Triangles and triple products of Laplace eigenfunctions

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Table of Contents

1 The problem and background

2 The main results

3 Sketch of proofs

4 Concluding remarks
$M$ is a Riemannian manifold, compact, $\partial M = \emptyset$. 

$\Delta_M$ is the Laplace-Beltrami operator on $M$. 

$\Delta_M e_j = -\lambda_j^2 e_j$ for $j = 1, 2, \ldots$ is an eigenbasis for $L^2(M)$. 

$\lambda_j$ is the frequency of $e_j$. 

Main objects 

Given two basis eigenfunctions $e_i, e_j$, their product is written $e_i e_j = \sum_k \langle e_i e_j, e_k \rangle e_k$. 

The coefficients $\langle e_i e_j, e_k \rangle$ are called eigenfunction triple products.
The Setting

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$$e_i e_j = \sum_k \langle e_i e_j, e_k \rangle e_k.$$

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Given two eigenfunctions $e_i$ and $e_j$, at which frequencies is the bulk of the spectral mass of the product $e_i e_j$ located?
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- Important for the validity of fast algorithms for electronic structure computing.
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- Related to questions about the algebraic structure of trigonometric polynomials.
- Here, the answer is related to counting configurations of triangles.
Prior results

Sarnak ‘94
\[ |\langle e_{2j}, e_k \rangle| = O_j(\lambda \epsilon k - \pi \lambda k/2) \] for some \( \epsilon > 0 \).

Bernstein-Reznikov ‘99
\[ |\langle e_{2j}, e_k \rangle| = O_j(\lambda \epsilon k - \pi \lambda k/2) \] for all \( \epsilon > 0 \).

Kontz-Stanton ‘04
Zelditch ‘12
Sarnak’s bounds for analytic manifolds.

Lu-Ying ‘15
Observed empirically that \( \langle e_i e_j, e_k \rangle \) tends to be supported in \( \lambda k \leq \lambda i + \lambda j \).

Lu-Steinerberger ‘18
\[ |\langle e_i e_j, e_k \rangle| = O(\lambda_{-\infty} k) \] for \( \lambda k \geq (\lambda i + \lambda j)^{1+\epsilon} \).

Lu-Sogge-Steinerberger ‘19
Steinerberger ‘19
Introduced the local correlation functional.
Prior results

Sarnak ’94

\[ |\langle e_j^2, e_k \rangle| = O_j(\lambda_k^\epsilon e^{-\pi \lambda_k/2}) \] for some \( \epsilon > 0 \).
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Sarnak’s bounds for analytic manifolds.
<table>
<thead>
<tr>
<th>Prior results</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Sarnak '94</strong></td>
</tr>
<tr>
<td><strong>Bernstein-Reznikov '99</strong></td>
</tr>
<tr>
<td><strong>Köntz-Stanton '04</strong></td>
</tr>
<tr>
<td><strong>Zelditch '12</strong></td>
</tr>
<tr>
<td><strong>Lu-Ying '15</strong></td>
</tr>
<tr>
<td>Reference</td>
</tr>
<tr>
<td>-------------------------------</td>
</tr>
<tr>
<td>Sarnak '94</td>
</tr>
<tr>
<td>Bernstein-Reznikov '99</td>
</tr>
<tr>
<td>Köntz-Stanton '04</td>
</tr>
<tr>
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<tr>
<td>Sarnak '94</td>
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<tr>
<td>Bernstein-Reznikov '99</td>
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<tr>
<td>Kötz-Stanton '04</td>
</tr>
<tr>
<td>Zelditch '12</td>
</tr>
<tr>
<td>Lu-Ying '15</td>
</tr>
<tr>
<td>Lu-Steinerberger '18</td>
</tr>
<tr>
<td>Lu-Sogge-Steinerberger '19</td>
</tr>
<tr>
<td>Steinerberger '19</td>
</tr>
</tbody>
</table>
A look at recent work

A priori, span \(\{e_i e_j: \lambda_i, \lambda_j \leq \lambda\}\) has dimension \(\approx \lambda^2 n\). (Lu and Ying ’15) Observed most of span \(\{e_i e_j: \lambda_i, \lambda_j \leq \lambda\}\) is contained in a space of dimension \(\approx \lambda^n\).

Theorem (Lu, Steinerberger ’18 and Lu, Sogge, Steinerberger ’19)

For all \(\epsilon > 0\),
\[
\sum \lambda_k \geq (\lambda_i + \lambda_j)^{1+\epsilon} |\langle e_i e_j, e_k \rangle|^2 = O(\epsilon (\lambda_i + \lambda_j)^{-\infty})
\]

Takeaway: For fixed \(i, j\), the triple products \(|\langle e_i e_j, e_k \rangle|\) start rapidly-decaying by the time \(\lambda_k \geq (\lambda_i + \lambda_j)^{1+\epsilon}\).
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Theorem (Lu, Steinerberger ’18 and Lu, Sogge, Steinerberger ’19)

For all $\epsilon > 0$,

$$\sum \lambda_k \geq \left( \lambda_i + \lambda_j \right)^{1+\epsilon} |\langle e_i e_j, e_k \rangle|^2 = O(\epsilon \left( \lambda_i + \lambda_j \right)^{-\infty})$$

Takeaway: For fixed $i, j$, the triple products $|\langle e_i e_j, e_k \rangle|$ start rapidly-decaying by the time $\lambda_k \geq \left( \lambda_i + \lambda_j \right)^{1+\epsilon}$. 
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**Theorem (Lu, Steinerberger '18 and Lu, Sogge, Steinerberger '19)**

*For all* \( \epsilon > 0 \),

\[
\sum_{\lambda_k \geq (\lambda_i + \lambda_j)^{1+\epsilon}} |\langle e_i e_j, e_k \rangle|^2 = O_{\epsilon}((\lambda_i + \lambda_j)^{-\infty})
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- Takeaway: For fixed \(i, j\), the triple products \(|\langle e_i e_j, e_k \rangle|\) start rapidly-decaying by the time \(\lambda_k \geq (\lambda_i + \lambda_j)^{1+\epsilon}\).
The main idea

Definition

\[ \mu = \sum_{i,j,k} |\langle e_i e_j, e_k \rangle|^2 \delta(\lambda_i, \lambda_j, \lambda_k). \]
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\[ \mu = \sum_{i,j,k} |\langle e_i, e_j, e_k \rangle|^2 \delta(\lambda_i, \lambda_j, \lambda_k). \]

- Contains information on the triple products.
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- Contains information on the triple products.
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- Independent of choice of basis.

Main Idea

The asymptotics of \( \mu \) are determined in part by the size of a configuration set of triangles with side lengths prescribed by \( \lambda_i, \lambda_j, \) and \( \lambda_k \).
How are triangles involved?

Let $T_n = \mathbb{R}^n / 2\pi \mathbb{Z}^n$ be the standard flat torus. Take basis elements $e_m(x) = (2\pi)^{-n/2} e^{i \langle x, m \rangle}$ for $m \in \mathbb{Z}^n$, having respective frequencies $\lambda_m = |m|$.

For $m, j, k \in \mathbb{Z}^n$, $\langle e_m e_j, e_k \rangle = \begin{cases} (2\pi)^{-n/2} & \text{if } k = m + j \\ 0 & \text{otherwise}. \end{cases}$

$\mu = (2\pi)^{-n/2} \sum (a, b, c) \# \{ (m, j) : |m| = a, |j| = b, |m + j| = c \} \delta(a, b, c)$
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\[ |j| = b \]

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\( \mu \) is supported only where \( \lambda_i, \lambda_j, \lambda_k \) are realizable as the side lengths of a triangle.
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Setup: Some definitions

Definition (triangle-good) \((a, b, c) \in \mathbb{R}^3\) is triangle-good if
\[ a < b + c, \ b < a + c, \text{ and } c < a + b. \]
We let \(\text{area}(a, b, c)\) denote the area of the triangle with side lengths \(a, b,\) and \(c\).

Definition (triangle-bad) \((a, b, c) \in \mathbb{R}^3\) is triangle-bad if
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We ignore \((a, b, c)\) specifying a degenerate triangle.
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We ignore \((a, b, c)\) specifying a degenerate triangle.
Let $\rho \in \mathcal{S}(\mathbb{R}^3)$, supp $\hat{\rho} \subset (-\text{inj } M, \text{inj } M)^3$, with $\int \rho = 1$. 
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**Theorem (triangle-bad)**

If \( \Gamma \) is a closed cone in \( \mathbb{R}^3 \setminus 0 \) consisting of triangle-bad points, then

\[
\rho \ast \mu(a, b, c) = O_{\Gamma}(|(a, b, c)|^{-\infty}) \quad \text{for all} \ (a, b, c) \in \Gamma.
\]
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**Theorem (triangle-good)**

If $\Gamma$ is a closed cone in $\mathbb{R}^3 \setminus 0$ consisting of triangle-good points, then

$$\rho * \mu(a, b, c) = C_n |M| abc \text{ area}(a, b, c)^{n-3} + O_{\Gamma}(|(a, b, c)|^{2n-4})$$

for $(a, b, c) \in \Gamma$. 

Interpretation

\[ \rho \ast \mu(a, b, c) = \sum_{i,j,k} \rho(a - \lambda_i, b - \lambda_j, c - \lambda_k) |\langle e_i e_j, e_k \rangle|^2. \]
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- $\rho \ast \mu(a, b, c) = \sum_{i,j,k} \rho(a - \lambda_i, b - \lambda_j, c - \lambda_k)|\langle e_i e_j, e_k \rangle|^2$.

- Main term in Theorem (triangle-good) arises as the measure of some configuration set of triangles in $\mathbb{R}^n$ with sidelengths $a, b, c$. 
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- The Leray measure of \( \{ (\xi, \eta) \in \mathbb{R}^{n+n} : (|\xi|, |\eta|, |\xi + \eta|) = (a, b, c) \} \) is

  \[ |S^{n-1}| |S^{n-2}| abc (2 \text{ area}(a, b, c))^{n-3}. \]
Interpretation

\[ \rho \ast \mu(a, b, c) = \sum_{i,j,k} \rho(a - \lambda_i, b - \lambda_j, c - \lambda_k) |\langle e_i e_j, e_k \rangle|^2. \]

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Theorem (triangle-bad) says essentially none of \( \mu \) lies in the ‘classically forbidden’ region where there are no such triangles.
Corollary

For every $\epsilon > 0$, for fixed $i, j,$

$$\sum_{\lambda_k \geq (1+\epsilon)(\lambda_i + \lambda_j)} |\langle e_i e_j, e_k \rangle|^2 = O_\epsilon((\lambda_i + \lambda_j)^{-\infty}).$$
What does the triangle-bad theorem tell us?

**Corollary**

For every $\epsilon > 0$, for fixed $i, j$,

$$\sum_{\lambda_k \geq (1+\epsilon)(\lambda_i + \lambda_j)} |\langle e_i e_j, e_k \rangle|^2 = O_{\epsilon}((\lambda_i + \lambda_j)^{-\infty}).$$

- If we take $\rho \geq 0$ with $\rho(0) \geq c > 0$, then

$$|\langle e_i e_j, e_k \rangle|^2 \leq c^{-1} \rho \ast \mu(\lambda_i, \lambda_j, \lambda_k).$$
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- Let $\Gamma_\epsilon = \{(a, b, c) : c \geq (1 + \epsilon)(a + b)\}.$
Corollary

For every $\epsilon > 0$, for fixed $i, j$, 

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- Theorem (triangle-bad) yields 

$$
|\langle e_i e_j, e_k \rangle|^2 = O_\epsilon(||(\lambda_i, \lambda_j, \lambda_k)||^{-\infty}) \quad \text{for} \quad \lambda_k \geq (1 + \epsilon)(\lambda_i + \lambda_j)
$$
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2. The main results
3. Sketch of proofs
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\[ \mu = \sum_{i,j,k} \left| \int_M e_i e_j e_k \, dV \right|^2 \delta(\lambda_i, \lambda_j, \lambda_k). \]
Translate to the language of FIOs

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\[ (\delta_\Delta, f|dV_{M^3}|^{1/2}) = \int_M f(x, x, x) \, dV_M(x). \]
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- First simplify by writing
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- Let \( \Delta = \{(x, x, x) : x \in M\} \subset M^3 \).

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  \[ (\delta_{\Delta}, f \, dV_{M^3} \, |^{1/2}) = \int_M f(x, x, x) \, dV_M(x). \]

- \[ \left| \int_M e_i e_j e_k \, dV_M \right|^2 = |(\delta_{\Delta}, \varphi_m)|^2 = (\delta_{\Delta} \otimes \delta_{\Delta}, \varphi_m \otimes \varphi_m). \]
The key composition

Given $t = (t_1, t_2, t_3) \in \mathbb{R}^3$, let $U : C^\infty_c(M^3 \times M^3) \to \mathcal{D}'(\mathbb{R}^3)$ with distribution kernel

$$U(t, x, y) = e^{it_1 \sqrt{-\Delta_M}}(x_1, y_1)e^{it_2 \sqrt{-\Delta_M}}(x_2, y_2)e^{it_3 \sqrt{-\Delta_M}}(x_3, y_3).$$
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Note,

$$U(t, x, y) = \sum_{m \in \mathbb{N}^3} e^{i \langle t, \lambda_m \rangle} \varphi_m(x) \overline{\varphi_m(y)} |dt|^{1/2}$$

where here $\lambda_m = (\lambda_i, \lambda_j, \lambda_k)$ and $x, y \in M^3$. 

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Key composition

\[
\check{\mu}(t)|dt|^{1/2} = (2\pi)^{-3} \sum_{m \in \mathbb{N}^3} e^{i\langle t, \lambda_m \rangle} (\delta_\Delta \otimes \delta_\Delta, \varphi_m \otimes \overline{\varphi_m})|dt|^{1/2}
\]

\[
= (2\pi)^{-3} U \circ (\delta_\Delta \otimes \delta_\Delta).
\]
The relevant dynamics on \( M \)

- Theorem (triangle-bad) follows if \( \text{WF}(\dot{\mu}) \) does not contain any triangle-bad covectors.
The relevant dynamics on $M$

- Theorem (triangle-bad) follows if $\text{WF}(\hat{\mu})$ does not contain any triangle-bad covectors.
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**Definition (geodesic triple)**

Let $x \in M$ and $\xi = (\xi_1, \xi_2, \xi_3) \in (T^*_x M)^3$. We say $(t, x, \xi)$ is a **geodesic triple** if all of the following hold.
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3. $\eta_1 + \eta_2 + \eta_3 = 0$, and...
4. $(y, \eta_j) = G^t_j(x, \xi_j)$ for $j = 1, 2, 3$. 
We apply a PDO cutoff to take out the ‘degenerate’ triangles. Then...
Conclusion of the proof of Theorem (triangle-bad)

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Lemma (the wavefront set of $\tilde{\mu}$)

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WF(\tilde{\mu}) \subset \{(t, \tau) \in \tilde{T}^*\mathbb{R}^3 : \text{there exists a geodesic triple } (t, x, \xi), \text{ and } \tau_j = |\xi_j|_x \text{ for each } j = 1, 2, 3\}.
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This concludes the proof of Theorem (triangle-bad).
Strategy for Theorem (triangle-good)

- **Note**

\[ \rho \ast \mu(\tau) = (2\pi)^3 (\eb{\bar{\mu}}, e^{-i \langle \cdot, \tau \rangle}). \]
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- Note
  \[ \rho \ast \mu(\tau) = (2\pi)^3 (\check{\mu}, \check{\rho}e^{-i\langle \cdot, \tau \rangle}). \]
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Lemma (the component of WF(\(\tilde{\mu}\)) over 0)

The wavefront set of the restriction of \(\tilde{\mu}\) to \((- \text{inj } M, \text{inj } M)^3\) is contained in

\[ \{(0, \tau) \in \dot{T}_x^* \mathbb{R}^3 : \tau \text{ is triangle-good}\}. \]

Furthermore, the composition \(WF'(U) \circ WF(\delta_{\Delta} \otimes \delta_{\Delta})\) is clean over this component.
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Furthermore, the composition \(\text{WF}'(U) \circ \text{WF}(\delta_\Delta \otimes \delta_\Delta)\) is clean over this component.

- We compute the symbol of the composition \(U \circ (\delta_\Delta \otimes \delta_\Delta)\).
The order of $U \circ (\delta_\Delta \otimes \delta_\Delta)$

\[ \text{ord} U \circ (\delta_\Delta \otimes \delta_\Delta) = \text{ord} U + \text{ord} \delta_\Delta \otimes \delta_\Delta + \frac{e}{2}. \]
The order of $U \circ (\delta_\Delta \otimes \delta_\Delta)$

- $\text{ord } U \circ (\delta_\Delta \otimes \delta_\Delta) = \text{ord } U + \text{ord } \delta_\Delta \otimes \delta_\Delta + \frac{e}{2}$.

- $\text{ord } U = -\frac{3}{4}$. 

- Lemma (the order of $U \circ (\delta_\Delta \otimes \delta_\Delta)$ at 0) 

- $\text{ord } U = -\frac{3}{4}$. 

- $\rho^* \mu(\tau)$ is polyhomogeneous of order $2n - 3 - \frac{3}{4} = 2n - 3$. 


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$$e = \dim \{(x, \xi_1, \xi_2, \xi_3) : x \in M, \xi_1 + \xi_2 + \xi_3 = 0, \quad |\xi_j|_x = \tau_j \text{ for } j = 1, 2, 3\} = 3n - 3.$$
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Lemma (the order of $U \circ (\delta_\Delta \otimes \delta_\Delta)$ at 0)

$\text{ord } U \circ (\delta_\Delta \otimes \delta_\Delta) = 2n - \frac{9}{4},$

and $\rho \ast \mu(\tau)$ is polyhomogeneous of order $2n - \frac{9}{4} - \frac{3}{4} = 2n - 3.$
The symbol of $U \circ (\delta_\Delta \otimes \delta_\Delta)$

- After applying the clean composition calculus...
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- Find, up to powers of $2\pi$, the principal symbol of $U \circ (\delta_\Delta \otimes \delta_\Delta)$ is
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The symbol of \( U \circ (\delta_\Delta \otimes \delta_\Delta) \)

- After applying the clean composition calculus...
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  \[
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  \]
- Theorem (triangle-good) follows by oscillatory testing,
  \[
  \rho \ast \mu(\tau) = (2\pi)^3(\hat{\mu}, \hat{\rho}e^{-i \langle \cdot, \tau \rangle})
  \]
Further questions

What happens at the degenerate interface? Can we refine bounds to\[
\sum \lambda_k \geq (1 + \epsilon)(\lambda_i + \lambda_j) \left| \langle e_i e_j, e_k \rangle \right|^2 \leq O((\lambda_i + \lambda_j)^{-\infty})\]
where \(\epsilon = \epsilon(\lambda_i + \lambda_j) \to 0\) quantitatively?

Scale \(\rho\) so that \(\rho T(\tau) = T(\rho(\tau))\). Can we get an Ivrii-type refinement for \(\rho T^* \mu(\tau)\) given some 'thinness' assumptions on the geodesic triples?
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- Scale \( \rho \) so that \( \rho_T(\tau) = T\rho(T\tau) \). Can we get an Ivrii-type refinement for

\[ \rho_T \ast \mu(\tau) \]

given some ‘thinness’ assumptions on the geodesic triples?
Thank you!