Project 1

Due Monday, October 30.

In this project, you will construct a model for the spread of a disease, under the hypothesis that survivors of the disease acquire lifetime immunity.

Let $p(a)$ denote the number of individuals whose age is a, let $s(a)$ denote the number of those of age a who have survived the disease and are therefore now immune, and let $u(a)$ denote the number of those of age a who have not yet contracted the disease and are therefore still susceptible. Let λ be the rate at which those susceptible become infected and let c denote the case fatality rate, i.e., the rate at which those infected die from the disease. Finally, let $\mu(a)$ denote the death rate due to all causes other than the disease.

(a) Find an expression for u' in terms of the above functions and constants.

(b) Find an expression for p' in terms of the above functions and constants.

(c) Let $y(a)$ denote the proportion of the population at age a that is susceptible to infection. Show that y satisfies an autonomous initial value problem.

(d) Determine the equilibria of the equation in (c) and analyze their stability. Interpret what this implies about the spread of the disease.

(e) Find y as a function of a.

(f) Given $c = \lambda = .125$, determine the proportion of 30-year-olds who have not been infected.

(g) (Mathematica) Graph solution curves corresponding to $\lambda = .125$ and $c = 0, .125, .5, .999$ on the same set of axes.

(h) In reality, both c and λ should depend on a. Modify the initial value problem in (c) to take into account that c and λ are functions of a, then solve it for y by first making the substitution $z = 1/y$. Express your answer in terms of

$$\Lambda(a) = \int_0^a \lambda(t)dt$$

(Of course your answer will also involve antiderivatives of other unknown functions.)

(i) (Mathematica) Using 2019 estimates, a model for the case fatality rate for measles is

$$c(a) = 0.0353091 - 0.0106933a + 0.0013788a^2 - 0.0000572a^3$$

Graph solution curves corresponding to this $c(a)$ and $\lambda = .001, .125, .5, .999$ on the same set of axes.