction.

Before we prove this more formally, first we prove an application.

**Corollary 1.2.** Let $M$ be a compact 1–manifold. Then $\partial M$ consists of an even number of points.

**Corollary 1.3.** Let $M$ be a compact manifold with boundary. Then there does not exist a smooth map $f : M \to \partial M$ with the property that $f|_{\partial M} = \text{id}$.

**Proof.** Suppose $f : M \to \partial M$ is such a map, and let $y \in \partial M$ be a regular value, which exists following Sard’s theorem. Then $f^{-1}(y) \subseteq M$ is a compact 1–manifold, and $\partial(f^{-1}(y)) = f^{-1}(y) \cap \partial M = \{y\}$. Since this is one point, it contradicts Corollary 1.2. □

**Corollary 1.4.** Let $f : B^n \to B^n$ be a smooth map from the closed ball $B^n$ to itself. Then $f$ has a fixed point.

**Proof.** Suppose not. Then define a map $g : B^n \to S^{n-1}$ as follows. To compute $g(x)$, draw a ray from $f(x)$ which passes through $x$, and let $g(x)$ be the point where this ray intersects $\partial B^n$. Then $g$ is a smooth map, and it is the identity on $\partial B^n$. □

We will prove a number of other applications following from Corollary 1.2. First we return to the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Fix $M \subseteq \mathbb{R}^n$. A parametrization by arclength of $M$ is a map $f : I \to M$, where $I \subseteq \mathbb{R}$ is an interval, $f$ is a diffeomorphism onto its image, $f(I) \subseteq M$ is open, and $Df(x) \in \mathbb{R}^n$ is a vector of unit length (here $x$ is the coordinate on $I$). (In particular, $I$ may have a closed endpoint only if that closed endpoint is mapped to a point in $\partial M$.)

**Lemma 1.5.** Let $f : I \to M$, $g : J \to M$ be two parametrizations by arclength. Then $f(I) \cap g(J)$ has at most two connected components. If it has two components then $M$ is diffeomorphic to $S^1$. If it has one component then there is a parametrization by arclength $h : K \to M$ with $h(K) = f(I) \cup g(J)$.

**Proof.** Consider the set $Q = f^{-1}(g(J)) \subseteq I$. Since $g(J) \subseteq M$ is an open set, $Q$ is an open subset of $I$, and therefore $Q$ is a disjoint collection of intervals. Let $x \in I$ be a point in the point-set boundary of $Q$, so there is a sequence $x_i \to x$ with $x_i \in Q$ but $x \notin Q$.

Consider the sequence $(g^{-1} \circ f)(x_i) \in J$. $g^{-1} \circ f$ is a partially defined map from $\mathbb{R}$ to $\mathbb{R}$ with derivative $\pm 1$ everywhere, the sequence $(g^{-1} \circ f)(x_i) \in J$ is Cauchy,
so it must converge to a point \( y \in \mathbb{R} \). But \( y \) cannot be in \( J \), because if it were then
\[ g(y) = \lim f(x_i) = f(x) \in M, \]
contradicting the fact that \( x \notin Q \).

Thus, every point in the point-set boundary of \( Q \subseteq I \) is associated to an endpoint of \( J \). This association must be injective, since \( g^{-1} \circ f : Q \to J \) is a diffeomorphism onto its image. Therefore, \( Q \) has at most two points in its boundary. Together with the endpoints of \( I \), we see that \( Q \subseteq \mathbb{R} \) has at most four points in its boundary. So, \( Q \) consists of no more than two intervals.

\[ g^{-1} \circ f : Q \to \mathbb{R} \]

is a function with derivative \( \pm 1 \) everywhere. Thus if \( Q \) is one interval, \( g^{-1} \circ f \) extends to a diffeomorphism \( L : \mathbb{R} \to \mathbb{R} \) so that \( L' = \pm 1 \) everywhere. In this case, \( L(Q) = L(I) \cap J \). Let \( K = L(I) \cup J \), and define \( h : K \to M \) by

\[
h(x) = \begin{cases} 
g(x) & \text{if } x \in J \\
(f \circ L^{-1})(x) & \text{if } x \in L(I),
\end{cases}
\]

and without much trouble we check that \( h \) is well-defined and injective.

Suppose now that \( Q \) has two components. We claim that in this case \( I \) and \( J \) are both open intervals. Indeed, \( Q \) is open as a subset of \( I \), and \( (g^{-1} \circ f)(Q) \) is an open subset of \( J \). Suppose instead that \( x_0 \in I \) is a closed endpoint. Since \( Q \) has four endpoints which are in bijective correspondence with the four endpoints of \( I \) and \( J \), \( x_0 \in Q \) also. But \( (g^{-1} \circ f) : Q \to J \) sends \( Q \) diffeomorphically onto its image, and \( (g^{-1} \circ f)(x_0) \) must be an interior point of \( J \) (since the endpoints of \( Q \) are in bijective correspondence with those of \( I \) and \( J \)). This contradicts the fact that \( (g^{-1} \circ f)(Q) \) is open in \( J \).

We now have an explicit picture: we can write \( I = (a, b), Q = (a, a+\varepsilon) \cup (b-\delta, b), J = (c, d), \) and either \( (g^{-1} \circ f)(Q) = (c, c+\delta) \cup (d-\varepsilon, d) \) or \( (g^{-1} \circ f)(Q) = (c, c+\varepsilon) \cup (d-\delta, d) \) depending on whether \( (g^{-1} \circ f)' \) is \( +1 \) or \( -1 \), respectively. Writing \( S^1 = \mathbb{R}/\mathbb{Z} \), with the constant \( l = b - a + d - c - \varepsilon - \delta \), define a function \( h : S^1 \to M \) by

\[
h(\theta) = \begin{cases} 
f(\theta + a) & \text{if } 0 < \theta < b - a \\
g(\theta + a - b + \delta + c) & \text{if } b - a - \delta < \theta < l + \varepsilon + \delta,
\end{cases}
\]

in the case where \( (g^{-1} \circ f)' \) is \( +1 \). In the case where \( (g^{-1} \circ f)' = -1 \), replace \( J \) by \( \overline{J} = -J \) and \( g \) by \( \overline{g}(x) = g(-x) \), this switches the cases. We easily check that \( h \) is a diffeomorphism onto its image. \( h(S^1) \subseteq M \) is open because it is a diffeomorphism onto its image, and it is closed since \( S^1 \) is compact. Since \( M \) is connected by assumption, \( h \) is a diffeomorphism.

\[ \square \]

We now complete the proof of the theorem. Firstly, if \( f : \mathbb{R} \to M \) is a parametrization by arclength, then the image of \( f \) must be closed, and so \( f \) is surjective. This then implies that we can find a maximal parametrization by arclength, say by using Zorn’s Lemma. Call this maximal parametrization \( f : I \to M \). If \( f \) is not surjective, then choose a point \( p \in M \setminus f(I) \) which is in the closure of \( f(I) \). Take a chart around \( p \), and reparametrize these coordinates be a parametrization by arclength \( g : J \to M \). Then \( g(J) \) must intersect \( f(I) \), since \( g(J) \) is an open set containing a limit point of \( f(I) \). If their intersection had only one component, we could extend it to a larger parametrization by arclength, contradicting maximality. Therefore they must intersect in two components, providing a diffeomorphism with \( S^1 \).

\[ \square \]
2. Mod-2 degree of a map

Let $f_0, f_1 : M \to N$ be two smooth maps of manifolds, where here $\partial M \neq \emptyset$. We say that $f_0$ is homotopic to $f_1$ if there is a smooth map $F : M \times [0, 1] \to N$, so that $F(\cdot, 0) = f_0$ and $F(\cdot, 1) = f_1$.

Now, let $f : M \to N$ be a map of closed (i.e. compact, boundaryless) manifolds, where dim $M = \dim N$. Let $y \in N$ be a regular value of $f$. Then $f^{-1}(y)$ is a compact 0–manifold, that is, a finite number of points. Define $\deg_2^n(f) \in \mathbb{Z}_2$ to be the number of points in $f^{-1}(y)$, modulo 2.

**Proposition/Definition 2.1.** Let $f : M \to N$ be a map of equidimensional closed manifolds, and let $y_0$ and $y_1$ be two regular points of $f$. Then $\deg_2^{y_0}(f) = \deg_2^{y_1}(f)$. We therefore denote this value by $\deg_2(f)$, called the mod-2 degree of $f$.

If $f_0$ and $f_1$ are two homotopic maps $M \to N$, then $\deg_2(f_0) = \deg_2(f_1)$.

**Proof.** Let $\varphi_t : N \to N$ be an isotopy so that $\varphi_t(y_0) = y_1$. And define $F : M \times [0, 1] \to N$ by $F(x, t) = (\varphi_t \circ f)(x)$. First, suppose that $y_0$ is a regular value of $F$. Then $F^{-1}(y)$ is a compact 1-manifold with boundary, and $\partial(F^{-1}(y)) = F^{-1}(y) \cap \partial(M \times [0, 1]) = (f^{-1}(y_0) \times \{0\}) \cup (f^{-1}(\varphi^{-1}_t)(y_0) \times \{1\})$. Thus, the total count of points in $\partial(F^{-1}(y))$ is equal to $\deg_2^{y_0}(f) + \deg_2^{y_1}(f)$ modulo 2, which is an even number since it is counting the number of points in the boundary of a compact 1–manifold.

Now, if $y_0$ is not a regular value for $F$, Sard’s theorem implies that there is a regular value in any neighborhood of $y_0$. On the other hand, the set of regular values is open, since $M \times [0, 1]$ is compact and the set of critical points is always a closed set. Since $\deg_2$ is locally constant among regular values of $f$, it suffices to prove the result for a point near $y_0$.

To prove that homotopic maps have the same degree, use the exact proof as above, now letting $F : M \times [0, 1] \to N$ be the homotopy between $f_0$ and $f_1$.

**Corollary 2.2.** Let $M$ be any closed manifold. Then the identity map on $M$ is not homotopic to a constant map.

**Proof.** $\deg_2(id) = 1$, but $\deg_2(constant) = 0$.

**Corollary 2.3.** $S^2$ is not diffeomorphic to $T^2 = S^1 \times S^1$.

**Proof.** Recall that any map $f : S^1 \to S^2$ is homotopic to a constant map. Let $g : S^1 \to T^2$ be the map $g(\theta) = (\theta, 1)$. To show that $S^2$ is not diffeomorphic to $T^2$ it suffices to show that $g$ is not homotopic to a constant map.

To do this, let $\pi : T^2 \to S^1$ be the projection $\pi(\theta, \varphi) = \theta$. If $g$ were homotopic to a constant, then $\pi \circ g : S^1 \to S^1$ would be homotopic to a constant as well. Since $\pi \circ g = \text{id}$, the previous corollary shows this is impossible.

3. Transversality

Recall that, if $f : M \to Q$ is a smooth map of manifolds and $N \subseteq Q$ is a submanifold, we say that $f$ is transverse to $N$, written $f \pitchfork N$, if $Df_x(TM_x) + TN_{f(x)} = TQ_{f(x)}$ at every point $x \in f^{-1}(Q)$. Here, we assume that $\partial N = \emptyset$, and if $\partial M \neq \emptyset$ then we also require that $f|_{\partial M} \pitchfork N$ to say that $f \pitchfork N$. Notice that $f \pitchfork \{y\}$ just means that $y$ is a regular value of $f$. With this in mind, we note that whenever $f \pitchfork N$, it follows that $f^{-1}(N) \subseteq M$ is a submanifold, with
\[\partial(f^{-1}(N)) = f^{-1}(N) \cap \partial M.\] This follows immediately from the implicit function theorem.

We prove two complimentary theorems. Together they say that any smooth map can be made transverse to any submanifold by a small perturbation.

**Proposition 3.1.** Let \( f : M \to Q \) be a smooth map. Then for some \( k \), there is a map \( F : M \times B^k \to Q \), so that \( DF \) is surjective everywhere, and \( F(\cdot, 0) = f \).

**Proof.** We assume that \( M \) is compact, otherwise we leave the proof as an exercise to the reader. For a given \( x \in M \), choose vector fields \( V_1, \ldots, V_n \) on \( Q \) so that \( TM_x + \text{span}\{V_1(x), \ldots, V_n(x)\} = TQ_{f(x)}. \) This condition is open in \( x \), meaning that the set \( x \) where it is satisfied defines an open set \( U_x \).

\( \{U_x\} \) is an open covering of a compact manifold, so it has some finite subcovering \( \{U_i\} \). Let \( W_1, \ldots, W_k \) be the collection of all the vector fields constructed as above, for all the sets \( U_i \) in our finite subcovering. Then \( TM_x + \text{span}\{W_1(x), \ldots, W_n(x)\} = TQ_{\tilde{f}(x)} \) at all points \( x \in M \). Since \( M \) is compact, we can assume that each \( W_j \) is compactly supported, by multiplying by a cutoff function if necessary. Let \( \varphi^j_i : Q \times \mathbb{R} \to Q \) be the flow of \( W_j \).

Let \( F : M \times \mathbb{R}^k \to Q \) be the map \( F(x, t_1, \ldots, t_k) = (\varphi^1_{t_1} \circ \cdots \circ \varphi^k_{t_k} \circ f)(x) \). Then \( \text{Im}(DF_{x,0,\ldots,0}) = TM_x + \text{span}\{W_1(x), \ldots, W_n(x)\} = TQ_{f(x)}. \) Therefore at each point \( x \in M \), there is a positive radius \( r_x \) so that \( DF_{x,t} \) is surjective for \( t \in B^k(r_x) \). Letting \( r \) be the infimum of these, we have defined \( F : M \times B^k \to Q \) which is a submersion everywhere.

**Proposition 3.2.** Let \( F : M \times S \to Q \) be a smooth map, and \( N \subseteq Q \) a submanifold. For every \( s \in S \) let \( f_s : M \to Q \) be the map \( f_s = F(\cdot, s) \). Suppose \( F \cap N \). Then the set of \( s \in S \) so that \( f_s \cap N \) is strongly dense, in the sense of Sard (i.e. it has full measure, or is the countable intersection of open, dense sets).

**Proof.** Let \( W = F^{-1}(N) \). Then \( W \) is a submanifold of \( M \times S \), since \( F \cap N \). Let \( \pi : M \times S \to S \) be the projection. We claim that, if \( s \) is a regular value of \( \pi|_W : W \to S \), then \( f_s \cap N \). This claim proves the proposition.

Suppose \( s \in S \) is a regular value, then \( D\pi(x,s)(TW(x,s)) = TS_s \), for all \( x \) so that \( (x, s) \in W \). Equivalently, this means that \( TW(x,s) + TM(x,s) = T(M \times S)(x,s) \), and so \( DF(x,s)(TM(x,s)) + TF(x,s)(TW(x,s)) = TQ_{f(x,s)} \). Since \( F \cap N \), we have that \( DF(x,s)(TM(x,s)) + TF(x,s)(TW(x,s)) = TQ_{f(x,s)} \), and so \( DF(x,s)(TM(x,s)) + TF(x,s)(TW(x,s)) + TN_{F(x,s)} = TQ_{f(x,s)}. \) But \( DF(x,s)(TW(x,s)) \subseteq TN_{F(x,s)} \) and so \( DF(x,s)(TW(x,s)) + TN_{F(x,s)} = TQ_{f(x,s)} \) and therefore \( DF(x,s)(TM(x,s)) + TN_{F(x,s)} = TQ_{f(x,s)} \). Since \( DF(x,s)(TM(x,s)) = Df_s \), this proves the result.

**Definition 3.3.** Suppose that \( M \) and \( N \) are closed manifolds, which are submanifolds of \( Q \), satisfying \( \dim M + \dim N = \dim Q \). Suppose \( M \cap N \). Then \( M \cap N \) is a finite number of points. Let \( \text{Int}_2(M,N) \subseteq \mathbb{Z}_2 \) denote the number of points in \( M \cap N \), modulo 2.

**Proposition 3.4.** Suppose that \( \tilde{M} \) is homotopic to \( \tilde{M} \), so that \( M \) and \( \tilde{M} \) are both transverse to \( N \). Then \( \text{Int}_2(M,N) = \text{Int}_2(\tilde{M},N) \).

\(^1\)We say that two manifolds are transverse if the inclusion map of one is transverse to the other.
Proof. Let $F : M \times [0,1] \to Q$ be a homotopy between the inclusion map $F(\cdot,0)$ and $F(M,1) = \tilde{M}$. First, suppose that $F \pitchfork N$. Then $F^{-1}(N)$ is a 1–manifold, and $\partial F^{-1}(N) = ((M \cap N) \times \{0\}) \cup ((\tilde{M} \cap N) \times \{1\})$. Since the number of points on the left is even and the number of points on the right counts $\text{Int}_2(M,N) - \text{Int}_2(\tilde{M},N)$, the result follows.

If $F$ is not transverse to $N$, use Propositions 3.1 and 3.2 above to find a map $F_s : M \times [0,1] \to Q$ which is $C^\infty$–close to $F$ and $F_s \pitchfork N$. Then $F_s(M,0)$ is $C^\infty$–close to $F(M,0) = M$, and so $\text{Int}_2(F_s(M,0),N) = \text{Int}_2(M,N)$. Then we apply the previous argument to $F_s$. □

Because of all the above propositions, we can define $\text{Int}_2(M,N)$ for any two submanifolds without a transversality assumption: Proposition 3.1 says the inclusion map $M \to Q$ fits into a family $F : M \times B^k \to Q$ which is transverse to $N$, Proposition 3.2 says that for most $s \in B^k$ these maps will be transverse to $N$, and Proposition 3.4 says that it doesn’t matter which of these maps we choose since they’re all homotopic.

**Proposition 3.5.** Let $f : M \to Q$ be an embedding of a manifold, and let $N$ be a submanifold. Assume $M$ and $N$ are closed, $\dim M + \dim N = \dim Q$, and $f \pitchfork N$. Let $P$ be a compact manifold satisfying $\partial P = M$. If $f$ extends to a smooth map $F : P \to Q$, then $\text{Int}_2(f(M),N) = 0$.

Proof. We can assume that $F \pitchfork N$. Then $F^{-1}(N)$ is a 1–manifold, and its boundary is $f(M) \cap N$. □

Let $\mathbb{CP}^2$ denote the space of all complex lines in $\mathbb{C}^3$, one easily shows that $\mathbb{CP}^2$ is a closed 4–manifold. As an application, we show that $\mathbb{CP}^2$ is not diffeomorphic to $S^4$.

**Corollary 3.6.** $\mathbb{CP}^2$ is not diffeomorphic to $S^4$.

Proof. Let $L \subseteq \mathbb{CP}^2$ denote a projective line, that is, the image of a plane in $\mathbb{C}^3$. We claim that $\text{Int}_2(L,L) = 1$. This is immediate since, if $\tilde{L}$ is another line, then $L \cap \tilde{L}$ is a single point (two planes in $\mathbb{C}^3$ intersect in a single line), and $\tilde{L}$ is homotopic to $L$.

$L$ itself is diffeomorphic to the space of all complex lines in $\mathbb{C}^2$. This space, $\mathbb{CP}^1$, is diffeomorphic to $S^2$, since both may be identified with $\mathbb{C} \cup \{\infty\}$. Therefore $\mathbb{CP}^2$ contains a 2–sphere whose self-intersection number is 1.

On the other hand, $S^4$ contains no such 2–sphere, because every map $f : S^2 \to S^4$ extends to a map $F : B^3 \to S^4$. □