Homework 5

Due Wednesday, March 18.

1. Suppose that $L/K/k$ are fields. Prove that

$$\operatorname{tr.deg} L/k = \operatorname{tr.deg} L/K + \operatorname{tr.deg} K/k.$$

2. A transcendence basis S for K/k is said to be *separating* if $K/k(S)$ is a separable algebraic extension.

(a) If k is perfect and $K = k(J)$ for some finite subset $J \subset K$, prove that K/k admits a separating basis.

(b) Give an example of an extension K/k with k perfect and $\operatorname{tr.deg} K/k = 1$ that does not admit a separating transcendence basis.

3. Call an extension K/k *separable* if $\operatorname{char} k = 0$ or $\operatorname{char} k = p$ and $K \otimes_k k^{1/p}$ is a reduced ring (i.e., has no nontrivial nilpotents). Recall that $k^{1/p} = \{\alpha \in \bar{k} : \alpha^p \in k\}$, where $\bar{k}$ is a fixed algebraic closure of k.

(a) Show that this agrees with our previous definition for K/k algebraic.

(b) Prove that K/k is separable if and only if for every finite $J \subset K$, $k(J)$ has a separating transcendence basis over k. (So every extension of a perfect field is separable.)

4. Suppose K/k with K algebraically closed and of finite transcendence degree over k. Prove that any nontrivial k-endomorphism of K is an automorphism.

5. We showed that $Aut_{\mathbb{Q}}\mathbb{C}$ is uncountable, and of course $Aut_{\mathbb{R}}\mathbb{C} \cong \mathbb{Z}/2\mathbb{Z}$. Determine $Aut_{\mathbb{Q}}\mathbb{R}$. (Observe that any endomorphism of $\mathbb{R}$ preserves signs.)