Homework 2

Due Friday, Feb. 7.

1. Give an example of a finite extension K/k with infinitely many intermediate fields.

2. Suppose $\operatorname{char} k = p$ and K/k is a finite extension. Let $K_s = \{\alpha \in K : \alpha \text{ is separable over } k\}$. Prove that K_s is a subfield of K which is the maximal intermediate field that is separable over k. Prove also that $[K : K_s] = p^n$ for some n.

3. With notation as in Problem 2, suppose in addition that K/k is normal. Prove that K_s is Galois over k and $\operatorname{Gal}(K_s/k) \cong \operatorname{Aut}_k(K)$.

4. Suppose $\operatorname{char} k \neq 2$ and $f \in k[X]$ has distinct, non-repeated roots $\alpha_1, \ldots, \alpha_n$ in some splitting field K. Let

$$\Delta = \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)\,.$$

Prove that if $G = \operatorname{Gal}(K/k)$ is viewed as a subgroup of S_n, then $G \subset A_n$ if and only if $\Delta \in k$.

5. Let K be the splitting field of $(X^2+2)(X^3-3)$ over $\mathbb{Q}$. Determine all $\mathbb{Q}$-isomorphism classes of subfields of K and find all those that are Galois over $\mathbb{Q}$.