MAXIMAL COUPLING OF EUCLIDEAN BROWNIAN MOTIONS

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ABSTRACT. We prove that the mirror coupling is the unique maximal Markovian coupling of two Euclidean Brownian motions starting from single points and discuss the connection between the uniqueness problems of Brownian coupling and mass transportation.

1. INTRODUCTION

Let $(E_1, \mathscr{B}_1, \mu_1)$ and $(E_2, \mathscr{B}_2, \mu_2)$ be two probability spaces. A coupling of the probability measures μ_1 and μ_2 is a probability measure μ on the product measurable space $(E_1 \times E_2, \mathscr{B}_1 \times \mathscr{B}_1)$ whose marginal probabilities are μ_1 and μ_2, respectively. We denote the set of coupling of μ_1 and μ_2 by $\mathscr{C}(\mu_1, \mu_2)$. Thus, loosely speaking, a coupling of two Euclidean Brownian motions on $\mathbb{R}^n$ starting from x_1 and x_2, respectively, is a $C(\mathbb{R}_+, \mathbb{R}^n \times \mathbb{R}^n)$-valued random variable (X_1, X_2) on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ such that the components X_1 and X_2 have the law of Brownian motion starting from x_1 and x_2, respectively. In this case, we say simply that (X_1, X_2) is a coupling of Brownian motions from (x_1, x_2).

In the present work we discuss the uniqueness problem of maximal couplings of Euclidean Brownian motion. As usual, the maximality of a coupling is defined as a coupling for which the coupling inequality becomes an equality. It is well known the mirror coupling is a maximal coupling. We show by an example that in general a maximal coupling is not unique. To prove a uniqueness result, we consider a more restricted class of couplings, that of Markovian couplings. In this class we show that the mirror coupling is the unique maximal coupling. This will be done by two methods. The first method is a martingale argument, which depends on the linear structure of the Euclidean state space. In the second method, we use the Markovian hypothesis to reduce the problem to a mass transport problem on the state space, whose solution is well known. This method is more interesting because it has the potential of generalization to more general settings (e.g., Brownian motion on a Riemannian manifold).

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2. Maximal coupling

Let

$$p\left(t,x,y\right)=\left(\frac{1}{2\pi t}\right)^{n/2}e^{-|x-y|^2/2t}$$

be the Gaussian heat kernel on $\mathbb{R}^n$. Here $|x|$ is the euclidean length of a vector $x\in\mathbb{R}^n$. Define the function

$$\phi_t(r)=\frac{2}{\sqrt{2\pi t}}\int_0^{r/2}e^{-\rho^2/2t}\,d\rho.$$

When $t=0$, we make the convention

$$\phi_0(r)=\begin{cases}0, & r=0,\\ 1, & r>0.\end{cases}$$

For $t>0$, this function is strictly concave. The probabilistic significance of this function is that it is the tail distribution of the first passage time of a one dimensional Brownian motion from 0 to $r/2$:

$$\mathbb{P}\left\{\tau_{r/2}\geq t\right\}=\phi_t(r).$$

It is also easy to verify that

$$\phi_t(|x_1-x_2|)=\frac{1}{2}\int_{\mathbb{R}^n}\left|p\left(t,x_1,y\right)-p\left(t,x_2,y\right)\right|dy. \tag{2.1}$$

Fix two distinct points x_1 and x_2 in $\mathbb{R}^n$. Let $X=(X_1,X_2)$ be a coupling of Euclidean Brownian motions from (x_1,x_2). This simply means that the laws of $X_1=\{X_1(t)\}$ and $X_2=\{X_2(t)\}$ are Brownian motions starting from x_1 and x_2, respectively. The coupling time $T(X_1,X_2)$ is the earliest time at which the two Brownian motions coincide afterwards:

$$T(X_1,X_2)=\inf\left\{t>0:X_1(s)=X_2(s)\text{ for all }s\geq t\right\}.$$

Note that $T(X_1,X_2)$ in general is not the first time the two processes meet and therefore is not a stopping time. The following well known coupling inequality gives a lower bound for the tail probability of the coupling time. Similar inequalities hold under more general settings (see Lindvall [5]).

Proposition 2.1. *Let (X_1,X_2) be a coupling of Brownian motions from (x_1,x_2). Then*

$$\mathbb{P}\left\{T(X_1,X_2)\geq t\right\}\geq\phi_t(|x_1-x_2|).$$

Proof. For any $A \in \mathscr{B}(\mathbb{R}^n)$, we have

$$\begin{aligned}\mathbb{P}\left\{T(X_1, X_2) > t\right\} &\geq \mathbb{P}\left\{X_1(t) \neq X_2(t)\right\} \\ &\geq \mathbb{P}\left\{X_1(t) \in A,\ X_2(t) \notin A\right\} \\ &\geq \mathbb{P}\left\{X_1(t) \in A\right\} - \mathbb{P}\left\{X_2(t) \in A\right\}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{P}\left\{T(X_1, X_2) > t\right\} &\geq \sup_{A \in \mathscr{B}(\mathbb{R}^n)} \int_A \left\{p\left(t, x_1, y\right) - p\left(t, x_2, y\right\} dy \\ &= \frac{1}{2}\int_{\mathbb{R}^n} \left|p\left(t, x_1, y\right) - p\left(t, x_2, y\right)\right| dy \\ &= \phi_t(|x_1 - x_2|).\end{aligned}$$

In the last step we have used (2.1) □

In view of the coupling inequality, the following definition is natural. A coupling (X_1, X_2) of Brownian motions from (x_1, x_2) is called maximal the coupling inequality is an equality for all $t > 0$, i.e.,

$$\mathbb{P}\left\{T(X_1, X_2) \geq t\right\} = \phi_t(|x_1 - x_2|).$$

It is a well known fact that the mirror coupling, which we will define shortly, is a maximal coupling.

Let H be the hyperplane bisecting the segment $[x_1, x_2]$:

$$H = \left\{x \in \mathbb{R}^n : \langle x - x_0, n\rangle = 0\right\},$$

where $x_0 = (x_1 + x_2)/2$ is the middle point and $n = (x_1 - x_2)/|x_1 - x_2|$ the unit vector in the direction of the segment. Let $\mathscr{R} :\ \mathbb{R}^n \to \mathbb{R}^n$ be the mirror reflection with respect to the hyperplane H:

$$\mathscr{R}x = x - 2\langle x - x_0, n\rangle n.$$

We now describe the mirror coupling. Let

$$\tau = \inf\left\{t \geq 0 : X_1(t) \in H\right\}$$

be the first hitting time of H by X_1. We know that

$$\mathbb{P}\left\{\tau \geq t\right\} = \phi_t(|x_1 - x_2). \tag{2.2}$$

A coupling (X_1, X_2) of Brownian motions from (x_1, x_2) is a mirror coupling (or X_2 is the mirror coupling of X_1) if X_2 is the mirror reflection of X_1 with respect to H before time τ and coincides with X_1 afterwards; namely,

$$X_2(t) = \begin{cases}\mathscr{R}X_1(t), & t \in [0, \tau]; \\ X_1(t), & t \in [\tau, \infty).\end{cases}$$

In this case the coupling time $T(X_1, X_2) = \tau$. From (2.2) the mirror coupling is a maximal coupling by definition, a well known fact (see, e.g., Lindvall [5]).

It was believed that the mirror coupling is the unique maximal coupling of euclidean Brownian motion. This, however, is not the case, as has been recently discovered by the authors and others (including Pat Fitzsimmons and Wilfrid Kendall). We describe Fitzsimmons' counterexample in dimension one1. Let

$$l = \sup\{t \le \tau :\ X_1(t) = x_1\}$$

be the *last* time the Brownian motion X_1 is at x_1 before it hits the middle plane H (i.e., before time τ). We let X_2 to be the time reversal of X_1 before time l, the mirror reflection of X_1 between l and τ and X_1 after τ; namely,

$$X_2(t) = \begin{cases} x_2 - x_1 + X_1(l-t), & t \in [0, l]; \\ x_1 + x_2 - X_1(t), & t \in [l, \tau]; \\ X_1(t), & t \in [\tau, \infty). \end{cases}$$

Of course X_2 is not the mirror coupling of X_1. On the other hand, by Williams' decomposition of Brownian path $\{X_1(t), 0 \le t \le \tau\}$ (see Revuz and Yor[6], 244–245 and 304–305), X_2 is a Brownian motion starting from x_2. The coupling time for (X_1, X_2) is again τ, which shows that the coupling is indeed maximal.

In order to recover the uniqueness, we need to consider a smaller class of couplings.

Definition 2.2. *Let $X = (X_1, X_2)$ be a coupling of Brownian motions. Let $\mathscr{F}_*^X = \{\mathscr{F}_t^X\}$ be the filtration of σ-algebras generated by X. We say that X is a Markovian coupling if for each $s \ge 0$, conditioned on the σ-algebra $\mathscr{F}_s^X$, the shifted process*

$$\{(X_1(t+s), X_2(t+s)),\ t \ge 0\}$$

is still a coupling of Brownian motions (now from $(X_1(s), X_2(s))$).

A few comments about this definition is in order. The condition that $X = (X_1, X_2)$ is a Markovian coupling only requires that, conditioned on $\mathscr{F}_s^X$, the law of each time-shifted component is that of a Brownian motion. In particular, (X_1, X_2) is a Markovian coupling as soon as each component is separately a Brownian motion with respect to a common filtration. This is the case if for instance $\mathscr{F}_*^{X_2} = \mathscr{F}_*^{X_1}$, i.e., the second Brownian motion is defined progressively (without looking forward) by the first Brownian motion. It should be pointed out that the definition does not imply that (X_1, X_2) is a Markov process. For the sake of comparison, we mention the definition of efficient couplings in Burdzy and Kendall [1] and that of ρ-optimal couplings in Chen [2].

The main result of this paper is the following.

Theorem 2.3. *Let $x_1, x_2 \in \mathbb{R}^n$. The mirror coupling is the only maximal Markovian coupling of n-dimensional Brownian motions starting from (x_1, x_2).*

We will give two proofs of this theorem. The first one is based on the fact that the Markovian condition implies that the joint process is a martingale. The second method is to use the Markovian condition to reduce the uniqueness problem of coupling of two process to the uniqueness of a mass transportation problem, whose solution is well known. This proof is more interesting from an analytic point of view. From the second proof it will be clear that that a stronger result holds, namely if a Markovian coupling is maximal at one fixed time t then it must be the mirror coupling up to time t.

3. Proof of the uniqueness using martingales

Without loss of generality we assume that the space dimension is one. Let $\mathscr{F}_* = \mathscr{F}_*^X$ be the filtration generated by the joint process $X = (X_1, X_2)$. The maximality hypothesis implies that each component is a Brownian motion with respect to $\mathscr{F}_*$. Therefore X is a continuous $\mathscr{F}_*$-martingale and so is the $X_1 - X_2$. Let

$$\sigma_t = \frac{1}{4}\langle X_1 - X_2\rangle(t).$$

By Lévy's criterion there is a Brownian motion W such that

$$X_1(t) - X_2(t) = 2W(\sigma_t).$$

Since both X_1 and X_2 are Brownian motion, by the Kunita-Watanabe inequality,

$$|\langle X_1, X_2\rangle_t| \leq \int_0^t \sqrt{d\langle X_1\rangle_s d\langle X_2\rangle_s} = t. \tag{3.1}$$

Hence,

$$\sigma_t = \frac{\langle X_1\rangle_t + \langle X_2\rangle_t - 2\langle X_1, X_2\rangle_t}{4} \leq t.$$

Now let

$$\tau_1 = \inf\{t \geq 0 : X_1(t) = X_2(t)\} \quad \text{and} \quad \tau_2 = \inf\{t \geq 0 : W(t) = 0\}.$$

It is clear that $T(X_1, X_2) \geq \tau_1$ and τ_2 is the first passage time of Brownian motion W from $|x_1 - x_2|/2$ to 0. The maximality of the coupling (X_1, X_2) means that $T(X_1, X_2)$ and τ_2 have the same distribution. On the other hand, by definition $\sigma_{\tau_1} = \tau_2$, hence

$$T(X_1, X_2) \geq \tau_1 \geq \sigma_{\tau_1} = \tau_2.$$

Since $T(X_1, X_2)$ and τ_2 have the same distribution, we must have

$$T(X_1, X_2) = \tau_2 = \sigma_{\tau_1} = \tau_1.$$

Therefore the coupling time coincides with the first meeting time of X_1 and X_2, and before they meet the equality must hold in the Kunita-Watanabe inequality (3.1), i.e., $\langle X_1, X_2\rangle_t = t$. It follows that for $0 \leq t \leq \tau_1$,

$$X_2(t) = X_2(0) + X_1(0) - X_1(t) = 2x_0 - X_1(t),$$

which simply means that X_2 is the mirror coupling of X_1.

4. Optimal Coupling of Gaussian distributions

The second proof we give in the next section, although a bit longer than the first one, has potentially wider applications to a general state space without a liner structure (a Riemannian manifold, for example). The basic idea is to use the maximal and the Markovian hypothesis to reduce the problem to the uniqueness of a very special mass transportation problem on the state space with a cost function determined by the transition density function. In this section we discuss this mass transportation problem. For general theory, see , Theorem 1.4 in Gangbo and McCann [3] and Section 4.3, Theorem 3 in Villani [8].

Given $t \geq 0$ and $x \in \mathbb{R}$ we use $N(x,t)$ to denote the Gaussian distribution of mean x and variance t. The density function is $p\,(t,x,z)$. A probability measure μ on $\mathbb{R}^2$ is called a coupling of $N(x_1,t)$ and $N(x_2,t)$ if they are the marginal distributions of μ. We use $\mathscr{C}(x_,x_2;t)$ to denote the set of such couplings. The mirror coupling $m(x_1,x_2;t)$, which we define shortly, is a distinguished member of $\mathscr{C}(x_1,x_2;t)$.

We may regard a coupling $\mu \in \mathscr{C}(x_1,x_2;t)$ as the joint distribution of a $\mathbb{R}^2$-valued random variable $Z=(Z_1,Z_2)$. Intuitively, in the mirror coupling Z_2 coincides with Z_1 as much as possible, and if this cannot be done then $Z_2 = \mathscr{R}Z_1$, the mirror image of Z_1. Thus the mirror coupling $m(x_1,x_2;t)$ can be described as follows:

$$\mathbb{P}\left\{Z_2 = Z_1 || Z_1 = z_1\right\} = \frac{p\,(t,x_1,z_1) \wedge p\,(t,x_2,z)}{p\,(t,x_1,z_1)},$$

$$\mathbb{P}\left\{Z_2 = x_1 + x_2 - Z_1 | Z_1 = z_1\right\} = 1 - \frac{p\,(t,x_1,z_1) \wedge p\,(t,x_2,z)}{p\,(t,x_1,z_1)}.$$

Equivalently we can also write

$$m(x_1,x_2,t)(dy_1,dy_2) = \delta_{y_1}(dy_2)h_0(y_1)dy_1 + \delta_{\mathscr{R}y_1}(dy_2)h_1(y_1)dy_1$$

where

$$h_0(z) = p\,(t,x_1,z) \wedge p\,(t,x_2,z),$$

and

$$h_1(z) = p\,(t,x_1,z) - h_0(z).$$

It is clear that $m(x_1,x_2;t)$ is concentrated on the union of the two sets

$$D = \left\{(z,z) : z \in \mathbb{R}\right\}, \quad L = \left\{(z,\mathscr{R}z) : z \in \mathbb{R}\right\},$$

on which it has the one dimensional densities $h_0(z)$ and $h_1(z)$, respectively.

Let ϕ be a nonnegative function on $[0,\infty)$ such that $\phi(0) = 0$. The transportation cost of a coupling $\mu \in \mathscr{C}(x_1,x_2;t)$ with the cost function ϕ is defined by

$$C_\phi(\mu) = \int_{\mathbb{R}^2} \phi(|x-y|)\mu(dy_1 dy_2).$$

The results we will need for studying couplings of euclidean Brownian motion are contained in the following two theorems.

Theorem 4.1. *Let ϕ be a strictly increasing strictly concave cost function. Let $m = m(x_1, x_2; t)$ be the mirror coupling. Then $C_\phi(\mu) \geq C_\phi(m)$ for all $\mu \in \mathscr{C}(x_1, x_2; t)$ and the equality holds if and only if $\mu = m$.*

Proof. Let

$$\mu_1 = p\,(t, x_1, z)\,dz, \quad \mu_2 = p\,(t, x_2, z)\,dz$$

be the probability measures for Z_1 and Z_2. Suppose that μ is a probability measure on $\mathbb{R}^2$ at which the minimum is attained. Let

$$D = \{(x, x) : x \in \mathbb{R}\}$$

be the diagonal in $\mathbb{R} \times \mathbb{R}$. We first show that the restriction of μ to D is

$$\mu\big|_D(dz) = \nu_0(dz) := h_0(z)dz, \tag{4.1}$$

where $h_0(z) := p\,(t, x_1, z) \wedge p\,(t, x_2, z)$ as before. First of all, since the marginal distributions of μ are $p\,(t, x_1, z)dz$ and $p\,(t, x_2, z)dz$, we must have $\mu|_D \leq \nu_0$. We need to show that equality holds.

We first explain the argument intuitively. We regard μ as a transport from the mass μ_1 to the mass μ_2. Suppose that the strict inequality holds at a point (y_0, y_0). From the fact that the first marginal distribution of μ is $p\,(t, x_1, z)dz > h_0(z)$ we see that there must be a point $y_2 \neq y_0$ such that (y_0, y_2) is in the support of μ. Similarly there must be a point $y_1 \neq y_0$ such that (y_1, y_0) is in the support of μ. This means that a positive mass is moved from y_1 to y_0 and then from y_0 to y_2. But then μ cannot be optimal because from the inequality

$$\phi(|y_1 - y_0|) + \phi(|y_0 - y_2|) > \phi(|y_1 - y_2|)$$

it is more efficient to transport the mass directly from y_1 to y_2.

To proceed rigorously, we write μ in the following forms:

$$\mu(dy_1 dy_2) = k_1(y_1, dy_2)\mu_1(dy_1) = k_2(y_2, dy_1)\mu_2(dy_2) \tag{4.2}$$

where k_1 and k_2 are appropriate Markov kernels on $\mathbb{R}$. Let

$$\nu_1 = \mu_1 - \nu_0, \quad \nu_2 = \mu_2 - \nu_0$$

and

$$\begin{aligned}\nu(dy_1 dy_2) = {} & \frac{1}{2}\delta_{y_1}(dy_2)\nu_0(dy_1) + \frac{1}{2}\int_{\mathbb{R}} k_2(y_0, dy_1)\, k_1(y_0, dy_2)\, \nu_0(dy_0) \\ & + \frac{1}{2}k_1(y_1, dy_2)\, \nu_1(dy_1) + \frac{1}{2}k_2(y_2, dy_1)\, \nu_2(dy_2).\end{aligned}$$

Then a straightforward calculation shows that ν is a coupling of a coupling of μ_1 and μ_2 and the following equality holds:

$$\begin{aligned}&\int_{\mathbb{R}\times\mathbb{R}} \phi(|y_1 - y_2|)\nu(dy_1 dy_2) - \int_{\mathbb{R}\times\mathbb{R}} \phi(|y_1 - y_2|)\mu(dy_1 dy_2) \\ &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \{\phi(|y_1 - y_2|) - \phi(|y_1 - y_0|) - \phi(|y_2 - y_0|)\} \times \\ &\qquad\qquad k_2(y_0, dy_2)\, k_1(y_0, dy_1)\, \nu_0(dy_0).\end{aligned}$$

Since ϕ is strictly concave and strictly increasing, the right side is always nonpositive and is equal to zero only if y_0 is equal to either y_1 or y_2 almost surely with respect to the measure in the last line of the above display. Thus with respect to the measure ν_0, either $k_1(y, \mathrm{cot})$ is concentrated on $\{y\}$ or $k_2(y, \cdot)$ is concentrated on $\{y\}$. Let A be the subset of $\mathbb{R}$ on which the former holds. Then we have from (4.2) that

$$\mu|_{D\cap A} \geq \mu_1 \geq \nu_0 \quad \text{and} \quad \mu|_{D\cap A^c} \geq \mu_2 \geq \nu_0.$$

It follows that $\mu|_D \geq \nu_0$ and therefore $\mu|_D = \nu_0$, which is what we wanted to prove.

We now investigate μ off diagonal. Recall that

$$\mu_1 = \nu_0 + \nu_1, \qquad \mu_2 = \nu_0 + \nu_2.$$

It is known from what we have shown that an optimal μ always leaves the part ν_0 unchanged. Moreover, the measures ν_1 and ν_2 are supported, respectively, on the two half intervals S_1 and S_2 separated by the point $(x_1 + x_2)/2$. In this case the intuitive idea of the proof is that transporting a mass from a point $y_1 \in S_1$ to $y_2 \in S_2$ costs the same as transporting the same mass from $\mathscr{R}y_1$ to $\mathscr{R}y_2$, but the two transports together are more expensive than the the two transports of y_1 to $\mathscr{R}y_1$ and of y_2 to $\mathscr{R}y_2$.

To make this argument rigorous, we first note that with the notation established in the first part of the proof μ can be written as

$$\mu(dy_1, dy_2) = \delta_{y_1}(dy_2)\nu_0(dy_1) + k_3(y_1, dy_2)\,\nu_1(dy_1).$$

Here $y_2 \in S_2$ almost surely with respect to $k_1(y_1, .)$ whenever $y_1 \in S_1$. Comparing the transportation costs of μ with that of the mirror coupling m yields the equality

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}} &\phi(|y_1 - y_2|)m(dy_1 dy_2) - \int_{\mathbb{R}\times\mathbb{R}} \phi(|y_1 - y_2|)\mu(dy_1 dy_2)\\ &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \{\phi(|y_1 + \mathscr{R}y_1|) + \phi(|y_2 + \mathscr{R}y_2|) - 2\phi(|y_1 + y_2|)\} \times \\ &\qquad\qquad\qquad\qquad k_3(y_1, dy_2)\nu_1(dy_1).\end{aligned}$$

Again, by the strict concavity of ϕ, the right side is always nonpositive and vanishes only if $y_2 = \mathscr{R}y_1$ almost surely with respect to the measure in the last line of the above display. This means that almost surely with respect to ν_1, the measure $k_3(y, \cdot)$ is concentrated on $\{\mathscr{R}y\}$. Combining the two parts, we see that the optimal coupling μ must be the mirror coupling. □

Theorem 4.2. *If the cost function is ϕ_s, then the cost of the mirror coupling $m(x_1, x_2; t)$ is $\phi_{s+t}(|x_1 - x_2|)$. Thus*

$$C_{\phi_s}(\mu) \geq \phi_{s+t}(|x_1 - x_2|)$$

and

$$C_{\phi_s}(m(x_1, x_2; t)) = \phi_{s+t}(|x_1 - x_2|).$$

Proof. Let (Z_1, Z_2) be mirror coupled, i.e, they have the law $m(x_1, x_2; t)$. Let $x_0 = (x_1 + x_2)/2$ and $r = (x_1 - x_2)/2$. Let

$$q(t, z_1, z) = p\,(t, z_1, z) - p\,(t, z_1, -z - 2x_0).$$

It is the density function of a Brownian motion B_t on (x_0, ∞) killed at the boundary. A straightforward computation shows that the random variable $|Z_1 - Z_2|/2$ has the density $q(t, r, z)$ on (x_0, ∞) and has a point mass at 0 of the size

$$1 - \int_{\mathbb{R}} q(t, r, z)\, dz.$$

Therefore $|Z_1 - Z_2| I_{\{|Z_1 - Z_2| > 0\}}$ and $2B_t I_{\{\tau > t\}}$ have the same law. On the other hand, $\phi_s(\rho) = \mathbb{P}_{\rho/2}\{\tau \geq s\}$ for a positive ρ, where τ is the first time of reaching 0 of the Brownian motion B starting from $\rho/2$. Using the Markov property of Brownian motion we have

$$\phi_{s+t}(2r) = \mathbb{P}_r\{\tau \geq s + t\} = \mathbb{E}_r\{\mathbb{P}_{B_t}[\tau \geq s]\,;\ \tau > t\}.$$

We have $\mathbb{P}_{B_t}[\tau \geq s] = \phi_s(2B_t)$. Therefore,

$$\phi_{s+t}(2r) = \mathbb{E}_r\{\phi_s(2B_t);\ \tau > t\} = \mathbb{E}_r \phi_s(|Z_1 - Z_2|).$$

□

For calculations related to the above proof, see Sturm[7], EXAMPLE 4.6.

5. SECOND PROOF OF THE UNIQUENESS

Let s and t be positive and assume that the coupling $X = (X_1, X_2)$ is maximal at time $s + t$. This means that

$$\phi_{s+t}(|x_1 - x_2|) = \mathbb{P}\{T(X_1, X_2) > s + t\}.$$

If $X_1(s + t) \neq X_2(t + s)$, certainly the coupling time $T(X_1, X_2) > s + t$, hence

$$\phi_{s+t}(|x_1 - x_2|) \geq \mathbb{P}\{X_1(s + t) \neq X_2(t + s)\}.$$

Since the coupling is Markovian, conditioned on $\mathscr{F}_s$ the random variables $X_1(s + t)$ and $X_2(s + t)$ have the Gaussian distribution of the variance t and means $X_1(s)$ and $X_2(s)$. Therefore there probability of not coincide is at least $1/2$ of the total variation of the difference of their distributions (see the proof of PROPOSITION 2.1, hence

$$\begin{aligned}\mathbb{P}\{X_1(s + t) \neq X_2(t + s) | \mathscr{F}_s\} &\geq \frac{1}{2} \int_{\mathbb{R}} |p\,(t, X_1(s), z) - p\,(t, X_2(s), z)|\, dz \\ &= \phi_t(|X_1(s) - X_2(s)|).\end{aligned}$$

It follows that

$$\phi_{s+t}(|x_1 - x_2|) \geq \mathbb{E}\,\phi_t(|X_1(s) - X_2(s)|). \tag{5.1}$$

We recognize that the right side is precisely cost of coupling of two Gaussian random variables with the same variance s and means x_1 and x_2 with the strictly concave cost function ϕ_t. By PROPOSITIONS 4.1 and 4.2, the minimum of this cost is attained only when $X_1(s)$ and $X_2(s)$ are mirrored coupled and in this case the total cost is equal to $\phi_{s+t}(|x_1 - x_2|)$. It follows that

$$\phi_{s+t}(|x_1 - x_2|) = \mathbb{E}\phi_t\left(|X_1(s) - X_2(s)|\right)$$

and $X_1(s)$ and $X_2(s)$ must be mirror coupled. To sum up, we have shown that if the coupling $X = (X_1, X_2)$ is maximal at a time t, then $X_1(s)$ and $X_2(s)$ must be mirror coupled Gaussian random variables for all $0 \leq s \leq t$.

Now suppose that $X = (X_1, X_2)$ is a maximal Markov coupling (for all time). Then by what we have proved, at any time t, we must have either $X_2(t) = X_1(t)$ or $X_2(t) = \mathscr{R}X_1(t) = x_1 + x_2 - X_1(t)$. Therefore before the first time they meet, we must always have the second alternative. It follows that the first time they meet must be the first passage time of X_1 to the middle point $(x_1 + x_2)/2$ and, by the maximality, they must coincide afterwards. This means exactly that X_2 is the mirror coupling of X_1, and the proof of the main THEOREM 2.3 is completed.

6. CONCLUDING REMARKS

Given two probability measures μ_1, μ_2 on $\mathbb{R}^n$, it is not clear if a maximal Markovian coupling always exists. It may happen that the probability $\mathbb{P}\{T(X_1, X_2) \geq t\}$ can be minimized for each fixed t but not at the same coupling for all t. In this respect, we can obtain some positive results by taking advantage of certain situations in which the unique minimizers are independent of the choice of strictly concave function ϕ. This is the case, for example, if $(\mu_1 - \mu_2)_+$ is supported on a half space and $(\mu_1 - \mu_2)_-$ is the reflection of $(\mu_1 - \mu_2)_+$ in the other half space, or if $(\mu_1 - \mu_2)_+$ is supported on an open ball and $(\mu_1 - \mu_2)_-$ is the spherical image of $(\mu_1 - \mu_2)_-$ (see EXAMPLE 1.5 in Gangbo and McCann [3]). Suppose that there is a measure m which uniquely minimizes the cost $C_\phi(\mu)$ for $\mu \in \mathscr{C}(\mu_1, \mu_2)$. It can be shown that the unique maximal Markovian coupling of Brownian motions starting from the initial distributions (μ_1, μ_2) and it is given by

$$\int_{\mathbb{R}} \mathscr{N}^{x_1, x_2}\, m(dx_1 dx_2),$$

where $\mathscr{N}^{x_1, x_2}$ denotes the law of the mirror coupling of Brownian motions from (x_1, x_2).

The second proof can be generalized to certain Riemannian manifolds with symmetry such as complete simply connected manifold of constant curvature (space forms). However, it is known whether a maximal coupling exists for Brownian motion on a general Riemannian manifold. The results Griffeath[4] on couplings of Markov chains seems to indicate that a maximal coupling always exists but may not be Markovian but we do not believe a proof of this statement exists in the literature. The body of works on couplings so far seems to point to the belief that a maximal coupling is in general non-Markovian.

REFERENCES

[1] Burdzy, K. and Kendall, W. S., Efficient Markovian couplings: examples and counterexamples, *Ann. Appl. Prob.*, **10** (2000) 362-409
[2] Chen, M. F., Optimal Markovianian couplings and applications, *Acta Math. Sinica* (New Series), **10**, no. 3 (1994), 260–275.
[3] Gangbo, M. and McCann, R. J., The geometry of optimal transportation, *Acta Math.*, **177** (1996), 113-161.
[4] Griffeath, D., A maximal coupling for Markov chains, *Z. Wahr.*, **31** (1975), 94–106.
[5] Lindvall, T., *Lectures on the Coupling Method*, John Wiley & Sons, Chichester and New York, 1992.
[6] Revuz, D. and Yor, M., *Continuous Martingales and Brownian Motion*, 2nd edition, Springer-Verlag (1994).
[7] Sturm. K. T., A semigroup approach to harmonic maps, preprint SFB 611 Bonn, **39** (2003).
[8] Villani, C., *Topics in mass transportation*, Graduate Studies in Mathematics, vol. 57, AMS, Providence, RI (2003).

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