Volume Growth and Escape Rate of Brownian Motion on a Cartan–Hadamard Manifold

Alexander Grigor'yan and Elton Hsu

Abstract We prove an upper bound for the escape rate of Brownian motion on a Cartan–Hadamard manifold in terms of the volume growth function. One of the ingredients of the proof is the Sobolev inequality on such manifolds.

1 Introduction

Let M be a geodesically complete noncompact Riemannian manifold. We denote by $d(x, y)$ the geodesic distance between x and y and by μ the Riemannian volume measure. We use $\mathbb{P}_x$ to denote the diffusion measure generated by the Laplace–Beltrami operator Δ. Let $X = \{X_t,\, t \in \mathbb{R}_+\}$ be the coordinate process on the path space $W(M) = C(\mathbb{R}_+, M)$. By definition, $\mathbb{P}_x$ is a probability measure on $W(M)$ under which X is a Brownian motion starting from x.

Fix a reference point $z \in M$, and let $\rho(x) = d(x, z)$. We say that a function $R(t)$ is an *upper rate function* for Brownian motion on M if

$$\mathbb{P}_z\{\rho(X_t) \leqslant R(t) \text{ for all sufficiently large } t\} = 1.$$

The purpose of this paper is to study the rate of escape of Brownian motion on M in terms of the volume growth function. Let us first point out that the notion of an upper rate function makes sense only if the lifetime of Brownian motion is infinite. In this case, the manifold M is called stochastically

Alexander Grigor'yan
University of Bielefeld, 33501 Bielefeld, Germany,
e-mail: grigor\@math.uni-bielefeld.de

Elton Hsu
Northwestern University, Evanston, IL 60208, USA,
e-mail: ehsu\@math.northwestern.edu

complete. The stochastic completeness is equivalent to the identity

$$\int_M p\,(t,x,y)\,d\mu(y) = 1,$$

where $p(t,x,y)$ is the minimal heat kernel on M, which is also the transition density function of Brownian motion on M.

Let $B(z,R)$ be the geodesic ball of radius R centered at z. It was proved [3] that M is stochastically complete if

$$\int^{\infty} \frac{r\,dr}{\log \mu(B(z,R))} = \infty. \tag{1.1}$$

The integral in (1.1) will be used in this paper to construct an upper rate function. Before we state the result, let us briefly survey the existing estimates of escape rate.

- The classical Khinchin law of the iterated logarithm says that for a Brownian motion in $\mathbb{R}^n$ with probability 1

  $$\limsup_{t\to\infty} \frac{\rho(X_t)}{\sqrt{4t\log\log t}} = 1$$

  (the factor 4 instead of the classical 2 appears because, in our setting, a Brownian motion is generated by Δ rather than $\frac{1}{2}\Delta$). It follows that for any $\varepsilon > 0$

  $$R(t) = \sqrt{(4+\varepsilon)t\log\log t} \tag{1.2}$$

  is an upper rate function.
- If M has nonnegative Ricci curvature, then (1.2) is again an upper rate function on M (see [9, Theorem 1.3] and [7, Theorem 4.2]).
- If the volume growth function is at most polynomial, i.e.,

  $$\mu(B(z,r)) \leqslant Cr^D$$

  for large enough r and some positive constants C and D, then the function

  $$R(t) = \text{const}\ \sqrt{t\log t} \tag{1.3}$$

  is an upper rate function (see [15, Theorem 5.1], [7, Theorem 1.1], and [9, Theorem 1.1]). Note that the logarithm in (1.3) is single in contrast to (1.2) and, in general, cannot be replaced by the iterated logarithm (see [1, 10]).
- If the volume growth function admits a sub-Gaussian exponential estimate

  $$\mu(B(z,r)) \leqslant \exp(Cr^{\alpha})$$

where $0 < \alpha < 2$, then the function

$$R(t) = \text{const } t^{\frac{1}{2-\alpha}}$$

is an upper rate function (see [7, Theorem 4.1]).

Note that (1.1) is satisfied if the volume growth function admits the Gaussian exponential estimate

$$\mu(B(z,r)) \leqslant \exp(Cr^2) \tag{1.4}$$

(under the condition (1.4), the stochastic completeness was also proved by different methods in [15], [12], and [2]). However, none of the existing results provided any estimates of escape rate under the condition (1.4), let alone under the volume growth function $\exp(Cr^2 \log r)$ and the like.

We construct an upper rate function under the most general condition (1.1). However, we assume, in addition, that M is a Cartan–Hadamard manifold, i.e., a geodesically complete simply connected Riemannian manifold of nonnegative sectional curvature. The property of Cartan–Hadamard manifolds that we use is the Sobolev inequality: if $N = \dim M$, then for any function $f \in C_0^\infty(M)$

$$\left(\int\limits_M |f|^{\frac{N}{N-1}} d\mu \right)^{\frac{N-1}{N}} \leqslant C_N \int\limits_M |\nabla f| d\mu, \tag{1.5}$$

where C_N is a constant depending only on N (see [11]). The Sobolev inequality allows us to carry through the Moser iteration argument in [14] and prove a mean value estimate for solutions of the heat equation on M, which is one of the ingredients of our proof.

Now, we state our main result.

Theorem 1.1. *Let M be a Cartan–Hadamard manifold. Assume that the following volume estimate holds for a fixed point $z \in M$ and all sufficiently large R :*

$$\mu(B(z,R)) \leqslant \exp(f(R)), \tag{1.6}$$

where $f(R)$ is a positive, strictly increasing, and continuous function on $[0, +\infty)$ such that

$$\int^{\infty} \frac{r\,dr}{f(r)} = \infty. \tag{1.7}$$

Let $\varphi(t)$ be the function on $\mathbb{R}_+$ defined by

$$t = \int\limits_0^{\varphi(t)} \frac{r\,dr}{f(r)}. \tag{1.8}$$

Then $R(t) = \varphi(C\,t)$ is an upper rate function for Brownian motion on M for some absolute constant C (for example, for any $C > 128$).

If we set $f(R) = \log \mu(B(z,R))$ for large R, then the condition (1.7) becomes identical to (1.1). Under this condition, Theorem 1.1 guarantees the existence of an upper rate function $R(t)$. This, in particular, means that in a finite time Brownian motion stays with probability one in a bounded set, which implies that the life time of Brownian motion is infinite almost surely. Hence the manifold M is stochastically complete. This recovers the above cited result that (1.1) on geodesically complete manifolds implies the stochastic completeness, although under the additional assumption that M is Cartan–Hadamard.

Let us show some examples.

- If
$$\mu(B(z,R)) \leqslant CR^D \tag{1.9}$$
for some constants C and D, then (1.6) holds with
$$f(R) = D\log R + \text{const}$$
and (1.8) yields
$$t \simeq \frac{\varphi^2}{2D\log\varphi}.$$
It follows that $\log t \simeq \log\varphi^2$ and
$$\varphi(t) \simeq \sqrt{Dt\log t}.$$
Hence the function
$$R(t) = \sqrt{CDt\log t}$$
is an upper rate function which matches the above cited results of [7, 9, 15].
- If $\mu(B(z,R)) \leqslant \exp(Cr^\alpha)$ for some $0 < \alpha < 2$, then (1.6) holds with $f(R) = Cr^\alpha$ and (1.8) yields $t \simeq \varphi(t)^{2-\alpha}$. Hence we obtain the upper rate function
$$R(t) = Ct^{\frac{1}{2-\alpha}}$$
which matches the above cited result [7].
- If
$$\mu(B(z,R)) \leqslant \exp(CR^2),$$
then $f(R) = CR^2$. Then (1.8) yields $t \simeq \ln\varphi(t)$. Hence we obtain the upper rate function
$$R(t) = \exp(Ct).$$
This result is new. Similarly, if
$$\mu(B(z,R)) \leqslant \exp(CR^2\log R),$$

then (1.8) yields $t \simeq \log\log\varphi$. Hence

$$R(t) = \exp(\exp Ct)).$$

The hypothesis that M is Cartan–Hadamard can be replaced by the requirement that the Sobolev inequality (1.5) holds on M. Furthermore, the method goes through also in the setting of weighted manifolds, when measure μ is not necessarily the Riemannian measure, but has a smooth positive density, say $\sigma(x)$, with respect to the Riemannian measure. Then, instead of the Laplace–Beltrami operator, one should consider the weighted Laplace operator

$$\Delta_\mu = \frac{1}{\sigma}\operatorname{div}(\sigma\nabla)$$

which is symmetric with respect to μ. Theorem 1.1 extends to the weighted manifolds that are geodesically complete and satisfy the Sobolev inequality (1.5).

This paper is organized as follows. Section 2 contains the proof of an upper bound for certain positive solutions of the heat equation. In Sect. 3, we prove the main result stated above. In Sect. 4, we compute the sharp upper rate function on model manifold and show that, for a certain range of volume growth functions, the upper rate function of Theorem 1.1 is sharp up to a constant factor in front of t.

2 Heat Equation Solution Estimates

In this section, we prove a pointwise upper bound of certain solutions of the heat equation on a Cartan–Hadamard manifold M (Theorem 2.3). It is an easy consequence of an L^2-bound for a general complete manifold and a mean value type inequality for a Cartan–Hadamard manifold. These two upper bounds are known, and we state them as lemmas.

For any set $A \subset M$ let A_r be the open r-neighborhood of A in M.

Lemma 2.1. *Let M be a geodesically complete Riemannian manifold. Suppose that $u(x,t)$ is a smooth subsolution to the heat equation in the cylinder $A_r \times [0,T]$, where $A \subset M$ is a compact set and $r, T > 0$ (see Fig. 1). Assume also that $0 \leqslant u(x,t) \leqslant 1$ and $u(x,0) = 0$ on A_r. Then for any $t \in (0,T]$*

$$\int_A u^2(x,t)\,d\mu(x) \leqslant \mu(A_r)\max\left(1, \frac{r^2}{2t}\right)\exp\left(-\frac{r^2}{2t} + 1\right).$$

For the proof see [6, Theorem 3] (see also [7, Proposition 3.6]). Note that no geometric assumption about M is made except for the geodesic completeness.

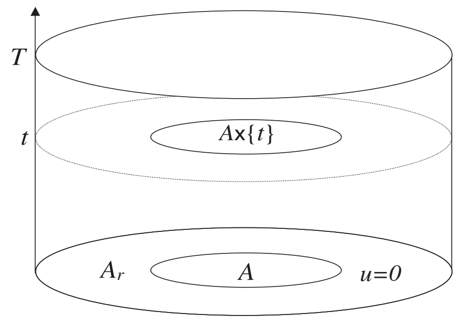

Fig. 1 Illustration to Lemma 2.1.

The proof exploits essentially the property of the geodesic distance function that $|\nabla d| \leqslant 1$.

Takeda [15] proved a similar estimate for $\int_A u(x,t)d\mu(x)$ by using a different probabilistic argument. However, it is more convenient for us to work with the L^2 rather than with L^1 estimate in view of the following lemma.

Lemma 2.2. *Let M be a Cartan–Hadamard manifold of dimension N. Suppose that $u(x,t)$ is a smooth nonnegative subsolution to the heat equation in a cylinder $B(y,r)\times[0,T]$, where $r,T>0$ (see Fig. 2). Then*

$$u(y,T)^2 \leqslant \frac{C_N}{\min(\sqrt{T},\, r)^{N+2}} \int_0^T \int_{B(y,r)} u^2(x,t)\, d\mu(x)dt, \tag{2.1}$$

where C_N is a constant depending only on N.

Proof. As was already mentioned above, a Cartan–Hadamard manifold admits the Sobolev inequality (1.5). By a standard argument, (1.5) implies the Sobolev–Moser inequality

$$\int_M |f|^{2+\frac{4}{N}}\, d\mu \leqslant C_N \left(\int_M |f|^2\, d\mu \right)^{2/N} \int_M |\nabla f|^2\, d\mu,$$

which leads, by the Moser iteration argument [14], to the mean value inequality (2.1). Note that the value of C_N may be different in all the above inequalities.

An alternative proof of the implication (1.5)⇒(2.1) can be found in [4] (see also [5, Theorem 3.1 and formula (3.4)]). □

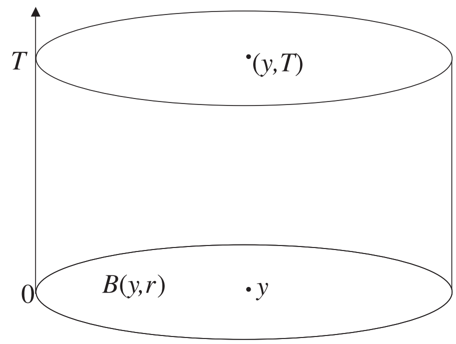

Fig. 2 Illustration to Lemma 2.2.

With these two preliminary results, we prove the main inequality in this section. This inequality gives an upper bound for probabilities of escaping times not exceeding a given upper bound.

Theorem 2.3. *Let M be a Cartan–Hadamard manifold of dimension N. Suppose that $u(x,t)$ is a smooth subsolution to the heat equation in a cylinder $B(y,2r)\times[0,T]$, where $r,T>0$. If $0\leqslant u\leqslant 1$ in this cylinder and $u(x,0)=0$ on $B(y,2r)$, then*

$$u(y,T)\leqslant C_N\sqrt{\mu(B(y,2r))}\frac{\max(\sqrt{T},r)}{\min(\sqrt{T},r)^{1+N/2}}\exp\left(-\frac{r^2}{4T}\right). \tag{2.2}$$

Proof. We use Lemma 2.1 with $A=\overline{B(y,r)}$. Then $A_r=B(y,2r)$ and for any $0<t\leqslant T$

$$\int\limits_{B(y,r)} u^2(x,t)d\mu(x)\leqslant \mu(B(y,2r))\max\left(1,\frac{r^2}{2t}\right)\exp\left(-\frac{r^2}{2t}+1\right)$$

(see Fig. 3). Hence

$$\int\limits_{T/2}^{T}\int\limits_{B(y,r)} u^2(x,t)d\mu(x)dt$$
$$\leqslant 2\mu(B(y,2r))T\max\left(1,\frac{r^2}{T}\right)\exp\left(-\frac{r^2}{2T}\right).$$

Applying Lemma 2.2 in the cylinder $B(y,r)\times[T/2,T]$, we obtain

$$u(y,T)^2 \leqslant C_N \mu(B(y,2r)) \frac{\max(T,r^2)}{\min(\sqrt{T},r)^{N+2}} \exp\left(-\frac{r^2}{2T}\right).$$

Hence (2.2) follows. □

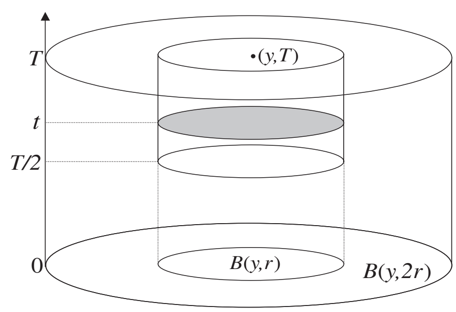

Fig. 3 Illustration to the proof of Theorem 2.3.

3 Escape Rate of Brownian Motion

We first explain the main idea of the proof. For any open set $\Omega \subset M$ we denote by τ_Ω the first exit time from Ω, i.e.,

$$\tau_\Omega = \inf\{t > 0 : X_t \notin \Omega\}.$$

Recall that $B(x,r)$ denotes the geodesic ball of radius r centered at x. Fix a reference point $z \in M$ and set $\rho(x) = d(x,z)$.

Let $\{R_n\}_{n=1}^\infty$ be a sequence of strictly increasing radii to be fixed later such that $\lim_{n\to\infty} R_n = \infty$. Consider the following sequence of stopping times:

$$\tau_n = \tau_{B(z,R_n)}.$$

Then $\tau_n - \tau_{n-1}$ is the amount of time the Brownian motion X_t takes to cross from $\partial B(z,R_{n-1})$ to $\partial B(z,R_n)$ for the first time (if $n=0$, then we set $R_0 = 0$ and $\tau_0 = 0$). Let $\{c_n\}_{n=1}^\infty$ be a sequence of positive numbers to be fixed later. Suppose that we can show that

$$\sum_{n=1}^\infty \mathbb{P}_z\{\tau_n - \tau_{n-1} \leqslant c_n\} < \infty. \tag{3.1}$$

Then, by the Borel–Cantelli lemma, with $\mathbb{P}_z$-probability 1 we have

$$\tau_n - \tau_{n-1} > c_n \quad \text{for all large enough } n. \tag{3.2}$$

For any $n \geqslant 1$ we set

$$T_n = \sum_{k=1}^{n} c_k.$$

From (3.2) it follows that for all sufficiently large n

$$\tau_n > T_n - T_0,$$

where T_0 is a large enough (random) number. In other words, we have the implication

$$t \leqslant T_n - T_0 \ \Rightarrow\ \rho(X_t) \leqslant R_n \text{ if } \ n \text{ is large enough.} \tag{3.3}$$

Let ψ be an increasing bijection of $\mathbb{R}_+$ onto itself such that

$$T_{n-1} - \psi(R_n) \to +\infty \text{ as } n \to \infty. \tag{3.4}$$

We claim that ψ^{-1} is an upper rate function. Indeed, for large enough t we choose n such that

$$T_{n-1} - T_0 < t \leqslant T_n - T_0.$$

If t is large enough, then also n is large enough so that, by (3.3),

$$\rho(X_t) \leqslant R_n$$

and, by (3.4),

$$T_{n-1} - \psi(R_n) > T_0.$$

It follows that

$$t > T_{n-1} - T_0 > \psi(R_n).$$

Hence

$$\rho(X_t) \leqslant R_n < \psi^{-1}(t),$$

which proves that ψ^{-1} is an upper rate function.

Now, let us find c_n such that (3.1) is true. By the strong Markov property of Brownian motion, we have

$$\mathbb{P}_z\{\tau_n - \tau_{n-1} \leqslant c_n\} = \mathbb{E}_z \mathbb{P}_{X_{\tau_{n-1}}}\{\tau_n \leqslant c_n\}. \tag{3.5}$$

Note that $X_{\tau_{n-1}} \in \partial B(z, R_{n-1})$. If a Brownian motion starts from a point $y \in \partial B(z, R_{n-1})$, then it has to travel no less than the distance

$$r_n = R_n - R_{n-1}$$

before it reaches $\partial B(z, R_n)$ (see Fig. 4). Hence

$$\mathbb{P}_y\{\tau_n \leqslant c_n\} \leqslant \mathbb{P}_y\{\tau_{B(y,r_n)} \leqslant c_n\}, \quad y \in \partial B(z, R_{n-1}).$$

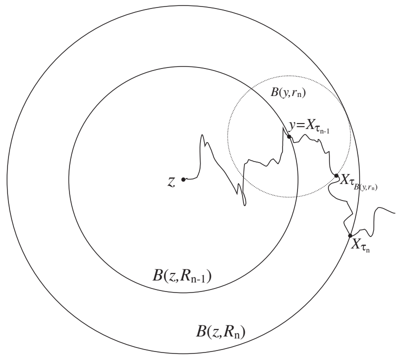

Fig. 4 Brownian motion X_t exits the ball $B(y, r_n)$ before $B(z, R_n)$.

From the above inequality and (3.5) we obtain

$$\mathbb{P}_z\{\tau_n - \tau_{n-1} \leqslant c_n\} \leqslant \sup_{y \in \partial B(z, R_{n-1})} \mathbb{P}_y\{\tau_{B(y,r_n)} \leqslant c_n\}. \tag{3.6}$$

For a fixed $y \in \partial B(z, R_{n-1})$ we consider the function

$$u(x,t) = \mathbb{P}_x\{\tau_{B(y,r)} \leqslant t\}.$$

Clearly, $u(x,t)$ is the solution of the heat equation in the cylinder $B(y,r) \times \mathbb{R}_+$. Furthermore, $0 \leqslant u \leqslant 1$ and

$$u(x, 0) = 0 \quad \text{for } x \in B(y, r).$$

The probability we wanted to estimate is the value of the solution at the center of the ball:

$$\mathbb{P}_y\{\tau_{B(y,r_n)} \leqslant c_n\} = u(y, c_n).$$

Applying the estimate (2.2) of Theorem 2.3 in the cylinder $B(y, r_n) \times [0, c_n]$ and noting that $B(y, r_n) \subset B(z, R_n)$ so that

$$\mu(B(y, r_n)) \leqslant \exp(f(R_n)),$$

we obtain

$$u(y, c_n) \leqslant C_N \exp(f(R_n)/2) \frac{\max(\sqrt{c_n}, r_n)}{\min(\sqrt{c_n}, r_n)^{1+N/2}} \exp\left(-\frac{r_n^2}{16 c_n}\right). \tag{3.7}$$

Now, we choose c_n to satisfy the identity

$$\frac{r_n^2}{16c_n} = f(R_n)$$

i.e.,

$$c_n = \frac{1}{16}\frac{r_n^2}{f(R_n)}.$$

Noticing that $c_n < r_n^2$ for large enough n, form (3.7) we obtain

$$\begin{aligned}
\mathbb{P}_y\{\tau_{B(y,r_n)} \leqslant c_n\} &\leqslant C_N \frac{r_n}{\sqrt{c_n}^{1+N/2}} \exp(-f(R_n)/2) \\
&= C_N r_n^{-N/2} f(R_n)^{\frac{2+N}{4}} \exp(-f(R_n)/2) \\
&\leqslant C_N r_n^{-N/2}.
\end{aligned}$$

Set now $R_n = 2^n$ so that $r_n = 2^{n-1}$. The above estimate together with (3.6) obviously implies

$$\sum_{n=1}^{\infty} \mathbb{P}_z\{\tau_n - \tau_{n-1} \leqslant c_n\} \leqslant C_N \sum_{n=1}^{\infty} r_n^{-N/2} < \infty,$$

i.e., (3.1).

Knowing the sequences $\{R_n\}$ and $\{c_n\}$, we can now determine a function ψ that satisfies (3.4). Indeed, we have

$$\begin{aligned}
T_n = c_1 + \ldots + c_n &= \frac{1}{16}\sum_{k=1}^{n} \frac{r_k^2}{f(R_k)} = \frac{1}{128}\sum_{k=1}^{n} \frac{R_{k+1}(R_{k+1} - R_k)}{f(R_k)} \\
&\geqslant \frac{1}{128}\sum_{k=1}^{n} \int_{R_k}^{R_{k+1}} \frac{rdr}{f(r)} = \frac{1}{128}\int_{R_1}^{R_{n+1}} \frac{rdr}{f(r)}.
\end{aligned}$$

Setting

$$\psi(r) = c\int_0^r \frac{rdr}{f(r)},$$

where $c < \frac{1}{128}$, and using (1.7), we obtain

$$T_n - \psi(R_{n+1}) \to \infty \text{ as } n \to \infty,$$

which is equivalent to (3.4). Therefore, ψ^{-1} is an upper rate function. Clearly, $\psi^{-1}(t) = \varphi(Ct)$, where φ is defined by (1.8) and $C = c^{-1}$, which completes the proof of our main result, Theorem 1.1.

4 Escape Rate on Model Manifolds

In this section, we compute sharp upper rate function on model manifolds and compare it to the one from Theorem 1.1. We first illustrate the method in a simple case when M is a hyperbolic space.

4.1 Constant curvature

Let M be the hyperbolic space $\mathbb{H}_K^N$ of dimension N and of the constant sectional curvature $-K$. Then $\mu(B(z,r)) \leqslant Ce^{(N-1)Kr}$ so that we can take $f(r) = (N-1)Kr$. Theorem 1.1 yields the following upper rate function:

$$R(t) = CK(N-1)t.$$

In this case, a sharp upper rate function can be computed as follows. The radial process $r_t = \rho(X_t)$ satisfies the identity (see [13])

$$r_t = \sqrt{2}\,W_t + (N-1)\int_0^t K \coth K r_s \, ds,$$

where W_t is a one-dimensional Brownian motion. We have

$$r_t \to \infty \;\text{ and }\; \frac{W_t}{t} \to 0$$

as $t \to \infty$. Therefore,

$$\frac{r_t}{t} \to (N-1)K.$$

Hence a sharp upper rate function is

$$R(t) = (1+\varepsilon)K(N-1)t,$$

where $\varepsilon > 0$.

4.2 General model manifolds

Here, M is not necessarily Cartan–Hadamard, but we do assume that M has a pole z, i.e., the exponential map $\exp : T_zM \to M$ is a diffeomorphism. Then the polar coordinates (ρ, θ) are defined on $M \setminus \{z\}$. The manifold M is said to be a *model* if the Riemannian metric of M is spherically symmetric, i.e., has the form

$$ds^2 = dr^2 + h(r)^2 d\theta^2, \tag{4.1}$$

where $h(r)$ is a smooth positive function of $r > 0$ and $d\theta^2$ is the canonical metric on $\mathbb{S}^{N-1}$ (note that θ varies in $\mathbb{S}^{N-1}$). For example, $\mathbb{R}^N$ is a model with $h(r) = r$ and the hyperbolic space $\mathbb{H}^N_K$ is a model with $h(r) = K^{-1} \sinh Kr$. The volume growth function of the metric (4.1) is

$$V(r) := \mu(B(z, r)) = \omega_N \int\limits_0^r h(s)^{N-1} ds,$$

where ω_N is the $(N-1)$-volume of the unit sphere $\mathbb{S}^{N-1}$. The Laplace operator of the metric (4.1) is represented in the polar coordinates as follows:

$$\Delta = \frac{\partial^2}{\partial r^2} + m(r) \frac{\partial}{\partial r} + \frac{1}{h^2(r)} \Delta_{\mathbb{S}^{n-1}},$$

where $\Delta_{\mathbb{S}^{n-1}}$ is the Laplacian in the variable θ with respect to the canonical metric of $\mathbb{S}^{N-1}$ and

$$m(r) := (N-1) \frac{h'}{h} = \frac{V''}{V'}.$$

The function $m(r)$ plays an important role in what follows. Clearly, m satisfies the identity

$$V'(r) = V'(r_0) \exp\Bigg(\int\limits_{r_0}^r m(s) ds \Bigg) \tag{4.2}$$

for all $r > r_0 > 0$. We assume in the sequel that

$$m(r) > 0 \ \text{ and } \ m'(r) \geqslant 0 \text{ for large enough } r \tag{4.3}$$

and

$$\int^\infty \frac{dr}{m(r)} = \infty. \tag{4.4}$$

For example, we have $m(r) = \frac{N-1}{r}$ in $\mathbb{R}^N$ and $m(r) = (N-1) K \coth Kr$ in $\mathbb{H}^N_K$. In neither case is the hypothesis (4.3) satisfied. On the other hand, if $V'(r) = \exp(r^\alpha)$, then $m(r) = \alpha r^{\alpha-1}$, and both (4.3) and (4.4) are satisfied provided that $1 \leqslant \alpha \leqslant 2$. If $V'(r) = \exp(r^2 \log^\beta r)$, then (4.3) and (4.4) hold for all $0 \leqslant \beta \leqslant 1$.

We claim that, under the condition (4.3), the Brownian motion on M is transient and, under the conditions (4.3)–(4.4), M is stochastically complete. We use the following well-known results (see [8]) that for model manifolds the transience is equivalent to

$$\int^{\infty} \frac{dr}{V'(r)} = \infty \tag{4.5}$$

and the stochastic completeness is equivalent to

$$\int^{\infty} \frac{V(r)}{V'(r)} dr = \infty. \tag{4.6}$$

Clearly, (4.3) implies $m(r) \geqslant c$ for some positive constant c and all large enough r. From (4.2) it follows that $V'(r)$ grows at least exponentially as $r \to \infty$, which implies (4.5). To prove (4.6), we observe that for large enough $r > r_0$

$$\begin{aligned} V(r) - V(r_0) &= \int_{r_0}^{r} V'(s) ds = \int_{r_0}^{r} \frac{V''(s)}{m(s)} ds \\ &\geqslant \frac{1}{m(r)} \int_{r_0}^{r} V''(s) ds = \frac{1}{m(r)} (V'(r) - V(r_0)). \end{aligned}$$

Therefore,

$$\frac{1}{m(r)} \leqslant \frac{V(r) - V(r_0)}{V'(r) - V'(r_0)} \sim \frac{V(r)}{V'(r)} \text{ as } r \to \infty.$$

Hence (4.6) follows from (4.4).

Let us define the function $r(t)$ by the identity

$$t = \int_0^{r(t)} \frac{ds}{m(s)}. \tag{4.7}$$

Our main result in this section is as follows.

Theorem 4.1. *Under the above assumptions, the function $r((1+\varepsilon)t)$ is the upper rate function for Brownian motion on M for any $\varepsilon > 0$, and is not for any $\varepsilon < 0$.*

Let us compare the function $r(t)$ with the upper rate function $R(t)$ given by Theorem 1.1, which is defined by the identity

$$\int_0^{R(t)} \frac{rdr}{\log V(r)} = Ct.$$

For "nice" functions $V(r)$ we have

$$\frac{V''}{V'} \simeq \frac{V'}{V} = (\log V)' \simeq \frac{\log V(r)}{r}, \tag{4.8}$$

which means that the functions $r(t)$ and $R(t)$ are comparable up to a constant multiple in front of t. For example, (4.8) holds for functions like $V(r) = \exp(r^\alpha)$ and $V(t) = \exp(r^\alpha \log^\beta r)$, where $\alpha > 0$, etc. On the other hand, it is easy to construct an example of $V(r)$ where $r(t)$ may be significantly less than $R(t)$ because one can modify a "nice" function $V(r)$ to make the second derivative $V''(r)$ very small in some intervals without affecting too much the values of V' and V. Then the function $r(t)$ in (4.7) will drop significantly, while $R(t)$ will not change very much.

Proof of Theorem 4.1. By the Ito decomposition, the radial process $r_t = \rho(X_t)$ satisfies the identity

$$r_t = \sqrt{2} W_t + \int_0^t m(r_s)\, ds, \tag{4.9}$$

where W_t is a one-dimensional Brownian motion (see [13]). Since the process X_t is transient, $r_t \to \infty$ as $t \to \infty$ with probability 1. Hence $m(r_t) \geqslant c$ for large enough t so that the second term on the right-hand side of (4.9) grows at least linearly in t. Since $W_t = o(t)$ as $t \to \infty$, we have with probability 1

$$r_t \sim \int_0^t m(r_s)\, ds \text{ as } t \to \infty. \tag{4.10}$$

Consider the function

$$u(t) = \int_0^t m(r_s) ds$$

From (4.10) it follows that for any $C > 1$ and large enough t

$$r_t \leqslant C u(t). \tag{4.11}$$

Hence, by the monotonicity of m,

$$m(r_t) \leqslant m(Cu(t)).$$

Since $\dfrac{du}{dt}(t) = m(r_t)$, we obtain the differential inequality for $u(t)$

$$\frac{du}{dt} \leqslant m(Cu(t)).$$

Solving it by separation of variables, for large enough t_0 and all $t > t_0$ we obtain

$$\int_{Cu(t_0)}^{Cu(t)} \frac{d\xi}{m(\xi)} \leqslant C(t-t_0).$$

Hence

$$\int_{0}^{Cu(t)} \frac{d\xi}{m(\xi)} \leqslant Ct + C_0, \tag{4.12}$$

where C_0 is a large enough (random) constant. Comparing (4.12) with (4.7) and using again (4.11), we obtain

$$r_t \leqslant Cu(t) \leqslant r(Ct + C_0) \leqslant r(C^2 t)$$

for large enough r with probability 1. Since $C > 1$ was arbitrary, this proves that $r((1+\varepsilon)t)$ is an upper rate function for any $\varepsilon > 0$. In the same way, one proves that $r_t \geqslant r(C^{-2}t)$ for large enough t so that $r((1-\varepsilon)t)$ is not an upper rate function. □

Acknowledgement. The research of the first author was supported by the German Research Council (grant no. SFB 701). The research of the second author was supported in part by the NSF (grant no. DMS-0407819). This work was done during a visit of the second author at the University of Bielefeld, which was supported by grant no. SFB 701.

References

1. Barlow, M.T., Perkins, E.A.: Symmetric Markov chains in $\mathbb{Z}^d$: how fast can they move? Probab. Th. Rel. Fields **82**, 95-108 (1989)
2. Davies. E.B.: Heat kernel bounds, conservation of probability and the Feller property. J. Anal. Math. **58**, 99-119, (1992)
3. Grigor'yan, A.: On stochastically complete manifolds (Russian). Dokl. Akad. Nauk SSSR **290**, no. 3, 534-537 (1986); English transl.: Sov. Math., Dokl. **34**, no. 2, 310-313 (1987)
4. Grigor'yan, A.: The heat equation on noncompact Riemannian manifolds (Russian). Mat. Sb. **182**, no. 1, 55-87 (1991); English transl.: Math. USSR–Sb. **72**, no. 1, 47-77 (1992)
5. Grigor'yan, A.: Heat kernel upper bounds on a complete noncompact manifold. Rev. Mat. Iberoam. **10**, no. 2, 395-452 (1994)
6. Grigor'yan, A.: Integral maximum principle and its applications. Proc. R. Soc. Edinb., Sect. A, Math. **124**, 353-362 (1994)
7. Grigor'yan, A.: Escape rate of Brownian motion on weighted manifolds. Appl. Anal. **71**, no. 1-4, 63-89 (1999)
8. Grigor'yan, A.: Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds. Bull. Am. Math. Soc. **36**, 135-249 (1999)

9. Grigor'yan, A., Kelbert M.: Range of fluctuation of Brownian motion on a complete Riemannian manifold. Ann. Prob. **26**, 78-111 (1998)
10. Grigor'yan, A., Kelbert M.: On Hardy–Littlewood inequality for Brownian motion on Riemannian manifolds. J. London Math. Soc. (2) **62**, 625-639 (2000)
11. Hoffman, D., Spruck, J.: Sobolev and isoperimetric inequalities for Riemannian submanifolds. Commun. Pure Appl. Math. **27**, 715–727 (1974); A correction to: "Sobolev and isoperimetric inequalities for Riemannian submanifolds." Commun. Pure Appl. Math. **28**, no. 6, 765–766 (1975)
12. Hsu, E.P.: Heat semigroup on a complete Riemannian manifold. Ann. Probab. **17**, 1248-1254 (1989)
13. Hsu, E.P.: Stochastic Analysis on Manifolds. Am. Math. Soc., Providence, RI (2002)
14. Moser J.: A Harnack inequality for parabolic differential equations. Commun. Pure Appl. Math. **17**, 101-134 (1964); Correction to "A Harnack inequality for parabolic differential equations." Commun. Pure Appl. Math. **20**, 231-236 (1967)
15. Takeda, M.: On a martingale method for symmetric diffusion process and its applications. Osaka J. Math **26**, 605-623 (1989)