Homework 3

Due Wednesday, Feb. 19.

1.

(a) Show that for any natural number n there are infinitely many primes congruent to 1 mod n. (This is a special case of a much deeper result, but don't use that. Instead, assume that $p_1, \ldots, p_k$ are the only such primes and study $\Phi_n(np_1 \ldots p_k)$.)

(b) Prove that for any finite abelian group G there is a Galois extension $K/\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q}) \cong G$. (With the word "abelian" removed this becomes a famous open problem, hence another opportunity for 5 bonus points.)

2. Suppose K/k is an algebraic extension such that every nonconstant polynomial in $k[X]$ has a root in K. Prove that K is algebraically closed. (Consider first the case where K/k is separable.)

3. Suppose K and L are extensions of k that are both contained in some common field F and suppose K/k is Galois.

(a) Prove that KL/L and $K/K \cap L$ are Galois and that $\mathrm{Gal}(KL/L) \cong \mathrm{Gal}(K/K \cap L)$ as topological groups.

(b) Suppose K' is any normal extension of k and that $\phi : K \to K'$ and $\psi : L \to K'$ are k-homomorphisms such that $\phi|K \cap L = \psi|K \cap L$. Prove that there is a k-homomorphism $\sigma : KL \to K'$ such that $\sigma|K = \phi$ and $\sigma|L = \psi$.

4.

(a) Prove that for every nonzero $n \in \mathbb{N}, \hat{\mathbb{Z}}/n\hat{\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$ as topological groups.

(b) Prove that $\hat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p$ as topological groups. (Recall that $\mathbb{Z}_p$ denotes the p-adic integers.)

(c) Determine all open subgroups of $\hat{\mathbb{Z}}$.

5. Fix a separable closure $\bar{k}_s$ of k and set $G = \mathrm{Gal}(\bar{k}_s/k)$. Let A denote the collection of all subextensions of $\bar{k}_s$ which are abelian over k. Let $k^{\mathrm{ab}} = \varinjlim_{A} K$.

(a) Prove that k^{ab} is Galois over k with $\mathrm{Gal}(k^{\mathrm{ab}}/k) \cong G/\bar{G}'$ as topological groups.

(b) Compute $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. You may assume the Kronecker-Weber Theorem.