Homework 8, due 4/17
Only your **four** best solutions will count towards your grade.

1. Let $E \to X$ be a complex vector bundle, with a Hermitian metric h, and let ∇ be a connection on E. We can view h as a section of the bundle $E^* \otimes \overline{E}^*$ (or equivalently as a bilinear map $E \times \overline{E} \to \mathbf{C}$), and ∇ induces a natural connection on this bundle. Show that ∇ is compatible with h if and only if $\nabla h = 0$.

2. Let $L \to X$ be a complex line bundle over X, with a connection ∇. Then the curvature F_∇ is a (complex valued) closed 2-form on X. Show that every closed 2-form cohomologous to F_∇ is the curvature of a connection on L.

3. Let $(L, h) \to X$ be a Hermitian line bundle, and ∇ a Hermitian connection on it. Show that the curvature F_∇ is a purely imaginary 2-form.

4. Show that the map
$$\nabla^2 : \mathcal{A}^k(E) \to \mathcal{A}^{k+2}(E)$$
is given by F_∇, acting on the form part by the exterior product, and on the E part through its endomorphism component. I.e. if locally $F_\nabla = \sum_{i,j} F_{ij} dx^i \wedge dx^j$ for endomorphisms F_{ij}, and $\alpha \otimes s$ is a section of $\mathcal{A}^k(E)$, where α is a k-form and s a section of E, then
$$\nabla^2 s = \sum_{i,j} dx^i \wedge dx^j \wedge \alpha \otimes F_{ij}(s).$$

5. Let X be a compact complex manifold and $D \subset X$ a codimension-one complex submanifold. Define the line bundle L as in Problem 2 from homework set 5 (usually denoted by $\mathcal{O}(D)$), and let s be a global holomorphic section of L whose zero set is D. Choose a Hermitian metric h on L, let ∇ be the Chern connection, and F_∇ its curvature. Show that for any closed form α we have
$$\int_X F_\nabla \wedge \alpha = \lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \partial \log |s|_h^2 \wedge \alpha,$$
where $D_\epsilon = \{x : |s(x)|_h < \epsilon\}$ is a small neighborhood of D, and $|s|_h^2 = h(s, s)$.

6. With the same notation as in the previous question, show that
$$\frac{\sqrt{-1}}{2\pi} \int_X F_\nabla \wedge \alpha = \int_D \alpha,$$
i.e. the first Chern class $c_1(L)$ represents the Poincaré dual of D.