Homework 4, due 4/26

1. Let $p, q, r \in S^2$ denote 3 points on the sphere. Let X denote the topological space obtained by gluing p, q, r together to a single point.
   (a) Describe X as a CW complex.
   (b) Using your description compute the homology $H_p(X)$ with $\mathbb{Z}_2$ coefficients.
2. Let M be a compact d-manifold, with a Morse function $f : M \to \mathbf{R}$. For each p denote by c_p the number of critical points of f with index p.
   (a) Show that the Euler characteristic of M is
   $$\chi(M) = c_0 - c_1 + c_2 - \ldots + (-1)^d c_d.$$
   (b) Using that $-f$ is also a Morse function with the same critical points as f (but with different indices), show that $\chi(M) = 0$ if the dimension d is odd.
3. Below is a simplicial complex K whose geometric realization is the Klein bottle (the vertices and edges around the boundary are identified according to the labels). Find a suitable discrete vector field on K with no non-trivial closed paths, to show that K is homotopy equivalent to a CW complex with one 0-cell, two 1-cells and one 2-cell.

   I.e. there should be only one vertex that is not the tail of an arrow, only two edges that are not the tail or tip of an arrow and only one face that is not the tip of an arrow.