Homework 9, due 5/1

Only your **four** best solutions will count towards your grade.

1. Let E be a complex vector bundle over a complex manifold X, and ∇ a connection on E. Show that the trace $\mathrm{tr}F_\nabla$ of the curvature defines a closed two-form on X.

2. In the setting of the previous question, show that if ∇' is another connection on E, then $[\mathrm{tr}F_{\nabla'}] = [\mathrm{tr}F_\nabla]$ in $H^2(X, \mathbf{C})$.

3. Let L be a holomorphic line bundle over a complex manifold X. Suppose that we have a sheaf homomorphism

   $$D : L \to \Omega_X \otimes L,$$

   satisfying the Leibniz rule $D(f \cdot s) = \partial f \otimes s + f \cdot D(s)$ for local holomorphic functions f and holomorphic sections s of L. Here Ω_X denotes the sheaf of holomorphic $(1,0)$-forms on X, and we are using L to denote the sheaf of holomorphic sections of L.

   Show that D can be extended to a connection

   $$\nabla : \mathcal{A}^0(L) \to \mathcal{A}^1(L)$$

   on L such that $\nabla s = Ds$ for holomorphic sections s, and the curvature of ∇ is a holomorphic $(2,0)$-form on X.

4. In the setting of the previous question, if in addition X is a compact Kähler manifold, show that the curvature of ∇ vanishes.

5. Let (E, h) be a Hermitian vector bundle over a complex manifold X, and let $p \in X$ be a point. Let ∇ be a unitary connection on E. Show that there exists a unitary frame for E in a neighborhood of p, such that the corresponding matrix of connection 1-forms A satisfies $A(p) = 0$.