SOME CURIOUS q-SERIES EXPANSIONS AND BETA INTEGRAL EVALUATIONS

GEORGE GASPER AND MICHAEL SCHLOSSER*

Dedicated to Dick Askey on the occasion of his 70th birthday

ABSTRACT. We deduce several curious q-series expansions by applying inverse relations to certain identities for basic hypergeometric series. After rewriting some of these expansions in terms of q-integrals, we obtain, in the limit $q \to 1$, some curious beta-type integral evaluations which appear to be new.

1. INTRODUCTION

Euler's beta integral evaluation (cf. [2, Eq. (1.1.13)])

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \Re(a), \Re(b) > 0, \tag{1.1}$$

is one of the most important and prominent identities in special functions. In Andrews, Askey and Roy's modern treatise [2], the beta integral (and its various extensions) runs like a thread through their whole exposition.

Concerning the evaluation of integrals of, say, elementary functions, there is no general procedure which will find the closed form evaluation if it exists. Being encountered with some explicit integral (in this paper all integrals are definite), if standard methods seem out of reach, it is usually wise to consult one of the compound volumes listing tables of integrals [7, 12, 14], hoping that the sought evaluation could be found in there. However, this does not always lead to success. In particular, several of the integral evaluations obtained in this paper (specifically, Theorems 5.1 and 5.2 and their specializations (1.2), (1.3), (5.4)) are apparently not included as entries in the standard references [7, 12, 14]. For instance, two special cases ($\alpha = \beta + 1$, and $\alpha = \beta$, respectively) of one of our main results (Theorem 5.1, which generalizes (1.1)) are the following beta-type integral evaluations:

$$\frac{\Gamma(\beta)\Gamma(\beta)}{2\,\Gamma(2\beta)} = (c-(a+1)^2)\int_0^1 \frac{(c-a(a+t))^{\beta}\,(c-(a+1)(a+t))^{\beta-1}}{(c-(a+t)^2)^{2\beta}}$$

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$$\times\, t^{\beta}\,(1-t)^{\beta-1}\,dt \quad (1.2)$$

and

$$\frac{\Gamma(\beta)\Gamma(\beta)}{\Gamma(2\beta)} = (c-(a+1)^2)\int_0^1 \frac{(c-a(a+t))^{\beta-1}\,(c-(a+1)(a+t))^{\beta-1}}{(c-(a+t)^2)^{2\beta}} \\ \times\,(c-(a-t)(a+t))\,t^{\beta-1}\,(1-t)^{\beta-1}\,dt, \quad (1.3)$$

where $\Re(\beta) > 0$. (These evaluations and others have been numerically verified using Mathematica.)

These integrals seem difficult to prove with standard methods, such as expanding all factors in terms of powers of t (by the binomial theorem) and integrating term-wise. Applying this procedure to (1.2) yields a five-fold sum that can be easily reduced to a four-fold sum, but then one is apparently stuck.

In the sequel, we will develop some machinery for proving our integral evaluations. First we derive, by inverse relations, new q-series expansions. We then rewrite these in terms of q-integrals. Finally, by letting $q \to 1$ we obtain the desired beta-type integral evaluations.

2. Preliminaries

2.1. **Hypergeometric and basic hypergeometric series.** For a complex number a, define the *shifted factorial*

$$(a)_0 := 1, \qquad (a)_k := a(a+1)\dots(a+k-1),$$

where k is a positive integer. Let r be a positive integer. The *hypergeometric ${}_rF_{r-1}$ series* with numerator parameters $a_1,\dots,a_r$, denominator parameters $b_1,\dots,b_{r-1}$, and argument z is defined by

$${}_rF_{r-1}\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_{r-1}\end{matrix};z\right] := \sum_{k\ge 0}\frac{(a_1)_k\dots(a_r)_k}{k!\,(b_1)_k\dots(b_{r-1})_k}\,z^k.$$

The ${}_rF_{r-1}$ series terminates if one of the numerator parameters is of the form $-n$ for a nonnegative integer n. If the series does not terminate, it converges when $|z|<1$, and also when $|z|=1$ and $\Re[b_1+b_2+\dots+b_{r-1}-(a_1+a_2+\dots+a_r)]>0$. See [3, 17] for a classic texts on (ordinary) hypergeometric series.

Let q (the "base") be a complex number such that $0<|q|<1$. Define the *q-shifted factorial* by

$$(a;q)_\infty := \prod_{j\ge 0}(1-aq^j) \qquad \text{and} \qquad (a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty}$$

for integer k. The *basic hypergeometric ${}_r\phi_{r-1}$ series* with numerator parameters $a_1,\dots,a_r$, denominator parameters $b_1,\dots,b_{r-1}$, base q, and argument

z is defined by

$$ {}_r\phi_{r-1}\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_{r-1}\end{matrix};q,z\right] := \sum_{k\geq 0} \frac{(a_1;q)_k\dots(a_r;q)_k}{(q;q)_k(b_1;q)_k\dots(b_{r-1};q)_k}\,z^k. $$

The ${}_r\phi_{r-1}$ series terminates if one of the numerator parameters is of the form q^{-n} for a nonnegative integer n. If the series does not terminate, it converges when $|z|<1$. For a thorough exposition on basic hypergeometric series (or, synonymously, *q-hypergeometric series*), including a list of several selected summation and transformation formulas, we refer the reader to [9].

We list three specific identities which we will utilize in this paper.

Lemma 2.1 (q-Kummer summation (cf. [9, Eq. (II.9)]))**.**

$$ {}_2\phi_1\left[\begin{matrix}a,b\\ aq/b\end{matrix};q,-\frac{q}{b}\right] = \frac{(-q;q)_\infty\,(aq;q^2)_\infty(aq^2/b^2;q^2)_\infty}{(-q/b;q)_\infty(aq/b;q)_\infty}, $$

provided $|q/b|<1$.

Proof. One may simply specialize Rogers' nonterminating very-well-poised ${}_6\phi_5$ summation (cf. [9, Eq. (II.20)])

$$ {}_6\phi_5\left[\begin{matrix}a,\ q\sqrt{a},-q\sqrt{a},b,c,d\\ \sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d\end{matrix};q,\frac{aq}{bcd}\right] $$
$$ = \frac{(aq;q)_\infty(aq/bc;q)_\infty(aq/bd;q)_\infty(aq/cd;q)_\infty}{(aq/b;q)_\infty(aq/c;q)_\infty(aq/d;q)_\infty(aq/bcd;q)_\infty}, $$

where $|aq/bcd|<1$, by setting $c=\sqrt{a}$ and $d=-\sqrt{a}$, hereby "cancelling off" the very-well-poised term. □

For a simple derivation of the following transformation from the q-binomial theorem (4.2), see [9, § 1.4].

Lemma 2.2 (Second iterate of Heine's transformation (cf. [9, Eq. (III.2)]))**.**

$$ {}_2\phi_1\left[\begin{matrix}a,b\\ c\end{matrix};q,z\right] = \frac{(c/b;q)_\infty\,(bz;q)_\infty}{(c;q)_\infty(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}abz/c,b\\ bz\end{matrix};q,\frac{c}{b}\right], $$

provided $|z|,|c/b|<1$.

Lemma 2.3 (An $(m+1)$-term ${}_3\phi_2$ summation)**.** *Let m be a nonnegative integer. Then*

$$ {}_3\phi_2\left[\begin{matrix}a,b,dq^m\\ c,d\end{matrix};q,\frac{cq^{-m}}{ab}\right] = \frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}a,b,q^{-m}\\ abq/c,d\end{matrix};q,q\right], \tag{2.1} $$

provided $|cq^{-m}/ab|<1$.

Proof. This can be obtained from [9, Eq. (III.34)], i.e. the three-term transformation

$$ {}_3\phi_2\left[\begin{matrix}a,b,c\\ d,e\end{matrix};q,\frac{de}{abc}\right] = \frac{(e/b;q)_\infty(e/c;q)_\infty}{(e;q)_\infty(e/bc;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}d/a,b,c\\ d,bcq/e\end{matrix};q,q\right] $$

$$+\frac{(d/a;q)_\infty(b;q)_\infty(c;q)_\infty(de/bc;q)_\infty}{(d;q)_\infty(e;q)_\infty(bc/e;q)_\infty(de/abc;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}e/b,e/c,de/abc\\ de/bc,eq/bc\end{matrix};q,q\right],$$

where $|de/abc|<1$, by first letting $a\to dq^m$, by which the coefficient of the second ${}_3\phi_2$ on the right-hand vanishes, and then suitably relabeling the parameters. ☐

A more direct proof of Lemma 2.3 proceeds by induction on m, using the q-Gauß summation (2.2) in the inductive basis, and the simple identity

$$\frac{1-dq^{m+k}}{1-dq^m}=q^k+\frac{1-q^k}{1-dq^m}$$

in the inductive step. The details are left to the reader.

Remark 2.4. We view (2.1) as a *summation* (versus a *transformation*) since the left-hand side is a nonterminating sum and the right hand-side contains a finite number of terms. For $m=0$ Lemma 2.3 reduces to the classical q-Gauß summation (cf. [9, Eq. (II.8)])

$$ {}_2\phi_1\left[\begin{matrix}a,b\\ c\end{matrix};q,\frac{c}{ab}\right]=\frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}, \tag{2.2}$$

whereas for $m=1$ it reduces to

$$ {}_3\phi_2\left[\begin{matrix}a,b,dq\\ c,d\end{matrix};q,\frac{c}{abq}\right]=\left(1-\frac{(1-a)(1-b)}{(1-abq/c)(1-d)}\right)\frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}.$$

2.2. **Inverse relations.** Let $\mathbb{Z}$ denote the set of integers and $F=(f_{nk})_{n,k\in\mathbb{Z}}$ be an infinite lower-triangular matrix; i.e. $f_{nk}=0$ unless $n\geq k$. The matrix $G=(g_{kl})_{k,l\in\mathbb{Z}}$ is said to be the *inverse matrix* of F if and only if

$$\sum_{l\leq k\leq n} f_{nk}g_{kl}=\delta_{nl}$$

for all $n,l\in\mathbb{Z}$, where δ_{nl} is the usual Kronecker delta.

The method of applying *inverse relations* [13] is a well-known technique for proving identities, or for producing new ones from given ones.

If $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ are lower-triangular matrices that are inverses of each other, then

$$\sum_{n\geq k} f_{nk}a_n=b_k \tag{2.3a}$$

if and only if

$$\sum_{k\geq l} g_{kl}b_k=a_l, \tag{2.3b}$$

subject to suitable convergence conditions. For some applications of (2.3) see e.g. [11, 13, 15].

Note that in the literature it is actually more common to consider the following inverse relations involving finite sums,

$$\sum_{k=0}^{n} f_{nk} a_k = b_n \qquad \textit{if and only if} \qquad \sum_{l=0}^{k} g_{kl} b_l = a_k. \tag{2.4}$$

It is clear that in order to apply (2.3) (or (2.4)) effectively, one should have some explicit matrix inversion at hand.

Lemma 2.5 (Krattenthaler [11]). *Let $(a_j)_{j\in\mathbb{Z}}$, $(c_j)_{j\in\mathbb{Z}}$ be arbitrary sequences and d an arbitrary indeterminate. Then the infinite matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ are inverses of each other, where*

$$f_{nk} = \frac{\prod_{j=k}^{n-1}(a_j - d/c_k)(a_j - c_k)}{\prod_{j=k+1}^{n}(c_j - d/c_k)(c_j - c_k)},$$

$$g_{kl} = \frac{(a_l c_l - d)(a_l - c_l)}{(a_k c_k - d)(a_k - c_k)} \frac{\prod_{j=l+1}^{k}(a_j - d/c_k)(a_j - c_k)}{\prod_{j=l}^{k-1}(c_j - d/c_k)(c_j - c_k)}.$$

Krattenthaler's matrix inverse is very general as it contains a vast number of other known explicit infinite matrix inversions. Several of its useful special cases are of (basic) hypergeometric type. The following special case of Lemma 2.5 is exceptional in the sense that although it involves powers of q, it is *not* to be considered a q-hypergeometric inversion. (More precisely, the following special case serves as a bridge between q-hypergeometric and certain non-q-hypergeometric identities. For some other such matrix inverses, see [15].)

Corollary 2.6 (MS [15, Eqs. (7.18)/(7.19)]). *Let*

$$f_{nk} = \frac{(1/b;q)_{n-k}\left(\frac{(a+bq^k)q^k}{c-a(a+bq^k)};q\right)_{n-k}}{(q;q)_{n-k}\left(\frac{(a+bq^k)bq^{k+1}}{c-a(a+bq^k)};q\right)_{n-k}},$$

$$g_{kl} = (-1)^{k-l} q^{\binom{k-l}{2}} \frac{(c-(a+bq^l)(a+q^l))}{(c-(a+bq^k)(a+q^k))} \frac{(q^{l-k+1}/b;q)_{k-l}\left(\frac{(a+bq^k)q^{l+1}}{c-a(a+bq^k)};q\right)_{k-l}}{(q;q)_{k-l}\left(\frac{(a+bq^k)bq^l}{c-a(a+bq^k)};q\right)_{k-l}}.$$

Then the infinite matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ are inverses of each other.

Proof. In Lemma 2.5 set $a_j \mapsto a + q^j$, $c_j \mapsto a + bq^j$ ($j \in \mathbb{Z}$), and $d \mapsto c$, and perform some elementary manipulations. □

3. Some curious q-series expansions

Corollary 2.6 was utilized in [15, Th. 7.16] to deduce from the classical q-Gauß summation a specific q-series expansion, the latter itself not belonging to the hierarchy of basic hypergeometric series. In particular, the following identity was obtained:

$$\frac{(b^2q;q)_\infty}{(bq;q)_\infty} = \sum_{k=0}^{\infty} \frac{(c-(a+1)(a+b))}{(c-(a+1)(a+bq^k))} \frac{(c-(a+bq^k)^2)}{(c-(a+b)(a+bq^k))} \\ \times \frac{(b;q)_k \left(\frac{(a+bq^k)}{c-a(a+bq^k)};q\right)_k \left(\frac{(a+bq^k)b^2q^{k+1}}{c-a(a+bq^k)};q\right)_\infty}{(q;q)_k \left(\frac{(a+bq^k)bq}{c-a(a+bq^k)};q\right)_\infty} (bq)^k, \quad (3.1)$$

where $|bq|<1$. We find this to be quite a curious expansion. For $c=0$ it reduces to a special case of the q-Gauß summation (2.2). On the other hand, for $a=0$ it reduces to

$$\frac{(b^2q;q)_\infty(b^2q/c;q)_\infty}{(bq;q)_\infty(b^3q/c;q)_\infty} = \sum_{k=0}^{\infty} \frac{(1-b^2q^{2k}/c)}{(1-b^2/c)} \frac{(b^2/c;q)_k(b;q)_k\,(b/c;q)_{2k}}{(q;q)_k(bq/c;q)_k\,(b^3q/c;q)_{2k}} (bq)^k,$$

a particular very-well-poised ${}_8\phi_7$ summation, which is equivalent to the $n\to\infty$ special case of the terminating very-well-poised ${}_{10}\phi_9$ summation,

$${}_{10}\phi_9\left[\begin{matrix} a,\, q\sqrt{a}, -q\sqrt{a}, \sqrt{b}, -\sqrt{b}, \sqrt{bq}, -\sqrt{bq}, a/b, a^2q^{n+1}/b, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/\sqrt{b}, -aq/\sqrt{b}, a\sqrt{q/b}, -a\sqrt{q/b}, bq, bq^{-n}/a, aq^{n+1} \end{matrix};q,q\right] \\ = \frac{(aq;q)_n\,(a^2q/b^2;q)_n}{(aq/b;q)_n\,(a^2q/b;q)_n}, \quad (3.2)$$

given in [9, Ex. 2.12]. This ${}_{10}\phi_9$ summation itself follows immediately from taking the limit $d\to 1$ in [9, Eq. (2.8.3)] which is Bailey's [4, p. 431], [5] transformation of a terminating, balanced, nearly-poised of the second kind ${}_5\phi_4$ series into a multiple of a particular terminating, balanced, very-well-poised ${}_{12}\phi_{11}$ series. (See [9] for the terminology.) This latter ${}_5\phi_4 \leftrightarrow {}_{12}\phi_{11}$ transformation is a consequence of the "WP-Bailey lemma", cf. [9, Eq. (2.8.2)] and [1, §§ 6 and 7].

The matrix inversion in Corollary 2.6 was also applied to both the classical q-Pfaff–Saalschütz summation and the 2-balanced ${}_3\phi_2$ summation, to derive two non-q-hypergeometric terminating summations, see [15, Ths. 7.34 and 7.38]. For illustration (and to correct some misprints which appeared in the printed version of [15]), we reproduce the first one of these, specifically, (7.35) of [15, Th. 7.34]:

$$\frac{(c^2q;q)_n}{(cq;q)_n} = \sum_{k=0}^{n} \frac{(b+(a-c)(a-1))}{(b+(a-c)(a-q^{-k}))} \frac{(b+(a-q^{-k})^2)}{(b+(a-1)(a-q^{-k}))} \\ \times \frac{(q^{-n};q)_k\,(c;q)_k \left(\frac{b+a(a-q^{-k})}{c(a-q^{-k})};q\right)_k}{(q;q)_k\,(q^{-n}/c;q)_k \left(cq\frac{b+a(a-q^{-k})}{(a-q^{-k})};q\right)_k} \frac{\left(cq\frac{b+a(a-q^{-k})}{(a-q^{-k})};q\right)_n}{\left(q\frac{b+a(a-q^{-k})}{(a-q^{-k})};q\right)_n} q^k. \quad (3.3)$$

The $n\to\infty$ case of (3.3) is equivalent to (3.1). For $a=0$, (3.3) is equivalent to (3.2).

For other terminating and nonterminating summations that were derived via inverse relations from classical ordinary and basic hypergeometric summations and do *not* belong to the hierarchy of (basic) hypergeometric series, such as identities of (q-)Abel, (q-)Rothe, or of the above type (as in (3.1) and (3.3)), see [15].

We commence with a new application of Corollary 2.6, which was missed in [15, § 7].

Theorem 3.1. *Let a, b, and c be indeterminate. Then*

$$\frac{(-bq;q)_\infty}{(-q;q)_\infty}=\sum_{k=0}^\infty \frac{(c-(a+1)(a+b))}{(c-(a+1)(a+bq^k))}\,\frac{(c-(a+bq^k)^2)}{(c-(a+b)(a+bq^k))}\\ \times\frac{(b;q)_k\left(\frac{(a+bq^k)}{c-a(a+bq^k)};q\right)_\infty}{(q;q)_k\left(\frac{(a+bq^k)bq}{c-a(a+bq^k)};q\right)_\infty}\,\frac{\left(\frac{(a+bq^k)b^2q^{k+2}}{c-a(a+bq^k)};q^2\right)_\infty}{\left(\frac{(a+bq^k)q^k}{c-a(a+bq^k)};q^2\right)_\infty}\,(-1)^k\,q^k. \quad (3.4)$$

For $a=0$, (3.4) reduces to a special case of Rogers' very-well-poised ${}_6\phi_5$ summation (cf. [9, Eq. (II.20)]). On the other hand, if $c=0$, we obtain (with $a\mapsto -1/a$)

$$\frac{(-bq;q)_\infty\,(abq;q)_\infty}{(-q;q)_\infty\,(a;q)_\infty}=\sum_{k=0}^\infty \frac{(b;q)_k}{(q;q)_k}\,\frac{(ab^2q^{2+k};q^2)_\infty}{(aq^k;q^2)_\infty}\,(-1)^k\,q^k, \quad (3.5)$$

which we could not find in this explicit form in the literature. Nevertheless, it is not difficult to find a conventional proof. Splitting the sum on the right hand side in two parts depending on the parity of k, (3.5) becomes

$$\frac{(-bq;q)_\infty\,(abq;q)_\infty}{(-q;q)_\infty\,(a;q)_\infty}=\frac{(ab^2q^2;q^2)_\infty}{(a;q^2)_\infty}\,{}_3\phi_2\left[\begin{matrix}b,bq,a\\ q,ab^2q^2\end{matrix};q^2,q^2\right]\\ -q\frac{(1-b)}{(1-q)}\,\frac{(ab^2q^3;q^2)_\infty}{(aq;q^2)_\infty}\,{}_3\phi_2\left[\begin{matrix}bq,bq^2,aq\\ q^3,ab^2q^3\end{matrix};q^2,q^2\right].$$

Now, this is just a special case of the nonterminating balanced ${}_3\phi_2$ summation [9, Eq. (II.24)].

Other *quadratic* identities similar to (3.5) (with infinite products in the summand) have been derived in [16, Th. 4.2, Cors. 5.4, 5.5 and 5.6].

Proof of Theorem 3.1. Let the inverse matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ be defined as in Corollary 2.6. Then (2.3a) holds for

$$a_n=(-bq)^n\quad\text{and}\quad b_k=(-bq)^k\frac{(-q;q)_\infty\left(\frac{(a+bq^k)q^{k+1}}{c-a(a+bq^k)};q^2\right)_\infty\left(\frac{(a+bq^k)b^2q^{k+2}}{c-a(a+bq^k)};q^2\right)_\infty}{(-bq;q)_\infty\left(\frac{(a+bq^k)bq^{k+1}}{c-a(a+bq^k)};q\right)_\infty}$$

by Lemma 2.1. This implies the inverse relation (2.3b), with the above values of a_n and b_k. After performing the shift $k\mapsto k+l$, and the substitutions $a\mapsto aq^l$, $c\mapsto cq^{2l}$, we get rid of l and eventually obtain (3.4). ∎

Next, we present two generalizations of (3.1).

Theorem 3.2. *Let a, b, and c be indeterminate. Then*

$$\frac{(z;q)_\infty}{(z/b;q)_\infty}=\sum_{k=0}^{\infty}\frac{(c-(a+1)(a+b))}{(c-(a+1)(a+bq^k))}\,\frac{(c-(a+bq^k)^2)}{(c-(a+b)(a+bq^k))}$$
$$\times\,{}_2\phi_1\left[\begin{matrix}1/b,z/b^2q\\ z/b\end{matrix};q,\frac{(a+bq^k)b^2q^{k+1}}{c-a(a+bq^k)}\right]$$
$$\times\frac{(b;q)_k\left(\frac{(a+bq^k)}{c-a(a+bq^k)};q\right)_k\left(\frac{(a+bq^k)b^2q^{k+1}}{c-a(a+bq^k)};q\right)_\infty}{(q;q)_k\left(\frac{(a+bq^k)bq}{c-a(a+bq^k)};q\right)_\infty}(z/b)^k, \quad (3.6)$$

provided $|z/b|<1$.

Clearly, (3.6) reduces to (3.1) when $z=b^2q$. At first glance, it seems that (3.6) is not at all related to (3.4) which also contains the base q^2. Notwithstanding, (3.6) is indeed more general than (3.4) and reduces to the latter for $z=-bq$. In this case the ${}_2\phi_1$ appearing in the summand of (3.6) becomes a ${}_1\phi_0$ with base q^2 (using $(1/b;q)_k(-1/b;q)_k=(1/b^2;q^2)_k$, etc.) which can be summed by virtue of (4.2).

Proof of Theorem 3.2. Let the inverse matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ be defined as in Corollary 2.6. Then (2.3a) holds for $a_n=z^n$ and

$$b_k=z^k\frac{(z/b;q)_\infty\left(\frac{(a+bq^k)b^2q^{k+1}}{c-a(a+bq^k)};q\right)_\infty}{(z;q)_\infty\left(\frac{(a+bq^k)bq^{k+1}}{c-a(a+bq^k)};q\right)_\infty}\,{}_2\phi_1\left[\begin{matrix}1/b,z/b^2q\\ z/b\end{matrix};q,\frac{(a+bq^k)b^2q^{k+1}}{c-a(a+bq^k)}\right]$$

by Lemma 2.2. This implies the inverse relation (2.3b), with the above values of a_n and b_k. After performing the shift $k\mapsto k+l$, and the substitutions $a\mapsto aq^l$, $c\mapsto cq^{2l}$, we get rid of l and eventually obtain (3.6). □

Theorem 3.3. *Let a, b, and c be indeterminate, and let m be a nonnegative integer. Then*

$$\frac{(b^2q;q)_\infty}{(bq;q)_\infty}=\sum_{k=0}^{\infty}\frac{(c-(a+1)(a+b))}{(c-(a+1)(a+bq^k))}\,\frac{(c-(a+bq^k)^2)}{(c-(a+b)(a+bq^k))}$$
$$\times\,{}_3\phi_2\left[\begin{matrix}1/b,\frac{(a+bq^k)q^k}{c-a(a+bq^k)},q^{-m}\\ 1/b^2,eq^k\end{matrix};q,q\right]\frac{(eq^m;q)_k}{(e;q)_k}$$
$$\times\frac{(b;q)_k\left(\frac{(a+bq^k)}{c-a(a+bq^k)};q\right)_k\left(\frac{(a+bq^k)b^2q^{k+1}}{c-a(a+bq^k)};q\right)_\infty}{(q;q)_k\left(\frac{(a+bq^k)bq}{c-a(a+bq^k)};q\right)_\infty}(bq^{1-m})^k, \quad (3.7)$$

provided $|bq^{1-m}|<1$.

Clearly, (3.7) reduces to (3.1) when $m=0$, or when $e\to\infty$.

Proof of Theorem 3.3. Let the inverse matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ be defined as in Corollary 2.6. Then (2.3a) holds for

$$a_n = \frac{(eq^m;q)_n}{(e;q)_n}\,(b^2q^{1-m})^n \qquad \text{and} \qquad b_k = {}_3\phi_2\left[\begin{matrix}1/b, \frac{(a+bq^k)q^k}{c-a(a+bq^k)}, q^{-m}\\ 1/b^2, eq^k\end{matrix}; q, q\right]$$

$$\times \frac{(eq^m;q)_k}{(e;q)_k}\,\frac{(bq;q)_\infty\left(\frac{(a+bq^k)b^2q^{k+1}}{c-a(a+bq^k)};q\right)_\infty}{(b^2q;q)_\infty\left(\frac{(a+bq^k)bq^{k+1}}{c-a(a+bq^k)};q\right)_\infty}\,(b^2q^{1-m})^k$$

by Lemma 2.3. This implies the inverse relation (2.3b), with the above values of a_n and b_k. After performing the shift $k\mapsto k+l$, and the substitutions $a\mapsto aq^l$, $c\mapsto cq^{2l}$, $e\mapsto eq^{-l}$, we get rid of l and eventually obtain (3.7). $\square$

4. q-Integrals

In the following we restrict ourselves to *real* q with $0<q<1$.

Thomae [18] introduced the q-integral defined by

$$\int_0^1 f(t)d_qt = (1-q)\sum_{k=0}^{\infty} f(q^k)q^k. \tag{4.1}$$

Later Jackson [10] gave a more general q-integral which however we do not need here.

By considering the Riemann sum for a continuous function f over the closed interval $[0,1]$, partitioned by the points q^k, $k\geq 0$, one easily sees that

$$\lim_{q\to 1^-}\int_0^1 f(t)d_qt = \int_0^1 f(t)dt.$$

It is well known that many identities for q-series can be written in terms of q-integrals, which then may be specialized (as $q\to 1$) to ordinary integrals. For instance, the q-binomial theorem (cf. [9, Eq. (II.3)])

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}z^k = \frac{(az;q)_\infty}{(z;q)_\infty}, \qquad |z|<1, \tag{4.2}$$

can be written, when $a\mapsto q^\beta$ and $z\mapsto q^\alpha$, as

$$\int_0^1 \frac{(qt;q)_\infty}{(q^\beta t;q)_\infty}t^{\alpha-1}d_qt = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}, \tag{4.3}$$

where

$$\Gamma_q(x) := (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty} \tag{4.4}$$

is the q-gamma function, introduced by Thomae [18], see also [2, § 10.3] and [9, § 1.11]. In fact, (4.3) is a q-extension of the beta integral evaluation (1.1).

Since the expansion of Proposition 3.1 involves an alternating series, it makes no sense to rewrite it as a q-integral; the limit $q\to 1$ would never produce a convergent integral. However, we can reasonably rewrite the expansions in Theorems 3.2 and 3.3 in terms of q-integrals. These will then be utilized in Section 5 to obtain new beta-type integral evaluations.

Starting with (3.6), if we replace z by $q^{\alpha+\beta}$, b by q^β, and multiply both sides of the identity by

$$(1-q)\frac{(q;q)_\infty}{(q^\beta;q)_\infty}, \tag{4.5}$$

we obtain the following generalization of the q-beta integral evaluation:

$$\begin{aligned}\frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} &= \int_0^1 \frac{(c-(a+1)(a+q^\beta))}{(c-(a+1)(a+q^\beta t))}\frac{(c-(a+q^\beta t)^2)}{(c-(a+q^\beta)(a+q^\beta t))}\\ &\quad\times {}_2\phi_1\left[\begin{matrix}q^{\alpha-\beta-1},q^{-\beta}\\ q^\alpha\end{matrix};q,\frac{(a+q^\beta t)q^{2\beta+1}t}{c-a(a+q^\beta t)}\right]\\ &\quad\times\frac{(qt;q)_\infty\left(\frac{(a+q^\beta t)}{c-a(a+q^\beta t)};q\right)_\infty\left(\frac{(a+q^\beta t)q^{2\beta+1}t}{c-a(a+q^\beta t)};q\right)_\infty}{(q^\beta t;q)_\infty\left(\frac{(a+q^\beta t)t}{c-a(a+q^\beta t)};q\right)_\infty\left(\frac{(a+q^\beta t)q^{\beta+1}}{c-a(a+q^\beta t)};q\right)_\infty}t^{\alpha-1}d_qt. \end{aligned}\tag{4.6}$$

Clearly, (4.6) reduces to (4.3) when either $c\to\infty$ or $a\to\infty$.

Similarly, starting with (3.7), if we replace b by q^β and multiply both sides of the identity by (4.5), we obtain the following q-beta-type integral evaluation:

$$\begin{aligned}\frac{\Gamma_q(\beta+1)\Gamma_q(\beta)}{\Gamma_q(2\beta+1)} &= \int_0^1 \frac{(c-(a+1)(a+q^\beta))}{(c-(a+1)(a+q^\beta t))}\frac{(c-(a+q^\beta t)^2)}{(c-(a+q^\beta)(a+q^\beta t))}\\ &\quad\times {}_3\phi_2\left[\begin{matrix}q^{-\beta},\frac{(a+q^\beta t)t}{c-a(a+q^\beta t)},q^{-m}\\ q^{-2\beta},et\end{matrix};q,q\right]\frac{(eq^m;q)_\infty}{(e;q)_\infty}\frac{(et;q)_\infty}{(eq^mt;q)_\infty}\\ &\quad\times\frac{(qt;q)_\infty\left(\frac{(a+q^\beta t)}{c-a(a+q^\beta t)};q\right)_\infty\left(\frac{(a+q^\beta t)q^{2\beta+1}t}{c-a(a+q^\beta t)};q\right)_\infty}{(q^\beta t;q)_\infty\left(\frac{(a+q^\beta t)t}{c-a(a+q^\beta t)};q\right)_\infty\left(\frac{(a+q^\beta t)q^{\beta+1}}{c-a(a+q^\beta t)};q\right)_\infty}t^{\beta-m}d_qt. \end{aligned}\tag{4.7}$$

This formula does not really extend (4.3) as there is only one "exponent parameter", β. However, for small m the ${}_3\phi_2$ appearing in the integrand can be expanded in explicit terms; the integrand thus has nearly "closed form". In particular, the ($e=0$ and) $m=0$ case of (4.7) is equal to the $\alpha=\beta+1$ special case of (4.6). More generally, for $e=0$ equation (4.7) is equivalent to the $\alpha=\beta+1-m$ special case of (4.6), due to the transformation (cf. [9, Eq. (III.7)])

$${}_2\phi_1\left[\begin{matrix}q^{-m},b\\ c\end{matrix};q,z\right]=\frac{(c/b;q)_m}{(c;q)_m}\,{}_3\phi_2\left[\begin{matrix}q^{-m},b,bzq^{-m}/c\\ bq^{1-m}/c,0\end{matrix};q,q\right].$$

5. Curious beta-type integrals

Observe that $\lim_{q\to 1^-}\Gamma_q(x)=\Gamma(x)$ (see [9, (1.10.3)]) and

$$\lim_{q\to 1^-}\frac{(q^\alpha u;q)_\infty}{(u;q)_\infty}=(1-u)^{-\alpha}$$

for constant u (with $|u|<1$), due to (4.2) and its $q\to 1$ limit, the ordinary binomial theorem.

We thus immediately deduce, as consequences of our q-integral evaluations from Section 4, some beta-type integral evaluations. Throughout it is implicitly assumed that the integrals are well defined, in particular that the parameters are chosen such that no poles occur on the path of integration $t\in[0,1]$ and the integrals converge.

The first beta integral evaluation is obtained from letting $q\to 1$ in (4.6).

Theorem 5.1. *Let $\Re(\beta),\Re(\alpha)>0$. Then*

$$\begin{aligned}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}&=(c-(a+1)^2)\int_0^1\frac{(c-a(a+t))^\beta\,(c-(a+1)(a+t))^{\beta-1}}{(c-(a+t)^2)^{2\beta}}\\&\quad\times{}_2F_1\left[\begin{matrix}\alpha-\beta-1,-\beta\\\alpha\end{matrix};\frac{(a+t)t}{c-a(a+t)}\right]t^{\alpha-1}\,(1-t)^{\beta-1}\,dt.\end{aligned}\tag{5.1}$$

We consider a few important special cases. First, it is clear that Theorem 5.1 reduces to the classical beta integral evaluation (1.1) when either $c\to\infty$ or $a\to\infty$. Other cases of interest concern the limits $c\to 0$ and $a\to 0$. If $c\to 0$ (5.1) reduces to

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}=a^\beta(a+1)^{\beta+1}\int_0^1\frac{t^{\alpha-1}(1-t)^{\beta-1}}{(a+t)^{2\beta+1}}\,{}_2F_1\left[\begin{matrix}\alpha-\beta-1,-\beta\\\alpha\end{matrix};\frac{-t}{a}\right]dt,\tag{5.2}$$

which can easily be recovered as a special case of Erdélyi's [6] fractional integral formula (see also [2, p. 112, Th. 2.9.1])

$$\begin{aligned}{}_2F_1\left[\begin{matrix}a,b\\c\end{matrix};x\right]&=\frac{\Gamma(c)}{\Gamma(\mu)\Gamma(c-\mu)}\int_0^1 t^{\mu-1}\,(1-t)^{c-\mu-1}\,(1-xt)^{\lambda-a-b}\\&\quad\times{}_2F_1\left[\begin{matrix}\lambda-a,\lambda-b\\\mu\end{matrix};xt\right]{}_2F_1\left[\begin{matrix}a+b-\lambda,\lambda-\mu\\c-\mu\end{matrix};\frac{(1-t)x}{1-xt}\right]dt.\end{aligned}\tag{5.3}$$

Indeed, (5.3) reduces to (5.2) when one does the replacements $\lambda\mapsto\alpha$, $\mu\mapsto\alpha$, $a\mapsto\beta+1$, $b\mapsto\alpha+\beta$, $c\mapsto\alpha+\beta$, and $x\mapsto -1/a$, and then simplifies. Some q-extensions of (5.3) and the other fractional integral representations for hypergeometric functions in [6] are derived in [8].

For $a\to 0$, (5.1) reduces to

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}=(c-1)c^\beta\int_0^1\frac{(c-t)^{\beta-1}}{(c-t^2)^{2\beta}}\,t^{\alpha-1}\,(1-t)^{\beta-1}$$

$$\times {}_2F_1\left[\begin{matrix}\alpha-\beta-1,-\beta\\ \alpha\end{matrix};\frac{t^2}{c}\right] dt, \quad (5.4)$$

which we were unable to find in the literature.

Some special cases of (5.1) where the ${}_2F_1$ in the integrand can be simplified are $\alpha=\beta+1$, which is (1.2), and $\alpha=\beta$, which is (1.3).

Next, we consider the beta-type integral evaluation obtained from letting $q\to 1$ in (4.7).

Theorem 5.2. *Let* $\Re(\beta)>\max(0,m-1)$. *Then*

$$\frac{\Gamma(\beta)\Gamma(\beta)}{2\,\Gamma(2\beta)} = (c-(a+1)^2)\int_0^1 \frac{(c-a(a+t))^{\beta}\,(c-(a+1)(a+t))^{\beta-1}}{(c-(a+t)^2)^{2\beta}}$$
$$\times {}_2F_1\left[\begin{matrix}-\beta,-m\\ -2\beta\end{matrix};\frac{c-(a+t)^2}{(c-a(a+t))(1-et)}\right]\left(\frac{1-et}{1-e}\right)^m t^{\beta-m}\,(1-t)^{\beta-1}\,dt. \quad (5.5)$$

Note that (5.5) can be further rewritten using Legendre's duplication formula

$$\Gamma(2\beta)=\frac{1}{\sqrt{\pi}}\,2^{2\beta-1}\Gamma(\beta)\Gamma(\beta+\tfrac12),$$

after which the left hand side becomes

$$\frac{\sqrt{\pi}}{4^{\beta}}\,\frac{\Gamma(\beta)}{\Gamma(\beta+\frac12)}.$$

If $e=0$, (5.5) is equivalent to the $\alpha-\beta-1=-m$ case of (5.1), due to Pfaff's transformation (cf. [2, p. 79, Eq. (2.3.14)])

$${}_2F_1\left[\begin{matrix}-m,b\\ c\end{matrix};x\right]=\frac{(c-b)_m}{(c)_m}\,{}_2F_1\left[\begin{matrix}-m,b\\ b+1-m-c\end{matrix};1-x\right].$$

If $c\to\infty$ or $a\to\infty$, then (5.5) reduces to the following beta-type integral evaluation:

$$\frac{\Gamma(\beta)\Gamma(\beta)}{2\,\Gamma(2\beta)}=\int_0^1 {}_2F_1\left[\begin{matrix}-\beta,-m\\ -2\beta\end{matrix};\frac{1}{1-et}\right]\left(\frac{1-et}{1-e}\right)^m t^{\beta-m}\,(1-t)^{\beta-1}\,dt. \quad (5.6)$$

For a conventional proof of (5.6), expand the ${}_2F_1$ in powers of $1/(1-et)$, interchange the order of summation and integration, and evaluate the integrals using the $\lambda=\mu=a$ special case of (5.3), interchange summations again, simplify by first using the Gauß summation (cf. [2, p. 66, Th. 2.2.2]) and then by using the binomial theorem. The details are left to the reader.

Other interesting cases of (5.5) are the specializations $c=0$ or $a=0$.

Finally, observe that by performing various substitutions we may change the form and path of integration of the above integrals. In particular, using $t\mapsto s/(s+1)$ these integrals then run over the half line $s\in[0,\infty)$.

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Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208-2730, USA

E-mail address: george@math.northwestern.edu

URL: http://www.math.northwestern.edu/~george

Institut für Mathematik der Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria

E-mail address: schlosse@ap.univie.ac.at

URL: http://www.mat.univie.ac.at/~schlosse