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Stochastic Local Gauss-Bonnet-Chern Theorem[1]

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The Gauss-Bonnet-Chern theorem for compact Riemannian manifold (without boundary) is discussed here to exhibit in a clear manner the role Riemannian Brownian motion plays in various probabilistic approaches to index theorems. The method with some modifications works also for the index theorem for the Dirac operator on the bundle of spinors, see Hsu.[7]

KEY WORDS: Riemannian Brownian motion; index theorems; differential forms, Gauss-Bonnet-Chern formula.

1. INTRODUCTION

The Gauss-Bonnet-Chern theorem is the index theorem for the Hodge-de Rham Laplacian on differential forms. The more refined local Gauss-Bonnet-Chern theorem was conjectured by McKean and Singer,[12] and proved by Patodi.[13] Later Gilkey[6] gave a proof based on the more comprehensive theory of local invariants, which has applications beyond proving index theorems.

Since the appearance of Bismut's probabilistic proof of the local Atiyah-Singer index theorem (Bismut[2]) (for the Dirac operator on spinor bundles), there have appeared many works to reprove various forms of index theorems by probabilistic and analytic methods, see Azencott,[1] Cycon *et al.*,[3] Elworthy,[4] Getzler,[5] Ikeda and Watanabe,[10] Léandre,[11] and Shigekawa *et al.*[16] The approach to the index theorem presented in this article is based on the author's proof which chronologically was one of

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the many proofs shortly after Bismut's work. The early manuscript was widely circulated and cited but never appeared in print for various reasons. Although the basic ideas underlying the proof remains unchanged, we have greatly streamlined and improved the presentation by taking advantage of the hindsights gained during the years since the early draft. One obvious feature of it is that it is a technically simple proof from the probabilistic point of view because it uses nothing beyond the definition of horizontal Brownian motion and Itô's formula. Another feature of the proof is that we state the result in such a way that the "fantastic cancellation" conjectured by McKean and Singer[12] occurs on the path level, hence the title of the article. In the traditional spirit of probability in relation to analysis, one obtains Patodi's local Gauss-Bonnet-Chern theorem by taking expectation. The purpose of this article is mainly pedagogical and the so-called stochastic local Gauss-Bonnet-Chern theorem is perhaps the only result in this article that has not appeared (at least not explicitly) in literature.

Let us briefly describe the contents of the article. In Section 2, we state without proof some results from linear algebra and differential geometry needed for the exposition, especially Weitzenböck's formula:

$$\Delta_M = \Delta_M^B + D^*R$$

where Δ_M is the Hodge-de Rham Laplacian on a Riemannian manifold, Δ_M^B Bochner's covariant Laplacian, and D^*R a fibre-wise, degree-preserving operator on $\bigwedge^* M$ (the space of differential forms on M) determined by the curvature tensor.

In Section 3, following McKean and Singer[12] we describe the relation between the heat kernel $e^{t\Delta_M/2}(x, y)$ and the Euler characteristic $\chi(M)$ of a compact Riemannian manifold M, namely

$$\chi(M) = \int_M \phi\{e^{t\Delta_M/2}(x, x)\}\, dx$$

where $\phi(T)$ is the supertrace of a degree-preserving linear operator T on the space of differential forms.

In Section 4, using the Weitzenböck formula and the Feynman-Kac formula we represent $e^{t\Delta_M/2}(x, x)$ in terms of Riemannian Brownian bridge:

$$e^{t\Delta/2}(x, x) = E^t_{x,x}\{U_t M_t\}\ p(t, x, x)$$

where $E^t_{x,x}$ denotes the expectation with respect to a Brownian bridge at x with time length t, U_t is the (stochastic) parallel transport along the Brownian bridge, and $\{M_s, 0 \leqslant s \leqslant t\}$ is the Feynman-Kac functional given by an ordinary differential equation along the horizontal lift $\{U_s, 0, \leqslant s \leqslant t\}$ of the Brownian bridge:

$$\frac{dM_s}{ds} = \frac{1}{2} M_s D^* \Omega_{U_s}, \qquad M_s = I$$

Here $D^*\Omega$ is the scalarization of the D^*R in the Weitzenböck formula. Some of the details for this section can be found in Ikeda and Watanabe,[9] [pp. 298–308].

In Section 5, we show how to define Riemannian Brownian bridges of lifetime t on a common probability space and prove that the limit

$$\lim_{t \to 0} \phi\{U_t M_t\} \; p(t, x, x) = e(x)$$

exists in $L^N(P)$ for any $N \geqslant 0$, and $e(x)$ is identified as the Euler form. This is the content of the stochastic local Gauss-Bonnet-Chern theorem. The local Gauss-Bonnet-Chern theorem follows from this by taking expectation.

In Section 6, the last section, we prove an estimate on the Brownian homology U_t which is used in the proof of our main result. This part represents the only technical portion of the article and can be safely skipped at the first reading, for the estimate is merely a rigorous statement of an easily acceptable intuition.

2. ALGEBRAIC AND DIFFERENTIAL GEOMETRIC PRELIMINARIES

Let V be a n-dimensional vector space over $\mathbb{R}$. Typically in this article $V = \mathbb{R}^n$ or $V = T_x^* M$, the cotangent space of a Riemannian manifold M at a point x. We use

$$\bigwedge^* V = \bigwedge^0 V \oplus \cdots \oplus \bigwedge^n V$$

to denote the exterior algebra of V. The set of linear maps from V to itself is denoted by $\mathrm{End}(V) = V^* \otimes_{\mathbb{R}} V$. Each $T \in \mathrm{End}(V)$ extends naturally to the exterior algebra $\bigwedge^* V$ by

$$\bigwedge^* T(v^1 \wedge \cdots \wedge v^k) = Tv^1 \wedge \cdots \wedge Tv^k \tag{2.1}$$

If no confusion is possible we write $\bigwedge^* T$ simply as T. There is another extension of T on $\bigwedge^* V$ as a derivation:

$$D^*T(v^1 \wedge \cdots \wedge v^k) = \sum_{j=1}^{k} v^1 \wedge \cdots \wedge Tv^j \wedge \cdots \wedge v^k$$

Note that $\bigwedge^* T$ and D^*T are degree-preserving, i.e., it sends each $\bigwedge^p V$ into itself. If T is a linear map from a vector space into itself, its (operator or matrix) exponential is $e^T = \sum_{i=0}^{\infty} T^i/i!$. For a $T \in \mathrm{End}(V)$, the relation between these two extensions of T is

$$\bigwedge^* e^T = e^{D^*T} \tag{2.2}$$

For a degree-preserving linear map $T: \bigwedge^* V \to \bigwedge^* V$, we define the supertrace $\phi(T)$ by

$$\phi(T) = \sum_{p=0}^{n} (-1)^p \operatorname{Tr}(T|_{\bigwedge^p V})$$

We have the following algebraic fact, whose proof can be found in Patodi,[13] [p. 235].

Lemma 1. Let $T_i \in \mathrm{End}(V)$, $1 \leqslant i \leqslant k$. If $k < \dim V$, then

$$\phi(D^*T_1 \circ \cdots \circ D^*T_k) = 0$$

If $k = \dim V$, then

$$\phi(D^*T_1 \circ \cdots \circ D^*T_k) = (-1)^k \times \text{the coefficient of } x_1 \cdots x_k \text{ in } \det\left(\sum_{i=1}^{k} x_i T_i\right).$$

We define a bilinear map

$$D^*: \mathrm{End}(V) \otimes_{\mathbb{R}} \mathrm{End}(V) \to \mathrm{End}\left(\bigwedge^* V\right)$$

by

$$D^*(T_1 \otimes T_2) = D^*T_1 \circ D^*T_2. \tag{2.3}$$

From the preceding lemma we have immediately the following result.

Corollary 1. Let $T_m \in \mathrm{End}(V)$, $1 \leqslant m \leqslant j$, and $S_k \in \mathrm{End}(V) \otimes_{\mathbb{R}} \mathrm{End}(V)$, $1 \leqslant k \leqslant i$. If $2i + j < \dim V$ then

$$\phi(D^* T_1 \circ \cdots \circ D^* T_j \circ D^* S_1 \circ \cdots \circ D^* S_i) = 0$$

Let M be a compact, oriented Riemannian manifold. We denote by $O(M)$ the bundle of orthonormal frames on M, and by $\pi: O(M) \to M$ the canonical projection. An element $u \in O(M)$ can be regarded as an isometry from $\mathbb{R}^n$ to $T_{\pi u} M$. The canonical horizontal vector field H_i, $1 \leqslant i \leqslant n$, on $O(M)$ is defined at $u \in O(M)$ by $ue_i = \pi_*(H_{i,u})$, where e_i is the unit coordinate vector in $\mathbb{R}^n$ along the x^i-axis.

The vector bundle of differential form is the associated bundle

$$\bigwedge\nolimits^* M = \bigwedge\nolimits^* \mathbb{R}^n \times_{O(n)} O(M)$$

where $O(n)$ acts on $\bigwedge^* \mathbb{R}^n$ as an isometry (see (2.1)). Since $u: \mathbb{R}^n \to T_{\pi u} M$ is an isometry, it can be extended naturally to an isometry from $\bigwedge^* \mathbb{R}^n$ to $\bigwedge^*_{\pi u} M$, the fibre of the vector bundle $\bigwedge^* M$ at πu. Let $\alpha \in \Gamma(\bigwedge^* M)$ be a differential form on M (a section of $\bigwedge^* M$). The $\bigwedge^* \mathbb{R}^n$-valued function $S_\alpha: O(M) \to \bigwedge^* \mathbb{R}^n$ defined by $S_\alpha(u) = u^{-1}\alpha(\pi u)$ is called the scalarization of α.

Let $d: \bigwedge^* M \to \bigwedge^* M$ be the exterior differentiation and d^* the formal adjoint of d with respect to the canonical pre-Hilbert structure on $\bigwedge^* M$ defined by

$$\|\alpha\|^2 = \int_M |\alpha(x)|^2 \, dx$$

where dx is the Riemannian volume measure on M. The second order elliptic operator $\Delta_M = dd^* + d^*d$ is the Hodge-de Rham Laplacian. We define the lift $\Delta_{O(M)}$ of Δ_M to $O(M)$ on the set of $O(n)$-invariant, $\bigwedge^* \mathbb{R}^n$-valued functions on $O(M)$ by

$$\Delta_{O(M)} S_\alpha = S_{\Delta_M \alpha}, \qquad \alpha \in \Gamma\left(\bigwedge\nolimits^* M\right)$$

Bochner's Laplacian is defined by $\Delta^B_M = \mathrm{Tr}\, \nabla^2$, where ∇ is the Levi-Civita connection extended to differential forms. By definition we have

$$S_{\Delta^B_M \alpha} = \Delta^B_{O(M)} S_\alpha, \qquad \alpha \in \Gamma\left(\bigwedge\nolimits^* M\right)$$

where $\Delta^B_{O(M)} = \sum_{i=1}^n H_i^2$ is Bochner's horizontal Laplacian on $O(M)$. The connection between the Hodge-de Rham Laplacian and Bochner's Laplacian is given by the Weitzenböck formula. Let

$$R \in \Gamma(T^*M \otimes T^*M) \to \Gamma(T^*M \otimes TM)$$

be the curvature tensor of M. At each $x \in M$, by the canonical identification of T_x^*M with T_xM, we can regard the curvature tensor R_x as an element in $\mathrm{End}(T_x^*M) \otimes_{\mathbb{R}} \mathrm{End}(T_xM)$. Therefore $D^*R_x\colon \bigwedge_x^* M \to \bigwedge_x^* M$ is well defined by (2.3). Finally at each point $u \in O(M)$, we define the scalarization $D^*\Omega_u \in \mathrm{End}(\bigwedge^* \mathbb{R}^n)$ of D^*R by

$$D^*\Omega_u = u^{-1} \circ D^*R_{\pi u} \circ u.$$

Proposition 1 (Weitzenböck formula). We have

$$\Delta_M = \Delta_M^B + D^*R,$$

or equivalently

$$\Delta_{O(M)} = \Delta^B_{O(M)} + D^*\Omega. \tag{2.4}$$

3. HEAT EQUATION AND THE EULER CHARACTERISTIC

Consider the heat equation on a differential form $\alpha = \alpha(t, x)$ on M:

$$\frac{\partial \alpha}{\partial t} = \frac{1}{2} \Delta_M \alpha, \qquad \alpha(0, \cdot) = \alpha_0, \tag{3.1}$$

where Δ_M is the Hodge-de Rham Laplacian on $\Gamma(\bigwedge^* M)$. Theory of parabolic equations shows that there is a smooth heat kernel

$$e^{t\Delta_M/2}(x, y) \colon \bigwedge_y^* M \to \bigwedge_x^* M$$

with respect to the Riemannian volume measure such that the solution of (3.1) is given by

$$\alpha(t, x) = \int_M e^{\Delta_M t/2}(x, y)\, \alpha_0(y)\, dy \tag{3.2}$$

Theory of elliptic equations shows that the spectrum of Δ is discrete,

$$\mathrm{Spec}(\Delta_M)\colon \lambda_0 = 0 \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots \to -\infty,$$

and we have the usual L^2-expansion of the heat kernel

$$e^{t\Delta_M/2}(x, y) = \sum_{n=0}^{\infty} e^{\lambda_n t/2} \phi_n(x) \otimes \phi_n(y)$$

Let $\mu_0 = 0 > \mu_1 > \mu_2 > \cdots$ be the *distinct* eigenvalues of Δ_M. Each eigenspace E_i is finite dimensional and

$$E_i = E_i^0 \oplus E_i^1 \oplus \cdots \oplus E_i^n$$

where E_i^p is the subspace of p-forms in E_p. Since $e^{t\Delta_M/2}(x, x) \colon \bigwedge_x^* M \to \bigwedge_x^* M$ is degree-preserving, the supertrace $\phi\{e^{t\Delta_M/2}(x, x)\}$ is defined. Then a simple computation shows that

$$\int_M \phi\{e^{t\Delta_M/2}(x, x)\}\, dx = \sum_{i=0}^{\infty} e^{\mu_i t/2} \sum_{p=0}^{n} (-1)^p \dim E_i^p$$

The key observation, due to McKean and Singer,[12] is that

$$\sum_{p=0}^{n} (-1)^p \dim E_i^p = \begin{cases} 0, & \text{if } \mu_i \neq 0, \\ \chi(M), & \text{if } \mu_i = 0 \end{cases}$$

where $\chi(M)$ is the Euler characteristic of M. The case $\mu_i \neq 0$ is a consequence of the fact that both d and d^* preserves the eigenspace E_i. The case $\mu_i = 0$ follows from the Hodge-de Rham theory, which expresses cohomology groups $H^*(M)$ of M in terms harmonic forms. We have now arrived at an expression of the Euler characteristic in terms of the heat kernel, namely for all $t > 0$,

$$\chi(M) = \int_M \phi\{e^{\Delta_M t/2}(x, x)\}\, dx \tag{3.3}$$

4. PROBABILISTIC REPRESENTATION OF THE HEAT KERNEL

We express the heat kernel $e^{t\Delta_M/2}(x, y)$ in terms of Brownian bridge. Consider the heat equation (3.1) and let $S_\alpha(t, \cdot)$ be the scalarization of $\alpha(t, \cdot)$. Then the equation is equivalent to the following equation for S_α:

$$\frac{\partial S_\alpha}{\partial t} = \frac{1}{2} \Delta_{O(M)} S_\alpha, \qquad S_\alpha(0, \cdot) = S_{\alpha_0} \tag{4.1}$$

We now give a probabilistic representation of the solution S_α. Fix a point $x \in M$ and a frame u over x, by which we identify the tangent space

T_xM with $\mathbb{R}^n$. Let $U=\{U_t, t\geqslant 0\}$ be the horizontal Brownian motion on $O(M)$ determined by the stochastic differential equation

$$dU_t = H_{U_t} \circ d\omega_t, \qquad U_0 = u$$

where $H_u = \{H_{i,u}\}$ are the canonical horizontal vector fields at u and $\omega = \{\omega_t, t\geqslant 0\}$ is a standard $\mathbb{R}^n$-valued Brownian motion. The diffusion process U on $O(M)$ is generated by Bochner's horizontal Laplacian $\Delta^B_{O(M)}/2$, hence the solution to the heat equation

$$\frac{\partial S}{\partial t} = \frac{1}{2}\Delta^B_{O(M)} S, \qquad S(0, \cdot) = S_0 \tag{4.2}$$

is given by

$$S(t, u) = E\{S_0(U_t)\}.$$

It is seen from this representation that $\Delta^B_{O(M)}$ and Δ^B_M are more naturally associated with Brownian motion than $\Delta_{O(M)}$ and Δ_M. But using the Weitzenböck formula we can write the solution to (4.1) in terms of a Feynman-Kac functional. Let M_t be the End($\bigwedge^* \mathbb{R}^n$)-valued multiplicative function defined by

$$\frac{dM_t}{dt} = \frac{1}{2} M_t D^* \Omega_{U_t}, \qquad M_t = I$$

Then the solution to (4.1) is given by

$$S_\alpha(t, u) = E\{M_t S_{\alpha_0}(U_t)\}$$

This is equivalent to

$$\alpha(t, x) = E_x\{M_t U_t^{-1} \alpha_0(\gamma_t)\} \tag{4.3}$$

The subscript x is added to emphasize the fact that $\gamma_t = \pi U_t$ is a Brownian motion on M starting at x. Note that we identify T_xM with $\mathbb{R}^n$ by the frame u, hence $\bigwedge^*_x M$ with $\bigwedge^* \mathbb{R}^n$. For a detailed discussion on this formula see Ikeda and Watanabe,[9] [pp. 298–308].

Comparing (4.3) with (3.2) we have the following representation of the heat kernel

$$e^{t\Delta_M/2}(x, y) = E^t_{x,y}\{M_t U_t^{-1}\}\ p(t, x, y)$$

where $E^t_{x,y}\{\cdot\} = E_x\{\cdot \mid \gamma_t = y\}$ is the expectation with respect to Brownian bridge from x to y with time length t, and $p(t, x, y)$ is the usual heat kernel on functions, i.e.,

$$P_x\{\gamma_t \in C\} = \int_C p(t, x, y)\, dy.$$

In particular, we have

$$e^{t\Delta_M/2}(x, x) = E^t_{x,x}\{MU_t^{-1}\}\; p(t, x, x) \tag{4.4}$$

Under the probability $P^t_{x,x}$, the process $\{U_s\}$ is the (stochastic) parallel transport along the Brownian bridge γ, which satisfies a stochastic equation of the form

$$dU_s = H_{U_s} \circ dW_s + h(t-s, U_s)\, ds, \qquad U_0 = u$$

where $h(s, u) = u^{-1}\nabla_y \log p(s, y, x)$.

5. STOCHASTIC LOCAL GAUSS-BONNET-CHERN THEOREM

To state a stochastic version of the local Gauss-Bonnet-Chern theorem, we need to put the collection of Brownian bridges at x of various time lengths $t > 0$ on a common probability space. Let $(W_o(\mathbb{R}^n), \mathscr{B}, P)$ be the standard Wiener probability space, where $W_o(\mathbb{R}^n)$ is the space of continuous functions from $[0, 1]$ to $\mathbb{R}^n$ starting at the origin, $\mathscr{B}$ the Borel σ-field on $W_o(\mathbb{R}^n)$ and P the Wiener measure. The coordinate process $\{\omega_s, 0 \leqslant s \leqslant 1\}$ is the canonical realization of Brownian motion on $\mathbb{R}^n$ starting from the origin. Let $\{U_{t,s}, 0 \leqslant s \leqslant 1\}$ be the solution of the following stochastic differential equation on $O(M)$:

$$dU_{t,s} = \sqrt{t}\, H_{U_{t,s}} \circ d\omega_s + t h(t(1-s), U_{t,s})\, ds, \qquad U_{t,0} = u \tag{5.1}$$

We rescale the process as follows:

$$U^t_s = U_{t,s/t}, \qquad \gamma^t_s = \pi U^t_s \tag{5.2}$$

Then $\{\gamma^t_s, 0 \leqslant s \leqslant t\}$ is a Brownian bridge at x; $U_t = U^t_t$ is an $\mathrm{End}(\mathbb{R}^n)$-valued random variable and is the parallel transport along the Brownian bridge at x. Define as before

$$\frac{dM^t_s}{ds} = \frac{1}{2} M^t_s D^*\Omega(U^t_s), \qquad M^t_0 = I \tag{5.3}$$

Then $M_t = M_t'$ is also an End($\mathbb{R}^n$)-valued random variable. It is now clear that

$$\chi(M) = \int_M E\{\phi(U_{t;\,x} M_{t;\,x})\}\ p(t, x, x)\, dx$$

where the subscript x indicates the base point of the Brownian bridge.

From (5.1), we have $U_t - I = O(\sqrt{t})$ as $t \to 0$. The key probabilistic fact here is that in fact $U_t - I = O(t)$. More precisely we have the following estimate, whose proof is relegated to Section 6.

Proposition 2. Let $K_t = t^{-1}\,|U_t - I|$. Then for each positive integer N, there is a constant C_N independent of $t \in [0, 1]$ such that $EK_t^N \leqslant C_N$.

We are in a position to prove our main result.

Theorem. The limit

$$e(x) = \lim_{t \to 0} \phi\{U_{t,x} M_{t,x}\}\ p(t, x, x) \tag{5.4}$$

exists in $L^N(P)$ for all $N > 0$. Furthermore, if $n = 2l$ is even,

$$e(x) = \frac{\phi\{(D^*\Omega_x)^l\}}{(4\pi)^l\, l!}$$

otherwise $e(x) = 0$.

Proof. As $t \to 0$, we have $p(t, x, x) \sim (2\pi t)^{-n/2}$. Thus we need to show that the following limit exists:

$$e(x) = \lim_{t \to 0} \left(\frac{1}{2\pi t}\right)^{n/2} \phi\{U_t M_t\} \tag{5.5}$$

The idea is to expand M_t and U_t into exponential series. Let $l = n/2$ and $[l]$ the largest integer not exceeding l. Let $l_* = [l] + 1$ for simplicity. We assume throughout the proof that $0 \leqslant t \leqslant 1$.

Let $u_t \in so(n)$ such that $U_t = \exp u_t$ in $O(n)$. Such u_t is unique if t is small because $U_t \to I$ as $t \to 0$. By (2.2) U_t, as an isometry on $\bigwedge^* \mathbb{R}^n$ (also denoted by $\bigwedge^* U_t$), is given by $U_t = \exp D^* u_t$. For any positive integer N,

$$U_t = \sum_{i=0}^{[l]} \frac{\{D^* u_t\}^i}{i!} + R_t$$

where the remainder $|R_t| \leqslant |u_t|^{l+1}/(l+1)!$ On the other hand, iterating (5.3) we have

$$M_s^t = \sum_{i=0}^{[l]} m_{i,s}^t + Q_s^t$$

where

$$m_{0,s}^t = I, \qquad m_{i,s}^t = \tfrac{1}{2} \int_0^s m_{i-1,\tau}^t D^* \Omega_{U_\tau^t} \, d\tau$$

Writing $m_{i,t}$ for $m_{i,t}^t$ we have

$$M_t = \sum_{i=0}^{[l]} m_{i,t} + Q_t$$

where the remainder satisfies $|Q_t| \leqslant (Ct)^{l+1}/(l+1)!$ for some constant C. Combining the two expansions we have

$$U_t M_t = \sum_{i+j \leqslant [l]} \frac{1}{j!} (D^* u_t)^j m_{i,t} + S_t \tag{5.6}$$

The remainder can be estimated as

$$|S_t| \leqslant C\{|u_t|^{2(l+1)} + |u_t|^{l+1} + t^{l+1}\} \tag{5.7}$$

From definition it is clear that $m_{i,t}$ is the limit of a sequence of linear combinations of the terms of the form $D^*S_1 \circ \cdots \circ D^*S_i$ with $S_k \in \mathrm{End}(\mathbb{R}^n) \times \mathrm{End}(\mathbb{R}^n)$. Hence by Proposition 1 we have

$$\phi\{(D^* u_t)^j m_{i,t}\} = 0, \qquad \text{if} \quad 2i + j < n$$

After taking the supertrace in (5.6), the only nonvanishing terms are $\phi(S_t)$ and the terms with $i + j \leqslant n + 2$ and $2i + j \geqslant n$, that is, $j = 0$ and $2i = n$. In this case $l = n/2$ must be an integer and

$$\phi(U_t M_t) = \frac{\phi(m_{l,t})}{l!} + \phi(S_t) \tag{5.8}$$

Suppose that $t^{-l}\phi(S_t) \to 0$ in $L^N(P)$. Then for n even, we have

$$e(x) = \lim_{t \to 0} \phi(U_t M_t) = \lim_{t \to 0} \left(\frac{1}{2\pi t}\right)^l \phi(m_{l,t}) = \frac{\phi\{(D^*\Omega)^l\}}{(4\pi)^l \, l!} \qquad \text{in } L^N(P)$$

For n odd, only the remainder in (5.8) stays and we obtain $e(x)=0$. Thus it is enough to show $t^{-l}\phi(S_t)\to 0$ in $L^N(P)$.

Let $F_t=\{\omega: |U_t(\omega)-I|\leqslant 1/2\}$. The probability of F_t^c is small by Proposition 2:

$$P(F_t^c)=P\{K_t>1/2t\}\leqslant(2t)^{lN+1}EK_t^{lN+1}, \qquad \forall N\geqslant 0 \tag{5.9}$$

Since $U_t=e^{u_t}$, on the set F_t,

$$|u_t|\leqslant \tfrac{3}{2}\,|\log U_t|\leqslant C\,|U_t-I|\leqslant CK_t t$$

From (5.8) it is clear that $\phi(S_t)$ is uniformly bounded, a fact we will use on F_t^c. Consider the cases F_t and F_t^c separately and using Proposition 2 we find

$$t^{-l}E\,|\phi(S_t)|^N\leqslant CtE(K_t+1)^{2(l+1)N}+Ct^{-lN}P(F_t^c)\leqslant C_1 t,$$

which completes the proof. □

Taking expectation in (5.4) and using (4.4) we obtain the local Gauss-Bonnet-Chern theorem:

$$\lim_{t\to 0}\phi\{e^{t\Delta/2}(x,x)\}=e(x)$$

Integrating over M and using (3.3), we obtain the Gauss-Bonnet-Chern theorem:

$$\chi(M)=\int_M e(x)\,dx$$

To identify the integrand $e(x)$ more explicitly, we let $\{e_i\}$ be the standard orthonormal basis for $T_xM=\mathbb{R}^n$ and $\{e_i^*\}$ the dual basis. We write the curvature tensor in this basis: $R=R_{ijkl}e_i^*\otimes e_j\otimes e_k^*\otimes e_l$. Then by Lemma 1 we have when n is even

$$e(x)=\frac{1}{(4\pi)^{n/2}\,(n/2)!}\sum_{I,J}\operatorname{sgn}(I,J)\,R_{i_1i_2j_1j_2}\cdots R_{i_{n-1}i_nj_{n-1}j_n}$$

where $I=\{i_1,\dots,i_n\}$ and $J=\{j_1,\dots,j_n\}$. The n-form $e=e(x)\,dx$ is the Euler form on M and is the unique form on M such that

$$\pi^*e=\frac{1}{(4\pi)^{n/2}\,(n/2)!}\sum_I \Omega_{i_1i_2}\wedge\cdots\wedge\Omega_{i_{n-1}i_n}$$

Here in the last expression $\Omega=\{\Omega_{ij}\}$ is understood to be the $o(n)$-valued curvature form on $O(M)$.

6. AN ESTIMATE ON BROWNIAN HOLONOMY

In this section we prove Proposition 2, which was used in the proof of our main theorem. In the course of the proof, we will need the following fact on the gradient of the heat kernel. There exists a constant C such that for all $(x, y, t) \in M \times M \times (0, 1]$:

$$|\nabla_y \log p(t, x, y)| \leqslant C \left\{ \frac{d(x, y)}{t} + \frac{1}{\sqrt{t}} \right\} \tag{6.1}$$

See Sheu[15] for a proof and Hsu[8] for a more recent treatment.

Let $\{U_{t,s}, 0 \leqslant s \leqslant 1\}$ be as defined in the last section and $\gamma_{t,s} = \pi U_{t,s}$. Recall that $U_s^t = U_{t,s/t}$ and $\gamma_s^t = \pi U_s^t = \gamma_{t,s/t}$. The process $\{\gamma_s^t, 0 \leqslant s \leqslant t\}$ is a Brownian bridge at x. Intuitively we have $d(\gamma_s^t, x) \to 0$ as $t \to 0$. We show more precisely that it goes to zero at the rate of $\sqrt{t}$.

Lemma 2. For any fixed $\alpha \in (0, 1/2)$, $t \in [0, 1]$, and positive integer N, there exist a random variable K_t and a constant C_N independent of t such that $EK_t^N \leqslant C_N$ and almost surely

$$d(\gamma_{t,s}, x) \leqslant K_t \sqrt{t}\, \{s \wedge (1-s)\}^\alpha, \qquad 0 \leqslant s \leqslant 1$$

Proof. Let O be a neighborhood of x covered by the Cartesian coordinate system $\{x^1, ..., x^n\}$ on the tangent space $T_x M$ with respect to the fixed basis at u. For each $i = 1, ..., n$, let f^i be a smooth function on M such that $f^i(x) = x^i$ for $x \in O$ and $|f^i|$ strictly positive outside O. Let $f = \sum_{i=1}^n |f^i|^2$. Then it is clear that there is a constant C such that

$$C^{-1} d(x, y)^2 \leqslant f(y) \leqslant C d(x, y)^2$$

From (6.1) and $h(t, u) = \nabla \log p(t, \pi u, x)$ we have

$$|h(t, u)| \leqslant C \left\{ \frac{\sqrt{f(y)}}{t} + \frac{1}{\sqrt{t}} \right\}, \qquad y = \pi u \tag{6.2}$$

Recall the Eq. (5.1) for $U_{t,s}$ and apply Itô's formula to $f(\gamma_{t,s})$. We have

$$\begin{aligned} f(\gamma_{t,s}) = {} & \sqrt{t} \int_0^s \langle \nabla^H F(U_{t,\tau}), d\omega_\tau \rangle + \frac{t}{2} \int_0^s \Delta_M f(\gamma_{t,\tau})\, d\tau \\ & + t \int_0^s \langle \nabla^H F(U_{t,\tau}), h(t(1-\tau), U_{t,\tau}) \rangle\, d\tau \end{aligned}$$

where $F(u)=f(\pi u)$ and $\nabla^H F=\{H_i F\}_{1\leqslant i\leqslant n}$, the so-called horizontal gradient. Using the estimate (6.2) we have for $0\leqslant s\leqslant 1/2$,

$$f(\gamma_{t,s})\leqslant t\Phi(t,s)+C\int_0^s f(\gamma_{t,\tau})\,d\tau$$

where

$$\Phi(t,s)=\left|\int_0^s \langle HF(U_{t,\tau}), d\omega_\tau\rangle+\frac{\sqrt{t}}{2}\int_0^s \Delta_M f(\gamma_{t,\tau})\,d\tau\right|^2+Cs^2$$

Hence,

$$\frac{f(\gamma_{t,s})}{t}\leqslant \Phi(t,s)+C\int_0^s \Phi(t,\tau)\,d\tau$$

Let K_t^2 be the supremum over $s\in[0,1/2]$ of the right-hand side of this inequality divided by $s^{2\alpha}$. Simple moment estimates on stochastic integrals show that for any N there is a constant C_N independent of t such that $EK_t^N\leqslant C_N$. The lemma for $0\leqslant s\leqslant 1/2$ now follows from (3.2) because

$$d(\gamma_{t,s},x)\leqslant C\sqrt{f(\gamma_{t,s})}\leqslant CK_t\sqrt{t}\,s^\alpha$$

The same conclusion holds for the interval $1/2\leqslant s\leqslant 1$ because the law of Brownian bridge is invariant under the time reversal $s\mapsto 1-s$. □

We now prove Proposition 2. We will use K_t to denote a general random variable with the property in the proposition. Let O be a neighborhood of x as in the preceding lemma. Let O_1 be another neighborhood of x whose closure is contained in O. Let $G\colon O(M)\to\mathbb{R}^{n^2}$ be a smooth function such that $G(u)=\{u_j^i\}$ on $\pi^{-1}(O_1)$ and zero on $\pi^{-1}(M\backslash O)$, where $u=\{u_j^i\,\partial/\partial x^i\}_{1\leqslant j\leqslant n}$. Then we have $G(U_{t,0})=I$ and $G(U_{t,1})=U_t$. Using Itô's formula, we have

$$\frac{U_t-I}{t}=\frac{1}{\sqrt{t}}\int_0^1 \langle\nabla^H G(U_{t,s}), d\omega_s\rangle+\frac{1}{2}\int_0^1 \Delta^B_{O(M)}G(U_{t,s})\,ds$$
$$+\int_0^1 \langle\nabla^H G(U_{t,s}), h(t(1-s), U_{t,s})\rangle\,ds \tag{6.3}$$

The key to the proof is the following fact

$$|\nabla^H G(u)|\leqslant Cd(\pi u,x). \tag{6.4}$$

To see this, we write H_i in the local Cartesian coordinates on O,

$$H_i = u_i^j \frac{\partial}{\partial x^j} - \Gamma_{kl}^q(y)\, u_i^k u_p^l \frac{\partial}{\partial u_p^q},$$

where $\Gamma_{kl}^q(y)$ are the Christoffel symbols at y [see Ikeda and Watanabe,[9] p. 280]. Inequality (6.4) follows immediately from the fact that the Christoffel symbols vanish at the origin of the Cartesian coordinates.

Now since $|\nabla^H G(U_{t,s})| \leqslant d(\gamma_{t,s}, x)$, using Lemma 2, we can show by simple estimates on stochastic integrals that the first term on the right-hand side of (6.3) is bounded by K_t (note the cancellation of $\sqrt{t}$). The second term is uniformly bounded. As for the third term, from (6.4), (6.2), and Lemma 2 with $\alpha = 1/4$, we have

$$\begin{aligned} &|\langle \nabla^H G(U_{t,s}), h(t(1-s), U_{t,s})\rangle| \\ &\quad \leqslant C d(\gamma_{t,s}, x) \left\{ \frac{d(\gamma_{t,s}, x)}{t(1-s)} + \frac{1}{\sqrt{t(1-s)}} \right\} \\ &\quad \leqslant \frac{K_t}{\sqrt{1-s}} \end{aligned}$$

Note again the cancellation of t and $\sqrt{t}$, respectively. It follows that the third term of the right-hand side of (6.3) is also bounded by K_t. This completes the proof. □

Remark 1. We can prove the following more precise results on $\gamma_{t,s}$ and U_t. Let $\psi: M \to \mathbb{R}^n$ be a Cartesian coordinate system in a neighborhood of x. Then

$$\frac{\psi(\gamma_{t,s})}{\sqrt{t}} \to Y_s \qquad \text{in } L^N(P)$$

where $\{Y_s, 0 \leqslant s \leqslant 1\}$ is the Euclidean Brownian bridge determined by

$$dY_s = d\omega_s - \frac{Y_s}{1-s}\, ds, \qquad Y_0 = 0$$

For the Brownian holonomy U_t, we have

$$\frac{U_t - I}{t} \to \frac{1}{8} \int_0^1 \Omega(Y_s, dY_s) \qquad \text{in } L^N(P)$$

where Ω is the ($o(n)$-valued) curvature form at the fixed frame u at x. Probabilistically these precise results are the only work we need beyond what we have done in this article in order to prove the Atiyah-Singer index theorem for the Dirac operator on the spinor bundle over a compact Riemannian manifold, see Hsu[7] for further details.

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