Homework 2

1. Let N be the nonmeasurable subset of $[0, 1]$ we constructed using rational equivalence.

- **a**. Prove that every measurable subset of N has measure 0.
- **b**. Prove that $m^*([0, 1] \sim N) = 1$.

2. Choose positive numbers c_k such that $\sum_{k=1}^{\infty} 2^{k-1} c_k < 1$. Starting with the unit interval, mimic the construction of the Cantor set by at the kth step deleting 2^{k-1} symmetrically arranged open subintervals, each of length c_k. Let C' denote the intersection of all the resulting sets.

- **a**. Prove $m(C') > 0$.
- **b**. Let U be the union of all open intervals which are deleted at the odd steps in the construction of C'. Prove that U is an open set with $m(\bar{U}) \neq m(U)$.

3. Suppose E is a measurable set. For each $k > 0$, define $U_k = \{x \in \mathbb{R} : d(x, E) < \frac{1}{k}\}$.

- **a**. If E is compact, prove that $m(E) = \lim_{k\to\infty} m(U_k)$.
- **b**. Find an example of an unbounded closed set and a bounded open set for which **a** does not hold.

4. Royden, Section 2.4, Exercise 22.

5. Royden, Section 2.6 Exercise 33.