CHARACTERIZATION OF BROWNIAN MOTION ON MANIFOLDS THROUGH INTEGRATION BY PARTS

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ABSTRACT. Inspired by Stein's theory in mathematical statistics, we show that the Wiener measure on the pinned path space over a compact Riemannian manifold is uniquely characterized by its integration by parts formula among the set of probability measures on the path space for which the coordinate process is a semimartingale. Because of the presence of the curvature, the usual proof will not be readily extended to this infinite dimensional setting. Instead, we show that the integration by parts formula implies that the stochastic anti-development of the coordinate process satisfies Lévy's criterion.

CONTENTS

1. INTRODUCTION

The basis of Stein's approach to the central limit theorem is the fact that the equation

$$\mathbb{E}f'(X) = \mathbb{E}Xf(X) \tag{1.1}$$

characterizes the standard normal distribution $N(0,1)$. More precisely, for a real-valued random variable X, if the above equality holds for all real-valued functions f such that both $xf(x)$ and $f'(x)$ are uniformly bounded, then X has the standard normal distribution:

$$\mathbb{P}\left\{X \le x\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy.$$

(1.1) is an integration by parts formula for the standard Gaussian measure μ because it moves the differentiation away from the function f.. Let

$$D = \frac{d}{dx}, \qquad D^* = -\frac{d}{dx} + x.$$

Then D^* is the adjoint of D with respect to μ:

$$\langle Dg, f \rangle_\mu = \langle g, D^* f \rangle_\mu .$$

It can be simply written as $\mathbb{E} D^* f(X) = 0$. In this article, we consider an infinite dimensional extension of this characterization of the standard Gaussian distribution.

Consider a real-valued Brownian motion $W = \{W_t,\ 0 \le t \le 1\}$ starting from zero and with the unit time horizon. It can be viewed as a random variable of infinite dimensional standard Gaussian distribution, which takes values in the path space $P_o(\mathbb{R}) = C_o([0,1], \mathbb{R})$, the space of real-valued continuous functions on $[0,1]$ starting from zero. The analog of the above integration by parts formula is the following identity by Malliavin

$$\mathbb{E}\, D_h F(W) = \mathbb{E}\left[F(W) \int_0^1 \dot{h}_s \, dW_s \right]$$

for a large class of functions F on $P_o(\mathbb{R})$ and directions h.

More generally, let M be a compact Riemannian manifold M and $o \in M$ a fixed point. The pinned path space over M is

$$P_o(M) = C_o([0,1], M).$$

Consider the probability space $(P_o(M), \mathscr{B}(P(M)), \nu)$, where ν is the Wiener measure. Let $X = \{X_t,\ 0 \le t \le 1\}$ be the coordinate process on $P_o(M)$, namely $X(\gamma)_t = \gamma_t$ for $\gamma \in P_o(M)$. Thus under the probability ν, the process X is the standard Brownian motion on M. The corresponding integration by parts formula, due to Bismut[1] and Driver[2], is

$$\mathbb{E} D_h F(X) = \mathbb{E}\left[F(X) \int_0^1 \left\langle \dot{h}_s + \frac{1}{2} \mathrm{Ric}_{U(X)_s} h_s, dW_s \right\rangle \right].$$

The purpose of this article is to show that this integration by parts formula characterizes Brownian motion among the set of M-valued semimartingales.

2. One dimensional case

The relation $\mathbb{E}f'(X) = \mathbb{E}\, Xf(X)$ for a standard Gaussian random variable can be verified directly by using the density function

$$\frac{\mu^X(dx)}{dx} = \frac{e^{-x^2/2}}{\sqrt{2\pi}}.$$

However, the following point of view is more fruitful. The differentiation operator $D = d/dx$ generates the the translation semigroup: $T_t x = x + t$. For the shifted Gaussian measure $\mu^{X+t} = \mu^X \circ T_t^{-1}$ we have

$$\frac{\mu^{X+t}(dx)}{\mu^X(dx)} = e^{tx - t^2/2}.$$

Now we have

$$\begin{aligned} \mathbb{E}f(X+t) &= \int_{\mathbb{R}} f(x)\mu^{X+t}(dx) \\ &= \int_{\mathbb{R}} f(x)e^{tx-t^2/2}\mu^X(dx) \\ &= \mathbb{E}\left[e^{tX-t^2/2}f(X)\right]. \end{aligned}$$

The relation $\mathbb{E}f'(X) = \mathbb{E}\, Xf(X)$ is then obtained by differentiating with respect to t and letting $t = 0$.

An operator D on a function space is called derivation if it satisfies

$$D(fg) = gDf + fDg.$$

For any derivation operator D, the adjoint operator has the form

$$D^* = -D + D^*1.$$

Thus finding an integration by parts formula is equivalent to calculating D^*1, the divergence of 1 (the unit vector). The general integration by parts formula takes the form

$$\langle Dg, f\rangle_\mu = -\langle g, Df\rangle_\mu + \langle g, (D^*1)f\rangle_\mu.$$

If we take $g = 1$ and let X be a random variable with the law μ, then we have

$$\mathbb{E}Df(X) = \mathbb{E}\, f(X)D_*1(X).$$

In our case, the underlying Gaussian measure μ has a density function $p(x) = e^{-x^2/2}/\sqrt{2\pi}$ with respect to the Lebesgue measure, which is invariant under the translation group $\{T_t,\ t \in \mathbb{R}\}$, and we have

$$D^*1(x) = -\frac{d}{dx}\ln p\,(x) = x.$$

In infinite dimensional situation we will not have a measure invariant under translations. Nevertheless, as we will see, the expression D^*1 (the derivative of the logarithmic "density") still makes sense.

The fact that the equation $\mathbb{E} f'(X) = \mathbb{E}\, X f(X)$ implies that X has the distribution $N(0,1)$ can be proved using Stein's equation. Let

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

be the distribution function of $N(0,1)$. The general Stein's equation has the form

$$f'(x) - x f(x) = h(x) - \int_{\mathbb{R}} h(z)\, d\Phi(z).$$

Take the special case $h(x) = I_{(-\infty,z]}(x)$ (z fixed). The equation

$$f'(x) - x f(x) = I_{(-\infty,z]}(x) - \Phi(z)$$

can be solved explicitly:

$$f(x) = \frac{1}{\Phi'(x)} \cdot \begin{cases} \Phi(x)(1-\Phi(z)), & x \le z; \\ \Phi(z)(1-\Phi(x)), & x \ge z. \end{cases}$$

It is easy to verify that both $x f(x)$ and $f'(x)$ are uniformly bounded. Using this f in $\mathbb{E} f'(X) = \mathbb{E}\, X f(X)$ we have immediately

$$\mathbb{P}\{X \le z\} = \Phi(z).$$

There is not a direct infinite dimensional analog of the above method, for the significance of Stein's equation in an infinite dimensional setting has thus far not been explored. We will resort to the well known Lévy's criterion in order to identify a Brownian motion among semimartingales.

Proposition 2.1. (Lévy's criterion) *Let $W = \{W_t,\ t \ge 0\}$ be a continuous semimartingale in $\mathbb{R}^n$ adapted to a filtration $\mathscr{F}_* = \{\mathscr{F}_t,\ t \ge 0\}$ of σ-fields. Then W is a Brownian motion if and only if*

(1) W is a continuous local $\mathscr{F}_*$-martingale;

(2) $\langle W^*, W\rangle_t = \left\{\left\langle W^i, W_j\right\rangle_t\right\}_{1\le i,j\le n} = I_n \cdot t$.

Here I_n is the $n \times n$ identity matrix.

3. Infinite product Gaussian measure

The one dimensional theory in the preceding section can be extended directly to the product Gaussian measure on $\mathbb{R}^{\mathbb{Z}_+}$. Probabilistically, this corresponds the case of a sequence $X = \{X_n\}$ of i.i.d. random variables with the standard Gaussian distribution $N(0,1)$.

We consider the set $\mathscr{C}$ of cylinder functions

$$F(x) = f(x^0, x^1, \cdots, x^n).$$

Here $x = (x_0, x_1, x_2, \cdots)$. Consider the gradient

$$DF(x) = (f_{x_0}(x), f_{x_1}(x), \cdots).$$

More conveniently, consider the set of directional derivatives:

$$D_l F(x) = \langle DF(x), l \rangle = \sum_{i=0}^{\infty} l_i f_{x_i}(x).$$

Each $l = (l_1, l_2, \cdots)$ is a direction of differentiation. It is easy to see that

$$\mathbb{E}\, D_l F(X) = \mathbb{E}\, \langle X, l \rangle\, F(X), \tag{3.1}$$

where $\langle X, l \rangle = \sum_{n=0}^{\infty} X_n l_n$ is the inner product in $l_2(\mathbb{Z}_+)$. This equation characterizes the product Gaussian measure on $\mathbb{R}^{\mathbb{Z}_+}$. To see this, let

$$Y_l = \langle X, l \rangle = l_0 X_0 + \cdots + l_n X_n$$

and

$$F(x) = f(l_0 x_0 + l_1 x_1 + \cdots + l_n x_n) = f(\langle x, l \rangle).$$

Then

$$D_l F(X) = |l|_2^2 f'(Y_l)$$

and (3.1) becomes

$$|l|_2^2\, \mathbb{E} f'(Y_l) = \mathbb{E} Y_l f(Y_l).$$

Hence by the one dimensional result, Y_l has the distribution $N(0, |l|_2^2)$, the Gaussian distribution of mean zero and variance $|l|_2^2$. It is an easy exercise to show that if $Y_l = \langle X, l \rangle$ has the law $N(0, |l|_2^2)$ for all $l \in l^2(\mathbb{Z}_+)$, then $X = (X_1, X_2, \ldots)$ is i.i.d. with the distribution $N(0, 1)$.

4. One dimensional Brownian motion

Consider the path space $P_o(\mathbb{R}) = C_o([0, 1], \mathbb{R})$ and the map $\Phi : P_o(\mathbb{R}) \to \mathbb{R}^{\mathbb{Z}_+}$ defined by

$$\Phi(W) = \{X_n,\ n \in \mathbb{Z}_+\} = \left\{ \int_0^1 e_n dW_t,\ n \in \mathbb{Z}_+ \right\}.$$

Here $\{e_n, n \in \mathbb{R}_+\}$ is an orthonormal basis for $L^2[0, 1]$, which we may take to be $e_0(t) = 1$ and

$$e_n(t) = \sqrt{2} \cos \pi n t, \quad n = 1, 2, \ldots.$$

Here we assume that $W = \{W_t,\ 0 \le t \le 1\}$ is a semimartingale so that the stochastic integration makes sense. Let

$$h_n(t) = \int_0^t e_n(s)\, ds.$$

The inverse $\Phi^{-1} : \mathbb{R}^{\mathbb{Z}_+} \to P_o(\mathbb{R})$ can be loosely described as

$$W = \Phi^{-1}(X) = \sum_{n=0}^{\infty} h_n X_n. \tag{4.1}$$

If ν is the Wiener measure ν on $P_0(\mathbb{R})$ (the law of Brownian motion W) and μ the product Gaussian measure on $\mathbb{R}^{\mathbb{Z}_+}$, then we have an isometry $\Phi : (P_o(\mathbb{R}), \nu) \to (\mathbb{R}^{\mathbb{Z}_+}, \mu)$ in the sense that $\mu = \nu \circ \Phi^{-1}$.

Recall that on $(\mathbb{R}^{\mathbb{Z}_+}, \mu)$ we have

$$\mathbb{E} D_l F(X) = \mathbb{E} \langle X, l \rangle F(X).$$

We use the isometry Φ to rewrite the above relation in the path space $P_o(\mathbb{R})$. Namely, let $W = \Phi^{-1} X$ and $G = F \circ \Phi$. Then

$$\mathbb{E} D_h G(W) = \mathbb{E}\, G(W) D_h^* 1(W),$$

where $D_h = \Phi_*^{-1} D_l$ is the pullback of D_l by Φ and $D_h^* 1(W) = \langle X, l \rangle$. We need to calculate D_h and $D_h^* 1$ on the path space $P_o(\mathbb{R})$ in terms of h and W. From (4.1) we see immediately that the pullback $D_h = \Phi_*^{-1} D_l$ is given by

$$D_h G(W) = \lim_{t \to 0} \frac{G(W + th) - G(W)}{t},$$

where

$$h(t) = \sum_{n=0}^{\infty} l_n h_n(t) = \int_0^t \left[\sum_{n=0}^{\infty} l_n e_n(s) \right] ds.$$

On the other hand,

$$\langle X, l \rangle = \sum_{n=0}^{\infty} l_n X_n = \sum_{n=0}^{\infty} l_n \int_0^1 e_n(s)\, dW_s = \int_0^1 \dot{h}(s)\, dW_s.$$

Introducing the Cameron-Martin space

$$\mathscr{H} = \left\{ h \in P_o(\mathbb{R}) :\ |h|_{\mathscr{H}} < \infty \right\},$$

where

$$|h|_{\mathscr{H}}^2 = \int_0^1 |\dot{h}_s|^2 ds$$

if the integral is defined and $|h|_{\mathscr{H}} = \infty$ otherwise, we can write symbolically

$$D_h^* 1(W) = \langle h, W \rangle_{\mathscr{H}} = \int_0^1 \dot{h}_s \, dW_s.$$

Now that we have found $D_h^* 1(W)$ as a stochastic integral, the integration by parts formula has been transplanted from the product space $\mathbb{R}^{\mathbb{Z}_+}$ to the path space $P_o(\mathbb{R})$ to read

$$\mathbb{E} D_h G(W) = \mathbb{E} \langle h, W \rangle_{\mathscr{H}} G(W).$$

We can prove the above integration by parts formula from another point of view. Let μ^W be the law of Brownian motion W and μ^{W+th} be the law of $W + th$ (Brownian motion shifted by th) with $h \in \mathscr{H}$. Then we have the Cameron-Martin-Maruyama-Girsanov theorem:

$$\frac{d\mu^{W+th}}{d\mu^W} = \exp\left[t \int_0^1 \dot{h}_s dW_s - \frac{t^2}{2} \int_0^1 |\dot{h}_s|^2 ds \right].$$

Differentiating the identity

$$\mathbb{E}\, G(W + th) = \mathbb{E}\left[G(W) \frac{d\mu^{W+th}}{d\mu^W} \right]$$

with respect to t and then letting $t = 0$ we have

$$\mathbb{E} D_h G(W) = \mathbb{E}\left[G(W) \int_0^1 \dot{h}_s \, dW_s \right].$$

This relation holds for all $h \in \mathscr{H}$ and nice functions G (e.g., cylinder functions). Note that h does not have to be deterministic, for it suffices to assume that it be $\mathscr{B}_*$-adapted and that $\mathbb{E}|h|^2_{\mathscr{H}} < \infty$, $\mathscr{B}_* = \{\mathscr{B}_s,\, 0 \le s \le 1\}$ being the standard Borel filtration on the path space $P_o(\mathbb{R})$.

Theorem 4.1. *Let W be a real-valued continuous semimartingale. It is a Brownian motion if and only if the integration by parts formula*

$$\mathbb{E} D_h G(W) = \mathbb{E} \langle h, W \rangle_{\mathscr{H}} G(W)$$

holds for and all cylinder functions G and all adapted $\dot{h}$ such that $\langle h, W \rangle_{\mathscr{H}} G(W)$ is integrable.

Proof. We have shown that the above integration by parts formula holds if W is a Brownian motion. To show the implication in the other direction, we can resort to the map Φ and work on the product space $\mathbb{R}^{\mathbb{Z}_+}$. Of course, this proof will not extend to the path space over a compact manifold (non-flat path space). A better way is to use Lévy's criterion.

Taking $G = 1$, we have $D_h G = 0$ and

$$\mathbb{E}\,\langle h, W\rangle_{\mathscr{H}} = \mathbb{E}\int_0^1 \dot{h}_s\, dW_s = 0$$

for all adapted $\dot{h}$ such that the stochastic integral is integrable. This implies that W is a (local) martingale from the fact that W, as a continuous semimartingale, is the sum of a local martingale and a process of local bounded variation.

Now we take $G(W) = W_1$. From the definition of D_h we have

$$D_h G(W) = \lim_{t\to 0} \frac{G(W+th) - G(W)}{t} = h_1.$$

The integration by parts formula becomes

$$\mathbb{E} h_1 = \mathbb{E}\,\langle h, W\rangle_{\mathscr{H}} W_1.$$

The right side is

$$\mathbb{E}\left[W_1 \int_0^1 \dot{h}_s\, dW_s\right] = \mathbb{E}\int_0^1 \dot{h}_s\, d\,\langle W, W\rangle_s.$$

Hence, equating the expressions, we have

$$\mathbb{E}\int_0^1 \dot{h}_s\, ds = \mathbb{E}\int_0^1 \dot{h}_s\, d\,\langle W, W\rangle_s.$$

This holds for all suitable adapted $\dot{h}$, from which we conclude immediately that $\langle W, W\rangle_t = t$. By Lévy's criterion W is a Brownian motion. □

5. Brownian motion on a Riemannian manifold

We briefly describe Brownian motion on a Riemannian manifold and its integration by parts formula. For a detailed discussion, the reader is referred to the monograph Hsu[3].

Let M be a compact Riemannian manifold (or more generally, a complete Riemannian manifold with uniformly bounded Ricci curvature) and o a fixed point on M. Let ν be the Wiener measure on the path space $P_o(M)$. On the probability space $(P_o(M), \mathscr{B}(P_o(M)), \nu)$, the coordinate process $\mathit{\Pi} = \{\mathit{\Pi}_t,\ 0 \le t \le 1\}$ is a Brownian motion on $\mathbb{R}^n$.

Let $\mathscr{O}(M)$ be the orthonormal frame bundle of M and $\pi : \mathscr{O}(M) \to M$ the canonical projection. Let $H_1, \cdots, H_n$ be the canonical horizontal vector field on $\mathscr{O}(M)$. Fix a frame $U_o \in \pi^{-1} o$ and consider the Itô type stochastic differential equation on $\mathscr{O}(M)$:

$$dU_t = \sum_{i=1}^{n} H(U_t) \circ dW_t^i.$$

Its unique solution is called a horizontal Brownian motion. The projection $X = \pi U$ is a Brownian motion on M starting from o, whose law is the Wiener measure ν on the path space $(P_o(M), \mathscr{B}(P_o(M)))$. The map $J : P_o(\mathbb{R}^n) \to P_o(M)$, which we call the Itô map, is the stochastic equivalent of the development map (rolling without slipping) in differential geometry. In differential geometry, the map J carries straight lines (with uniform speed) to geodesics. In the context of stochastic analysis, it carries a euclidean Brownian motion on $\mathbb{R}^n$ to a Riemannian Brownian motion on M. As in the case in differential geometry, J is invertible and the inverse image $W = J^{-1}X$ is again obtained by solving an Itô type stochastic differential equation driven by X. The Wiener measure (the law of Brownian motion X on M) is $\nu = \mu \circ J^{-1}$ and we have an isometry:

$$J : \ (P_o(\mathbb{R}^n), \nu) \to (P_o(M), \nu).$$

Since $W = J^{-1}X$ is obtained from X by solving an Itô type stochastic differential equation, W is well defined when X is a semimartingale. We have the following basic fact (see Hsu[3]).

Proposition 5.1. *X is a Brownian motion on M if and only if its stochastic anti-development $W = J^{-1}X$ is a Brownian motion on $\mathbb{R}^n$.*

Let $x \in M$ and T_xM the tangent space at x. Let $\mathrm{Ric}_x : \ T_xM \to T_xM$ be the Ricci transform at x of the Levi-Civita connection of M. Let $u \in \mathscr{O}(M)$ be an orthonormal frame at $x = \pi u$. Then $u : \mathbb{R}^n \to T_xM$ is a linear isometry. The scalarized Ricci transform at u is

$$\mathrm{Ric}_u \overset{\text{def}}{=} u^{-1}\mathrm{Ric}_x u : \ \mathbb{R}^n \to \mathbb{R}^n.$$

In [1],Bismut proved his famous integration by parts formula

$$\mathbb{E}\left\langle \nabla f(X_t), h_t \right\rangle = \mathbb{E}\left[f(X_t) \int_0^1 \left\langle \dot{h}_s + \frac{1}{2}\mathrm{Ric}_{U(X)_s} h_s, dW_s \right\rangle \right]$$

for : for $h \in \mathscr{H}$, $f \in C^\infty(M)$ and $0 \le t \le 1$. This formula can be extended to a more general integration by parts formula for directional derivative operators D_h on the path space $P_o(M)$.

We briefly recall the definition of D_h. For a path $h \in P_o(\mathbb{R}^n)$ and a path $\gamma \in P_o(M)$ for which the horizontal lift $U(\gamma)$ makes sense, we define $D_h(\gamma)_s = U(\gamma)_s h_s$. We can regard D_h as a vector field on the path space $P_o(M)$. The directional derivative D_h along the direction h can be defined as

$$D_h G(\gamma) = \lim_{t\to 0} \frac{G(\zeta_h^t \gamma) - G(\gamma)}{t},$$

where $\left\{\zeta_t^h,\ t \in \mathbb{R}\right\}$ is the flow generated by the vector field D_h. It takes a fair amount of effort to define D_h in this matter. However for a cylinder function $G(\gamma) = g(\gamma_{s_1}, \cdots, \gamma_{s_l})$ with $0 \le s_1 < s_2 < \cdots < s_l \le 1$ and $g \in C^\infty(M^l)$, $D_h G(\gamma)$ can be simply defined by the formula:

$$D_h G(\gamma) = \sum_{i=1}^{l} \left\langle U(\gamma)_{s_i}^{-1} \nabla^i f(\gamma), h_{s_i} \right\rangle.$$

Starting from Bismut's integration by parts formula, we can show by induction on the number of time dependencies the following integration by parts formula for the Wiener measure ν on the path space $P_o(M)$: for all cylinder functions G and $h \in \mathscr{H}$,

$$\mathbb{E}\, D_h G(X) = \mathbb{E}\left[G(X) \int_0^1 \left\langle \dot{h}_s + \frac{1}{2}\mathrm{Ric}_{U(X)_s} h_s, dW_s \right\rangle \right].$$

This form of integration by parts formula was first proved in Driver[2]. It is the proper generalization in the path space $P_o(M)$ of the relation $\mathbb{E} f'(X) = \mathbb{E}\, X f(X)$ for a standard Gaussian variable X.

In the next section we show that this equation characterizes Brownian motion in the set of semimartingales.

6. Characterization of Riemannian Brownian motion

In this section we prove the main result of the article.

Theorem 6.1. *Let X be an M-valued semimartingale on a filtered probability space $(\Omega, \mathscr{F}_*, \mathbb{P})$. Then it is a Brownian motion on M if and only if*

$$\mathbb{E}\, D_h G(X) = \mathbb{E}\left[G(X) \int_0^1 \left\langle \dot{h}_s + \frac{1}{2}\mathrm{Ric}_{U(X)_s} h_s, dW_s \right\rangle \right]$$

for all cylinder functions G and all $\mathscr{F}_*$-adapted $\dot{h} \in \mathscr{H}$ such that the random variables on the two sides are integrable. Here W and U are the anti-development of X in $\mathbb{R}^n$ and the horizontal lift of X in $\mathscr{O}(M)$, respectively.

Proof. Recall that X is a Brownian motion on M if and only if the continuous semimartingale $W = J^{-1}X$ is a Brownian motion on $\mathbb{R}^n$. All we need to show is that the semimartingale W satisfies Lévy's criterion.

To show that W is a local martingale, we again let $G \equiv 1$. From $D_h 1 = 0$ we have

$$\mathbb{E} \int_0^1 \left\langle \dot{h}_s + \frac{1}{2} \mathrm{Ric}_{U(X)_s} h_s, dW_s \right\rangle = 0.$$

Let $W = M + A$ be the Doob-Meyer decomposition of the semimartingale and introduce the notation

$$|W|_s = \sum_{1 \le i,j \le n} | \left\langle M^i, M^j \right\rangle |_s + \sum_{1 \le i \le n} |A^i|_s. \tag{6.1}$$

Consider the set $\mathscr{F}(W)$ of adapted $h = \{h_s,\ 0 \le s \le 1\}$ such that

$$\mathbb{E} \left[\int_0^1 |\dot{h}_s|^2 d|W|_s + \left(\int_0^1 |\dot{h}_s|\, d|W|_s \right)^2 \right] < \infty.$$

[For a function real-valued function f of locally bounded variation we use $|f|_t$ to denote its total variation on $[0, t]$.] It is easy to verify that if $g \in \mathscr{F}(W)$, the unique solution h of the ordinary differential equation

$$\dot{h}_s + \frac{1}{2} \mathrm{Ric}_{U(X)_s} h_s = \dot{g}_s, \quad 0 \le s \le 1$$

is again in $\mathscr{F}(W)$. Therefore,

$$\mathbb{E} \int_0^1 \langle \dot{g}_s, dW_s \rangle = 0$$

for all $g \in \mathscr{F}(W)$. This easily implies immediately that W is a local martingale.

The verification that the quadratic variation

$$\langle W^*, W \rangle_t = I_n \cdot t, \qquad I_n = (n \times n) - \text{identity matrix}$$

is more difficult. First of all for a fixed $0 < u < t \le 1$ we take the function G to be

$$G(X) = W_t - W_u = (J^{-1}X)_t - (J^{-1}X)_u.$$

Of course, G is not a cylinder function on $P_o(M)$. However, from general approximation theory for stochastic differential equations (see Ikeda and Watanabe[5]) one can approximate $G(X) = W_t - W_u$ by a sequence of cylinder functions in a very strong sense. More precisely, suppose that $h \in \mathscr{F}(W)$, i.e., it satisfies the condition

$$\mathbb{E} \int_0^1 |\dot{h}_s|^2\, d|W|_s < \infty,$$

see (6.1). Let

$$l_1(X) = \int_0^1 \left\langle \dot{h}_s + \frac{1}{2}\mathrm{Ric}_{U(X)_s} h_s, dW_s \right\rangle .$$

Then there is a sequence of cylinder functions G_n such that

$$\lim_{n\to\infty} G_n(X) l_1(X) = G(X) l_1(X)$$

and

$$\lim_{n\to\infty} D_h G_n(X) = D_h G(X),$$

both limits taking place in $L^2(P_o(M), \mathscr{B}(P_o(M)), \nu)$. Here $D_h G(X)$ is obtained by calculating formally the pushforward of the vector field D_h through the map J^{-1}:

$$D_h G(X) = D_h (J^{-1}X)_t - (J^{-1}X)_u = (J_*^{-1} D_h)_t - (J_*^{-1} D_h)_u.$$

This calculation can be found in [2] and [4] when W is assumed to be a Brownian motion but only slight modifications are needed if W is only assumed to be a local martingale. We have

$$\begin{aligned}(J_*^{-1}D_h)_t - (J_*^{-1}D_h)_u &= \int_u^t \langle \theta(h)_s, dW_s\rangle \\ &\quad + \int_u^t \left\{ \dot{h}_s\, ds + \frac{1}{2}\sum_{1\le i,j\le n} \Omega_j(U(X)_s)(He_i, Hh_s)\, d\left\langle W^i, W^j\right\rangle_s \right\}.\end{aligned}$$

Here Ω is the scalarized curvature tensor (or the curvature form on the orthonormal frame bundle $\mathscr{O}(M)$) and $Hf = \sum_{i=1}^n f_i H_i$ is the horizontal vector defined by $f \in \mathbb{R}^n$. The explicit expression of the integrand $\theta(h)$ involves the curvature tensor and W is not important for our purpose. What is important is that under the condition imposed on h the stochastic integral on the right side is a martingale.

Now we have

$$\mathbb{E} D_h G_n(X) = \mathbb{E}\left[G_n(X) \int_0^1 \left\langle \dot{h}_s + \frac{1}{2}\mathrm{Ric}_{U(X)_s} h_s, dW_s \right\rangle \right].$$

We take the limit as $n \to \infty$. On the left side we have

$$\mathbb{E}\int_u^t \left\{ \dot{h}_s\, ds + \frac{1}{2}\sum_{1\le i,j\le n} \Omega_j(U(X)_s)(He_i, Hh_s)\, d\left\langle W^i, W^j\right\rangle_s \right\}.$$

On the right side we have

$$\mathbb{E}\left[(W_t - W_u) \int_0^1 \left\langle \dot{h}_s + \frac{1}{2}\mathrm{Ric}_{U_s} h_s, dW_s \right\rangle \right].$$

Because W is a local martingale, the above expression is equal to

$$\mathbb{E}\int_u^t d\left\langle W^*, W\right\rangle_s \left\{\dot{h}_s + \frac{1}{2}\mathrm{Ric}_{U_s} h_s\right\}.$$

Equating the two sides, the resulting equality has the following form:

$$\mathbb{E}\int_u^t \left\{I_n ds - d\left\langle W^*, W\right\rangle_s\right\}\dot{h}_s + \mathbb{E}\int_u^t C_{ijk}(s)\, h_s^k d\left\langle W^i, W^j\right\rangle = 0. \tag{6.2}$$

Here $C_{ijk} = \left\{C_{ijk}(s),\ 0 \le s \le 1\right\}$ is continuous and adapted, whose actual expression is not needed in the following discussion. Using

$$h_s = \int_0^s \dot{h}_\tau\, d\tau$$

and changing the order of integration we find that the second term is the sum of

$$\mathbb{E}\int_0^u \dot{h}_\tau^k\, d\tau \int_u^t C_{ijk}(s)\, d\left\langle W^i, W^j\right\rangle_s$$

and

$$\mathbb{E}\int_u^t \dot{h}_\tau^k\, d\tau \int_\tau^t C_{ijk}(s)\, d\left\langle W^i, W^j\right\rangle_s.$$

Note that the last term here and the first term in (6.2) do not involve the values $\dot{h}_\tau$ for $0 \le \tau \le u$, hence we must have

$$\mathbb{E}\left[\int_u^t C_{ijk}(s)\, d\left\langle W^i, W^j\right\rangle_s \middle| \mathscr{F}_\tau\right] = 0$$

for all $0 \le \tau \le u \le t \le 1$. This fact in term implies in turn that both terms in rftwot2 vanish. It follows that

$$\mathbb{E}\int_u^t \left\{I_n ds - d\left\langle W^*, W\right\rangle_s\right\}\dot{h}_s = 0,$$

which implies that $\left\langle W^*, W\right\rangle_t = I_n \cdot t$. By Lévy's criterion W W is a Brownian motion on $\mathbb{R}^n$. By Proposition 2.1 X is a Brownian motion on M. $\square$

7. Concluding remarks and acknowledgements

We have shown that the integration by parts formula for the Wiener measure on the path space over a Riemannian manifold characterizes uniquely the Wiener measure among the set of probability measures on the path space under which the coordinate process is a semimartingale. This may be regarded as the first step towards exploring in the context of stochastic analysis the circle of ideas surrounding the well known Stein-Chen technique in mathematical statistics.

The success of the Stein method in the theory of central limit theorems may point to a possible parallel theory in the current context. In particular, one wonders whether it is possible to introduce a useful concept of distance between a semimartingale and a Brownian motion on a fixed Riemannian manifold. In addition, in view of the importance of Stein's equation, it may also be worthwhile to explore this equation in an infinite dimensional setting.

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