Homework 4

Due Friday, March 6.

1. Suppose $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint G. Prove that F is *continuous*, i.e., that for any direct system $\{X_i\}$ in $\mathcal{C}$,

$$F(\varinjlim X_i) = \varinjlim F(X_i)$$

whenever $\varinjlim X_i$ exists.

2.

(a) Suppose R and S are (not necessarily commutative) rings and N is an (R, S)-bimodule. Prove that $\mathrm{Hom}_S(N, -)$ and $- \otimes_R N$ are adjoint functors.

(b) Deduce Frobenius reciprocity from (a). Also deduce the alternative version, $\mathrm{Hom}_G(W, \mathrm{Ind}_H^G V) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G W, V)$.

3. Let $H < G$, let W be a representation of H, and set $V = \mathrm{Ind}_H^G W$. Prove that

$$\chi_V(g) = \frac{1}{|H|} \sum_{g_1 \in G} \chi_W(g_1 g g_1^{-1})$$

where we extend χ_W to G by making it 0 on $G \setminus H$.

4. Determine the irreducible representations of Q_8 (describe them explicitly) and find the character table.

5. If R is a commutative ring and S is a subring, we say that $\alpha \in R$ is *integral* over S if α satisfies a monic polynomial in $S[X]$.

(a) Prove that α is integral over S if and only if $S[\alpha]$ is a finitely-generated S-module. (One direction is easy. For the other, suppose $u_1, \ldots, u_n$ generate $S[\alpha]$ as an S-module, where $u_1 = 1$. If $\alpha u_i = \sum a_{ij} u_j$, set $A = (a_{ij})$ and apply $\alpha I - A$ to the vector $(u_1, \ldots, u_n)$.)

(b) Prove that $\{\alpha \in R : \alpha \text{ is integral over } S\}$ is a subring of R.

(c) Prove that $\{\alpha \in \mathbb{Q} : \alpha \text{ is integral over } \mathbb{Z}\} = \mathbb{Z}$.