Homework 5, due 9/30

1. Let $\Omega = D(0,2) \setminus \{-1,1\}$. Find a closed curve $\gamma : [0,1] \to \Omega$ such that $n(\gamma, a) = 0$ for all $a \in \Omega$, but γ is not contractible to a constant in Ω. You can just describe the curve in words, you do not need to give a formula.

2. Let $\Omega \subset \mathbf{C}$ be connected, $f : \Omega \to \mathbf{C}$ non-constant, holomorphic. Let γ be a closed curve in Ω, homotopic to a constant in Ω. Suppose that $a \in \Omega$ satisfies $n(\gamma, a) > 0$, and $n(\gamma, z) \geq 0$ for all $z \in \Omega$.

   Prove that $f(\Omega)$ contains the connected component of $f(a)$ in $\mathbf{C} \setminus \mathrm{Im}(f \circ \gamma)$. This gives another proof of the open mapping theorem.

3. Prove that for each $w \in D(0,1)$, the equation $z^5(z-2) = w$ has exactly five solutions in $D(0,1)$, counted with multiplicities.

4. Let $g(z) = (z^2-1)^{-1}$.

   (a) Show that there is no holomorphic function $f : \mathbf{C} \setminus \{-1,1\} \to \mathbf{C}$ such that $f' = g$.

   (b) Is there a holomorphic function $f : \mathbf{C} \setminus \overline{D(0,1)}$ with $f' = g$?

5. Let $\omega_1, \omega_2 \in \mathbf{C}$ be linearly independent over $\mathbf{R}$, and let

   $$L = \{n_1\omega_1 + n_2\omega_2 \,:\, n_1, n_2 \in \mathbf{Z}\}.$$

   Let $f : \mathbf{C} \to \mathbf{C}$ be meromorphic, and doubly periodic, i.e. $f(z+l) = f(z)$ for all $l \in L$ and $z \in \mathbf{C}$ where f is holomorphic.

   For $z_0 \in \mathbf{C}$ denote by $P(z_0)$ the parallelogram

   $$P(z_0) = \{z_0 + t_1\omega_1 + t_2\omega_2 \,:\, t_1, t_2 \in [0,1]\}.$$

   (a) Show that if f has no poles, then f is constant.

   (b) If $\partial P(z_0)$ contains no zeros or poles of f, show that

   $$\int_{\partial P(z_0)} f\, dz = 0.$$

   (c) If $\partial P(z_0)$ contains no zeros or poles of f, prove that there are the same number of zeros and poles in $P(z_0)$, counted with multiplicity.

   (d) Suppose $\partial P(z_0)$ contains no zeros or poles of f. Let $z_1, \ldots, z_n$ be the zeros in $P(z_0)$, and $w_1, \ldots, w_n$ be the poles in $P(z_0)$, repeated according to multiplicities. Considering the integral of $zf'(z)/f(z)$ prove that

   $$\sum_{k=1}^{n} (z_k - w_k) \in L.$$