Homework 7

1. Compute all homology groups of:

   - a genus n torus
   - $S^n \vee S^n$
   - $S^1 \times \cdots \times S^1$

2. In this exercise, you will work out the details of the proof that $H_n(X) \cong H_n(X^{(1)})$.

   a) Define $\phi_n : C_n(X) \to C_n(X^{(1)})$ by $\phi_0(v) = v$ and $\phi_n(\sigma) = [b, \phi_{n-1}(\partial\sigma)]$, where b is the barycenter of σ and $n \geq 1$. Prove that ϕ is a chain map.

   b) Define $f : X^{(1)} \to X$ by $f(b) = v_0$ if b is the barycenter of $[v_0, \dots, v_n]$. Prove that if $b_0, \dots, b_k$ span a simplex in $X^{(1)}$, then $f(b_0), \dots, f(b_k)$ span a simplex in X. Thus, f defines a simplicial map.

   c) Prove that $f_n \circ \phi_n$ is the identity on $C_n(X)$.

   d) Prove that $\phi \circ f_*$ is chain homotopic to the identity, as follows:

   i) Define h_0 to be 0 on vertices of X and $h_0(b) = [b, v_0]$ if b is the barycenter of $[v_0, \dots, v_n]$ in X. Show that h_0 satisfies the chain homotopy condition between $\phi_0 f_0$ and Id.

   ii) For $n \geq 1$, assume h_{n-1} has been defined and that for each simplex σ of $X^{(1)}$, $h_{n-1}(\sigma) \in C_n(\Delta)$, where Δ is the maximal simplex of X containing σ. Prove that

   $$\partial(\phi_n f_n - \mathrm{Id} - h_{n-1}\partial) = 0$$

   iii) Conclude that given any $\sigma \in C_n(X^{(1)})$, there is a $\tau \in C_{n+1}(X^{(1)})$ such that

   $$\partial\tau = \phi_n f_n(\sigma) - \sigma - h_{n-1}(\partial\sigma)$$

   and set $h_n(\sigma) = \tau$.

3. Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of topological spaces. Prove that $(g \circ f)_* = g_* \circ f_*$.