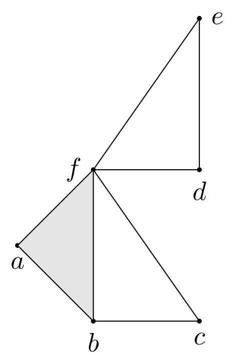
Midterm practice problems.

1. Let K be the simplicial complex shown in the diagram.



   (a) Write down bases for the $0, 1, 2$-chains in K, and write down the matrices representing the boundary maps ∂_2, ∂_1.

   (b) Compute the dimensions of $H_0(K), H_1(K), H_2(K)$.

   (c) Check your answer to (b) using the Euler-Poincaré formula for the Euler characteristic.

   (d) Let $K_0 \subset K$ be the subcomplex $K_0 = \{c, d, e\}$, i.e. K_0 just consists of 3 vertices. Compute the relative homology $H_1(K, K_0)$.

   (e) Let L be the simplicial complex consisting of the vertices a, b, f, and the 3 edges between them (i.e. $L = \{[a], [b], [f], [ab], [bf], [af]\}$). We have seen that $H_1(L) = \mathbb{Z}_2$ ($|L|$ is homeomorphic to a circle). Then map $F : \mathrm{Vert}(L) \to \mathrm{Vert}(K)$, with $F(a) = a, F(b) = b, F(c) = c$ is a vertex map, which extends to a simplicial map $F : L \to K$. What is the induced linear map $F_* : H_1(L) \to H_1(K)$?

2. Let K denote a simplicial complex consisting of a single 3-simplex (together with all of its faces). Let p, q be two vertices of K, and denote by $K_0 = \{[p], [q]\}$ the subcomplex consisting of these two vertices.

   (a) Use the long exact sequence in homology for the pair (K, K_0) to compute $H_1(K, K_0)$.

   (b) What does your answer mean geometrically?

3. Let K be a simplicial complex and $K_0 \subset K$ a subcomplex. Form the simplicial complex $\widetilde{K}$ as follows: add a new vertex o to K, and for any simplex $[u_0, \ldots, u_p]$ of K_0, add in a new simplex $[o, u_0, \ldots, u_p]$. In other words, $\widetilde{K}$ is obtained from K by taking the union $\widetilde{K} = K \cup CK_0$ of K with the cone over K_0, glued along K_0.

   Use the excision theorem, and the long exact sequence for a pair to show that $\widetilde{H}_p(\widetilde{K}) \cong H_p(K, K_0)$. You can use the fact, shown in homework, that $\widetilde{H}_p(CK_0) = 0$ for all p.

4. Let M be a Möbius strip, and S^1 its boundary circle. By representing M, S^1 as simplicial complexes, find the relative homology $H_p(M, S^1)$, for $p = 0, 1, 2$. (You should get the reduced homology of $\mathbb{R}P^2$, using the previous problem, since $\mathbb{R}P^2$ can be obtained by gluing a disk to a Möbius strip along its boundary.)