RESEARCH STATEMENT

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My mathematical research is mainly in geometric analysis. Currently I am interested in studying the geometry of Kähler manifolds by using the Gromov-Hausdorff convergence technique. For instance, the resolution of Yau-Tian-Donaldson conjecture was solved in this spirit.

1. THE UNIFORMIZATION CONJECTURE AND RELATED PROBLEMS

A major motivation of my current research is the uniformization conjecture of Yau. To state the conjecture, recall the uniformization theorem in one complex variable: every simply connected Riemann surface is isomorphic to the complex plane, the unit disk, or the Riemann sphere. From a geometric point of view, one can say if a complete Riemannian surface has positive curvature, then it is isomorphic to the complex plane or the Riemann sphere. It is natural to generalize such result to higher dimensions. If we consider the complex setting, it is natural to ask for a classification of Kähler manifolds with positive curvature. In higher dimensions there are different notions of curvature. Here let us consider the bisectional curvature. For a Kähler manifold, we say the bisectional curvature is positive, if $\sec(X, Y) + \sec(X, JY) > 0$ for all nonzero real vectors $X, Y$. Here $J$ is the almost complex structure.

In the compact case, Mori and Siu-Yau solved the famous Frankel conjecture that a compact Kähler manifold with positive bisectional curvature is biholomorphic to $CP^n$.

In the noncompact case, we have the following uniformization conjecture:

**Conjecture 1.** [Yau, 1974] Let $M$ be a complete noncompact Kähler manifold with positive bisectional curvature, then $M$ is biholomorphic to $CP^n$.

**Conjecture 2.** [Yau] Let $M$ be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Then the ring $O_P(M)$ is finitely generated.

**Conjecture 3.** [Yau] Let $M^n$ be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Then for any $d > 0$, $\dim(O_d(M)) \leq \dim(O_d(C^n))$. 
Conjecture 3 is the Kähler counterpart of another conjecture of Yau (solved by Colding-Minicozzi [7]) on dimension estimates for harmonic functions in the Riemannian case. Conjecture 3 was confirmed by Ni [42] with the assumption that $M$ has maximal volume growth. Later, by using Ni’s method, Chen-Fu-Le-Zhu [14] removed the extra condition. In [27], the author gave an alternative proof of conjecture 3 with a weaker assumption that the holomorphic sectional curvature is nonnegative, i.e., $\text{sec}(X, JX) \geq 0$ for all $X$. The tool is the three circle theorem which will be stated in section 3.

For conjecture 2 not much was known before. In [30], the author was able to confirm conjecture 2 in the general case:

**Theorem 1.1.** Let $M^n$ be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Then the ring $O_p(M)$ is finitely generated.

During the course of the proof, we obtained a partial result for conjecture 1:

**Theorem 1.2.** Let $M^n$ be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Assume that $M$ has maximal volume growth. Then $M$ is biholomorphic to an affine algebraic variety.

Here maximal volume growth means $\text{Vol}(B(p, r)) \geq Cr^{2n}$ for some $C > 0$. Sometimes this condition is also called noncollapsing. Theorem 1.2 seems to be the first uniformization type result without assuming an upper curvature bound.

If one wishes to prove conjecture 1 by considering $O_p(M)$, it is important to know when $O_p(M) \neq \mathbb{C}$. In [42], Ni proposed the following interesting conjecture:

**Conjecture 4.** Let $M^n$ be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Assume that $M$ has positive bisectional curvature at one point $p$. Then the following three conditions are equivalent:

1. $O_p(M) \neq \mathbb{C}$;
2. $M$ has maximal volume growth;
3. There exists a constant $C$ independent of $r$ so that $\frac{1}{B(p, r)} \int S \leq \frac{C}{r^2}$. Here $S$ is the scalar curvature and $\int$ means the average.

In the case of complex dimension one, the conjecture is known to hold. For higher dimensions, Ni proved that (1) implies (3) in [42]. The proof used the heat flow method. Then in [45], Ni and Tam proved that (3) also implies (1). Their proof employed the Poincare-Lelong equation and the heat flow method. Thus, it remained to prove that (1) and (2) are equivalent. Ni [43] and Ni-Tam [44] also made important contribution on equivalence of (1) and (2). In [28][29], the author proved that (1) is equivalent to (2). As a consequence, we obtain that

**Theorem 1.3.** Conjecture 4 is true.

In the first version of [33], building on the resolution of conjecture 2 the author was able to prove

**Theorem 1.4.** Let $M^n (n \leq 3)$ be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Assume $M$ has maximal volume growth, then $M$ is biholomorphic to $\mathbb{C}^n$. In fact, we can find $n$ polynomial growth holomorphic functions $f_1, \ldots, f_n$ which serve as the biholomorphism.

More recently, in the second version of [33] in 2017, the dimension restriction was removed. This confirms conjecture 1 (uniformization) under the assumption that $M$ has maximal volume growth.
Remark 1.1. The uniformization under maximal volume growth condition was also obtained by Lee and Tam \cite{37} recently. Their method is very different from ours.

Let $f$ be a polynomial growth holomorphic function on $M$. We define the degree of $f$ be the infimum of $d > 0$ so that $f \in O_d(M)$. We have the following interesting corollary:

**Corollary 1.1.** Under the same assumption as above, if $f$ is a nonconstant polynomial growth holomorphic function on $M$ with minimal degree, then $df \neq 0$ at any point.

The proof of results above is very different from earlier work. Here we make use of several different techniques: the Gromov-Hausdorff convergence theory developed by Cheeger-Colding \cite{2, 3, 4, 5} and Cheeger-Colding-Tian \cite{6}; the heat flow method by Ni \cite{42} and Ni-Tam \cite{44, 45}; the Hörmander $L^2$ estimate of $\partial$ \cite{20, 21}; and the three circle theorem in \cite{27} by the author.

**Further questions**

We noted that the uniformization conjecture is open. In fact, it is widely open: without giving extra conditions, one can hardly prove anything nontrivial about complete noncompact Kähler manifolds with positive bisectional curvature. For instance, it is unknown whether or not such manifolds are simply connected.

Here are two long-standing conjectures that are weaker than the uniformization conjecture.

**Conjecture 5** (Siu). Let $M$ be a complete noncompact Kähler manifold with positive bisectional curvature. Then $M$ is Stein.

**Conjecture 6** (Greene-Wu, Siu). Let $M$ be a complete noncompact Kähler manifold with positive sectional curvature. Then $M$ is biholomorphic to $\mathbb{C}^n$.

2. **Gromov-Hausdorff limits of Kähler manifolds with curvature bounded below**

The structure of Gromov-Hausdorff limits of Riemannian manifolds with Ricci curvature bounded below has been studied extensively since the seminal work of Cheeger-Colding \cite{2, 3, 4, 5}, with a great deal of recent progress by Colding, Cheeger and Naber. In the Kähler setting, the recent breakthrough work of Donaldson-Sun \cite{22} has led to many important advances. They proved in particular that the Gromov-Hausdorff limit of a sequence of non-collapsed, polarized Kähler manifolds, with 2-sided Ricci curvature bounds, is a normal projective variety. In \cite{34}, we generalized this statement, removing the assumption of an upper bound for the Ricci curvature.

**Theorem 2.1.** Given $n, d, v > 0$, there are constants $k_1, N > 0$ with the following property. Let $(M^n_i, L_i, \omega_i)$ be a sequence of polarized Kähler manifolds such that

- $L_i$ is a Hermitian holomorphic line bundle with curvature $\sqrt{-1}\omega_i$,
- $\text{Ric}(\omega_i) > -\omega_i$, $\text{vol}(M_i) > v$, and $\text{diam}(M_i, \omega_i) < d$,
- The sequence $(M^n_i, \omega_i)$ converges in the Gromov-Hausdorff sense to a limit metric space $X$.

Then each $M^n_i$ can be embedded in a subspace of $\mathbb{C}P^N$ using sections of $L_i^{k_1}$, and the limit $X$ is homeomorphic to a normal projective variety in $\mathbb{C}P^N$. Taking a subsequence and applying suitable projective transformations, the $M_i \subset \mathbb{C}P^N$ converge to $X$ as algebraic varieties.

The overall approach follows Donaldson-Sun \cite{22}, and a key step is the following partial $C^0$-estimate, conjectured originally by Tian for Fano manifolds.
Theorem 2.2. Given $n, d, v > 0$, there are $k_2, b > 0$ with the following property. Suppose that $(M, L, \omega)$ is a polarized Kähler manifold with $\text{Ric}(\omega) > -\omega$, $\text{vol}(M) > v$ and $\text{diam}(M, \omega) < d$. Then for all $p \in M$, the line bundle $L^k$ admits a holomorphic section $s$ over $M$ satisfying $|s|_{L^2} = 1$, and $|s(p)| > b$.

Tian conjectured this result under a positive lower bound for the Ricci curvature, with $L = K_M^{-1}$, and proved it in the two-dimensional case. Donaldson-Sun [22] showed the result with two-sided Ricci curvature bounds, but arbitrary polarizations, and later several extensions of their result were obtained by others. The result assuming a lower bound for the Ricci curvature, with $L = K_M^{-1}$ was finally shown by Chen-Wang [19]. The improvement in our result is that we allow for general polarizations.

Now let $(M^n_m, q_i, d_i)$ be a sequence of pointed complete Riemannian manifolds with $\text{Ric} > -1$. By the Gromov compactness theorem, we may assume that the sequence converges in the pointed Gromov-Hausdorff sense to a metric length space $(Y, q, d)$. Let us also assume that the sequence is non-collapsing, that is, $\text{vol}(B(q_i, 1)) > v > 0$. A basic question in metric differential geometry is on the regularity of $(Y, q, d)$. The work of Cheeger-Colding [3] shows that $(Y, q, d)$ is a length space of Hausdorff dimension $m$, and at each $p \in Y$ it admits possibly non-unique tangent cones $Y_p$, which are metric cones. A point $p \in Y$ is regular if every tangent cone at $p$ is isometric to $\mathbb{R}^m$, and is singular otherwise. We will denote by $\mathcal{R} \subset Y$ the set of regular points. In addition, for small $\epsilon > 0$, denote by $\mathcal{R}_\epsilon \subset Y$ the set of points $p$ so that $\omega_m - \lim_{r \to 0} \frac{\text{Vol}(B(p, r))}{r^m} < \epsilon$. Here $\omega_m$ is the volume of the unit ball in $\mathbb{R}^m$. Then $\mathcal{R} = \cap_{\epsilon > 0} \mathcal{R}_\epsilon$. Note that $\mathcal{R}_\epsilon$ is an open set, while in general $\mathcal{R}$ may not be open. Cheeger-Colding [3] showed that the Hausdorff dimension of $Y \setminus \mathcal{R}$ is at most $m - 2$, with more quantitative estimates obtained by Cheeger-Naber [16]. Moreover, in a recent deep work of Cheeger-Jiang-Naber [15], it was shown that for small $\epsilon$ the set $Y \setminus \mathcal{R}_\epsilon$ has bounded $(m - 2)$-dimensional Minkowski content and is $m - 2$ rectifiable. These results show that the singular set behaves well from the perspective of geometric measure theory. On the other hand, the topology of the singular set could be rather complicated. In a recent paper, Li and Naber [36] show that even assuming non-negative sectional curvature, non-collapsed limit spaces can have singular sets that are Cantor sets.

Now let us assume in addition that the $N^n_m$ is a sequence of polarized Kähler manifolds. Then, as we saw above, the limit $Y$ is naturally identified with a projective variety. When the metrics along the sequence are Kähler-Einstein, Donaldson-Sun [22] showed that the metric singular set of $Y$ is the same as the complex analytic singular set of the corresponding projective variety. In our setting this is not necessarily the case, however we have the following.

Theorem 2.3. Let $(X, d)$ be a Gromov-Hausdorff limit as in Theorem 2.2. Then for any $\epsilon > 0$, $X \setminus \mathcal{R}_\epsilon$ is contained in a finite union of analytic subvarieties of $X$. Furthermore, the singular set $X \setminus \mathcal{R}$ is equal to a countable union of subvarieties.

The key to the theorem is a complex analytic characterization of the metric regularity of the limit space $X$. We proved that a point on $X$ is metric regular if and only if it is complex analytic regular and the Lelong number for certain positive closed $(1, 1)$ current is zero. Then the theorem follows from a famous result of Yum-Tong Siu.

Remark 2.1. Since the singular set could be dense (think about a convex surface in $\mathbb{R}^3$), the countable union in Theorem 2.3 cannot be replaced by a finite union.

Remark 2.2. In view of Li-Naber [36], this result shows that the behavior of singularities in the Kähler case is sharp in contrast with the Riemannian case. On the one hand, the
metric singularities in the Kähler case might seem flexible, since one can perturb Kähler potentials locally. On the other hand, analytic sets are very rigid, and so in particular Theorem 2.3 implies the following: if we perturb the Kähler metric inside a holomorphic chart and assume that the geometric assumptions are preserved, then the metric singular set can change by at most a countable set of points.

For the technical part, the following proposition in [34] plays a fundamental role:

**Theorem 2.4.** There exists $\epsilon > 0$, depending on the dimension $n$ with the following property. Suppose that $B(p, \epsilon^{-1})$ is a relatively compact ball in a (not necessarily complete) Kähler manifold $(M^n, p, \omega)$, satisfying $\text{Ric}(\omega) > -\epsilon \omega$, and

$$d_{GH}(B(p, \epsilon^{-1}), B_{\mathbb{C}^n}(0, \epsilon^{-1})) < \epsilon.$$

Then there is a holomorphic chart $F : B(p, 1) \rightarrow \mathbb{C}^n$ which is a $\Psi(\epsilon|n)$-Gromov-Hausdorff approximation to its image. In addition on $B(p, 1)$ we can write $\omega = i\partial \bar{\partial} \phi$ with $|\phi - r^2| < \Psi(\epsilon|n)$, where $r$ is the distance from $p$.

Basically theorem 2.4 is a holomorphic $\epsilon$-regularity result. This is an extension of proposition 1.3 of [30], where the bisectional curvature lower bound was assumed. In particular this leads to a complex manifold structure on the set $\mathcal{R}_\epsilon$ above, for sufficiently small $\epsilon$.

Here are two interconnected difficulties for theorem 2.4. 1. The lack of positivity. It is well-known that to solve $\bar{\partial}$-equation, one needs some pointwise positivity of curvature. Here, however, there is no natural positivity, from the geometric assumptions. 2. The weakness of regularity. If the Ricci curvature has two side bounds, then we have the $C^{1,\alpha}$ regularity. However, when we only assume Ricci lower bound, it is not even clear whether there is a bilipschitz regularity.

To overcome the difficulties, we need to do regularization. When the bisectional curvature has a lower bound ([30]), we considered the heat flow. In the Ricci curvature lower bound case, this method breaks down fundamentally, since the Ricci lower bound is too weak to control complex Hessian. Instead, we regularize the metric. That is, we use the Ricci flow. New difficulties appear. For example, since the metric is not assumed to be complete, it is not clear whether the Kähler condition will be preserved. Eventually we managed to overcome all difficulties. One key tool is Perelman’s pseudolocality.

We give two further applications of this result. The first result shows that under Gromov-Hausdorff convergence to a smooth Riemannian manifold, the scalar curvature functions converge as measures. Here we state a simple corollary of this.

**Corollary 2.1.** Given any $\epsilon > 0$, there is a $\delta > 0$ depending on $\epsilon, n$ satisfying the following. Let $B(p, 1)$ be a relatively compact unit ball in a Kähler manifold $(M^n, \omega)$ satisfying $\text{Ric} > -1$, and $d_{GH}(B(p, 1), B_{\mathbb{C}^n}(0, 1)) < \delta$. Then $|\int_{B(p, 1)} S| < \epsilon$, where $S$ is the scalar curvature of $\omega$.

**Remark 2.3.** To the best of the author’s knowledge, previously it was not even clear whether the integral of scalar curvature is uniformly bounded.

The other application is the following, which was proved previously by the author [30] under the assumption of non-negative bisectional curvature.

**Proposition 2.1.** There exists $\epsilon > 0$ depending on $n$, so that if $M^n$ is a complete non-compact Kähler manifold with $\text{Ric} \geq 0$ and $\lim_{r \to \infty} r^{2n} \text{vol}(B(p, r)) \geq \omega_{2n} - \epsilon$, then $M$ is biholomorphic to $\mathbb{C}^n$. Here $\omega_{2n}$ is the volume of the Euclidean unit ball.
In [31][34], Gromov-Hausdorff limits of Kähler manifolds with bisectional curvature lower bound and noncollapsed volume is studied.

**Theorem 2.5.** Let \((M_\infty, p_\infty)\) be the pointed Gromov-Hausdorff limit of a sequence of complete (compact or noncompact) Kähler manifolds \((M^n_i, p_i)\) with bisectional curvature lower bound \(-1\) and \(\text{vol}(B(p_i, 1)) \geq v > 0\). Then \((M_\infty, p_\infty)\) is homeomorphic to a normal complex analytic space with singular set of complex codimension at least 4. Furthermore, the metric singularity is equal to a countable union of analytic subvarieties.

In another joint work [35] with Gabor Székelyhidi, we studied the analyticity of the tangent cones for Gromov-Hausdorff limits of Kähler manifolds with Ricci curvature lower bound and noncollapsed volume. Note here no projectivity is assumed. The following result is obtained:

**Theorem 2.6.** Let \((X, p)\) be a noncollapsed Gromov-Hausdorff limit of complete (compact or noncompact) Ricci flat Kähler manifolds of dimension \(n\). Let \(V\) be a tangent cone at \(p\).

Then \(V\) is homeomorphic to an affine algebraic variety.

**Remark 2.4.** Such result was obtained by Donaldson-Sun [23] when all Kähler manifolds are polarized.

Recently, we are almost certain that such result can be further extended to the case when the sequence of Kähler manifolds have only Ricci lower bound and noncollapsed volume.

**Further questions**

It is interesting to ask whether the limit space \(X\) in theorem 2.6 is a complex analytic space. As we see above, this basically removes the polarization assumption in Donaldson-Sun’s theorem.

3. **Three circle theorem**

The classical Hadamard three circle theorem says that the logarithm of the modulus of a holomorphic function in the disk is a convex function of \(\log r\). It is natural to wonder whether one can generalize this theorem to complete Kähler manifolds.

**Definition 3.1.** Let \(M\) be a complete Kähler manifold. Given a point \(p \in M\), a holomorphic function \(f\) defined on some ball around \(p\), and a radius \(r > 0\), put \(M_f(r) = \sup |f(x)|\) for \(x \in B(p, r)\). We say \(M\) satisfies the three circle theorem if \(\log M_f(r)\) is a convex function of \(\log r\). In other words, for \(r_1 < r_2 < r_3\),

\[
\log \left(\frac{r_3}{r_1}\right) \log M_f(r_2) \leq \log \left(\frac{r_3}{r_2}\right) \log M_f(r_1) + \log \left(\frac{r_2}{r_1}\right) \log M_f(r_3).
\]

In [27], the author classified Kähler manifolds satisfying the three circle theorem:

**Theorem 3.1.** Let \(M\) be a complete Kähler manifold. Then \(M\) satisfies the three circle theorem if and only if the holomorphic sectional curvature is nonnegative.

The proof is elementary. However, it is an important tool in uniformization type problems in section 1. The theorem basically helps us to bound and construct polynomial growth holomorphic functions.
4. **Three-manifolds with nonnegative Ricci curvature**

Let $M$ be a complete manifold with nonnegative Ricci curvature. Then it is a fundamental problem in geometry to find restrictions on the topology of $M$. In the two dimensional case, the Ricci curvature is the same as the Gaussian curvature $K$. It is well known that if $K \geq 0$, then the surface is covered by $C$ or $S^2$.

Let us consider 3-manifolds with nonnegative Ricci curvature. Using Ricci flow, Hamilton [25] classified compact 3-manifolds with nonnegative Ricci curvature. He proved that the universal cover is diffeomorphic to $S^3$, $S^2 \times \mathbb{R}$ or $\mathbb{R}^3$. In the latter two cases, the metric is a product on each $\mathbb{R}$ factor. For the noncompact case, Anderson-Rodriguez [1] and Shi [48] classified these manifolds when the sectional curvature have an upper bound. Zhu [61] proved that if the manifold has volume growth like $r^3$, then the manifold is contractible. In [50], by means of minimal surfaces, Schoen and Yau proved that a complete noncompact 3-manifold with positive Ricci curvature is diffeomorphic to $\mathbb{R}^3$. They also announced the classification for the nonnegative Ricci curvature case.

In [32], the author was able to obtain a complete classification for the noncompact case, following the method of Schoen and Yau [50].

**Theorem 4.1.** Let $M^3$ be a complete noncompact three manifold with nonnegative Ricci curvature. Then either $M$ is diffeomorphic to $\mathbb{R}^3$ or the universal cover is isometric to $N^2 \times \mathbb{R}$ where $N^2$ is a complete 2-manifold with nonnegative curvature.

5. **Compactification of certain noncompact Kähler manifolds with nonnegative Ricci curvature**

Recall three long standing conjectures of Yau (the first is just the uniformization conjecture):

**Conjecture 7.** Let $M$ be a complete noncompact Kähler manifold with positive bisectional curvature, then $M$ is biholomorphic to $\mathbb{C}^n$.

**Conjecture 8.** Let $M$ be a complete noncompact Ricci flat Kähler manifold with finite topological type, then $M$ can be compactified complex analytically.

**Conjecture 9.** Let $M$ be a complete noncompact Kähler manifold with positive Ricci curvature. Then $M$ is biholomorphic to a Zariski open set of a compact Kähler manifold.

The moral in these conjectures is that when the Ricci curvature is positive (nonnegative), one might be able to compactify the manifold complex analytically.

By extending some techniques in section 1, we obtain in [29] the following:

**Theorem 5.1.** Let $(M^n, p) (n \geq 2)$ be a complete noncompact Kähler manifold with nonnegative Ricci curvature and maximal volume growth. Let $r(x) = d(x, p)$. Then

(I) $M$ is biholomorphic to a Zariski open set of a Moishezon manifold, if for some $\epsilon > 0$, the bisectional curvature $BK \geq -\frac{C}{r^2}$. If fact, on $M$, the ring of polynomial growth holomorphic functions is finitely generated.

(II) If $BK \geq -\frac{C}{r^2}$ and $M$ has a unique tangent cone at infinity, then $M$ is biholomorphic to a Zariski open set of a Moishezon manifold.

(III) $M$ is quasiprojective, if the Ricci curvature is positive and $|Rm| \leq \frac{C}{r^2}$.

Combining part (II) with some argument of Donaldson-Sun, we obtain

**Corollary 5.1.** Let $M$ be a complete noncompact Kähler-Ricci flat manifold with maximal volume growth. Assume the curvature has quadratic decay. Then $M$ is a crepant resolution.
of a normal affine algebraic variety. Furthermore, there exist two step degenerations from that affine variety to the unique metric tangent cone of $M$ at infinity.

Remark 5.1. In [52], Tian classified complete noncompact Kähler-Ricci flat manifolds with maximal volume growth and curvature faster than quadratic decay. Tian and Yau [53][54] also constructed many examples of complete Ricci-flat Kähler manifolds with maximal volume growth and quadratic curvature decay.


Theorem 1.1 is a generalization of several known results. For instance, part I is a generalization of theorem 1.2 in section 1.

In a series of papers, R. Conlon and H. Hein systematically studied asymptotically conical Calabi-Yau manifolds. In some sense, the corollary is a generalization of some of their results.

In [40], Mok proved the following.

**Theorem 5.2.** Let $M^n$ be a complete noncompact Kähler manifold with positive Ricci curvature and maximal volume growth. Assume $|Rm| \leq \frac{C}{r^2}$ and $\int_M Ric^n < +\infty$, then $M$ is biholomorphic to a quasi-projective variety.

Part III removes the assumption that $\int_M Ric^n < +\infty$.

The strategy of part I and II is to consider polynomial growth holomorphic functions. We shall follow the argument in [29]. However, there are some differences.

1. In this paper, we construct plurisubharmonic functions by using elliptic method. In [30], the parabolic method of Ni-Tam [44] was adopted.

2. The original three circle theorem in [27] does not work in this paper. In part I, we just use the extended version in [27]. In part II, we apply Donaldson-Sun’s three circle theorem [23].

3. In the setting of [30], polynomial growth holomorphic functions separate points and tangents. This is no longer true in part I and II, due to the possibility of compact subvarieties.

In some sense, part III resembles part I and II. However, the argument is very different. We basically follow the argument of Mok [40]. The strategy is to consider pluri-anticanonical sections with polynomial growth. The key new result is a uniform multiplicity estimate for pluri-anticanonical sections. This provides the dimension estimate for polynomial growth pluri-anticanonical sections, without the extra assumption $\int_M Ric^n < +\infty$.

**References**


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