Homework 10, due 12/5

1. Let X be a compact Riemann surface, and for any (1,0)-form $\theta \in \Omega_X^{1,0}$, define the norm $\|\theta\|$ by
$$\|\theta\|^2 = i \int_X \theta \wedge \overline{\theta}.$$
From the previous homework we know that this is a non-negative real number, which vanishes only if $\theta = 0$. Denote by $[\theta]$ the equivalence class of θ in $\Omega^{1,0}/(\operatorname{im}\partial)$.

   Show that if $\alpha \in [\theta]$ has minimal norm among the elements in the class $[\theta]$, then $\overline{\partial}\alpha = 0$, i.e. α is a holomorphic one-form. (Note that this gives another approach to proving the isomorphism $H^{0,1} = \overline{H^{1,0}}$ from class.)

2. Show that there is no non-constant holomorphic map $f : \mathbf{P}^1 \to X$, where $X = \mathbf{C}/\Lambda$ is a complex torus. {*Hint: use f to define a holomorphic one-form on $\mathbf{P}^1$*}

3. Let $\Lambda_1 = \{m_1 w_1 + m_2 w_2 : m_1, m_2 \in \mathbf{Z}\}$ for some $w_1, w_2 \in \mathbf{C}$ satisfying $\operatorname{Im}(w_1/w_2) > 0$. Similarly define Λ_2 using $u_1, u_2 \in \mathbf{C}$ satisfying $\operatorname{Im}(u_1/u_2) > 0$.

   Show that the torus X/Λ_1 is biholomorphic to X/Λ_2 if and only if $\Lambda_1 = c\Lambda_2$ for some $c \in \mathbf{C}$. {*Hint: think about how a biholomorphism relates the holomorphic one-forms.*}