Some Systems of Multivariable Orthogonal Askey-Wilson Polynomials

George Gasper* and Mizan Rahman†

Abstract

In 1991 Tratnik derived two systems of multivariable orthogonal Wilson polynomials and considered their limit cases. q-Analogues of these systems are derived, yielding systems of multivariable orthogonal Askey-Wilson polynomials and their special and limit cases.

1. Introduction. In [10] Tratnik extended the Wilson [12] polynomials

$$w_n(x;a,b,c,d) = (a+b)_n(a+c)_n(a+d)_n \times {}_4F_3\left[\begin{matrix}-n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d\end{matrix}; 1\right] \tag{1.1}$$

to the multivariable Wilson polynomials (in a different notation)

$$\begin{aligned} W_{\mathbf{n}}(\mathbf{x}) &= W_{\mathbf{n}}(\mathbf{x}; a,b,c,d,a_2,a_3,\ldots,a_s) \\ &= \left[\prod_{k=1}^{s-1} w_{n_k}(x_k; a+\alpha_{2,k}+N_{k-1}, b+\alpha_{2,k}+N_{k-1}, a_{k+1}+ix_{k+1}, a_{k+1}-ix_{k+1})\right] \\ &\quad \times w_{n_s}(x_s; a+\alpha_{2,s}+N_{s-1}, b+\alpha_{2,s}+N_{s-1}, c, d), \end{aligned} \tag{1.2}$$

where, as elsewhere,

$$\mathbf{x} = (x_1,\ldots,x_s),\ \mathbf{n} = (n_1,\ldots,n_s),\ \alpha_{j,k} = \sum_{i=j}^{k} a_i,\ \alpha_k = \alpha_{1,k}, \tag{1.3}$$

$$N_{j,k} = \sum_{i=j}^{k} n_i,\ N_k = N_{1,k},\ \alpha_{k+1,k} = N_{k+1,k} = 0, \quad 1 \le j \le k \le s.$$

These polynomials are of total degree N_s in the variables $y_1,\ldots,y_s$ with $y_k = x_k^2$, $k = 1,2,\ldots,s$, and they form a complete set for polynomials in these variables.

In Askey and Wilson [1], [2] the notations $W_n(x^2;a,b,c,d)$ and $p_n(-x^2)$ are used for the polynomials in (1.1) and their orthogonality relation is given. Tratnik [10, (2.5)] proved that the $W_{\mathbf{n}}(\mathbf{x})$ polynomials satisfy the orthogonality relation

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} W_{\mathbf{n}}(\mathbf{x})W_{\mathbf{m}}(\mathbf{x})\rho(\mathbf{x})\, dx_1\cdots dx_s = \lambda_{\mathbf{n}}\,\delta_{\mathbf{n},\mathbf{m}} \tag{1.4}$$

* Dept. of Mathematics, Northwestern University, Evanston, IL 60208

† School of Mathematics and Statistics, Carleton University, Ottawa, ON, K1S 5B6, CANADA. This work was supported, in part, by an NSERC grant #A6197.

for $\mathrm{Re}(a,b,c,d,a_2,\ldots,a_s)>0$ with

$$(1.5)\qquad \begin{aligned}\rho(\mathbf{x}) &= \Gamma(a+ix_1)\Gamma(a-ix_1)\Gamma(b+ix_1)\Gamma(b-ix_1)\\ &\times\left[\prod_{k=1}^{s-1}\frac{\Gamma(a_{k+1}+ix_{k+1}+ix_k)\Gamma(a_{k+1}-ix_{k+1}-ix_k)}{\Gamma(2ix_k)}\right.\\ &\times\left.\frac{\Gamma(a_{k+1}+ix_{k+1}-ix_k)\Gamma(a_{k+1}-ix_{k+1}+ix_k)}{\Gamma(-2ix_k)}\right]\\ &\times\frac{\Gamma(c+ix_s)\Gamma(c-ix_s)\Gamma(d+ix_s)\Gamma(d-ix_s)}{\Gamma(2ix_s)\Gamma(-2ix_s)},\end{aligned}$$

$$(1.6)\qquad \begin{aligned}\lambda_{\mathbf{n}} &= (4\pi)^s\left[\prod_{k=1}^{s} n_k!\,(N_k+N_{k-1}+2\alpha_{k+1}-1)_{n_k}\right.\\ &\times\left.\frac{\Gamma(N_k+N_{k-1}+2\alpha_k)\Gamma(n_k+2a_{k+1})}{\Gamma(2N_k+2\alpha_{k+1})}\right]\\ &\times\Gamma(a+c+\alpha_{2,s}+N_s)\Gamma(a+d+\alpha_{2,s}+N_s)\Gamma(b+c+\alpha_{2,s}+N_s)\\ &\times\Gamma(b+d+\alpha_{2,s}+N_s),\end{aligned}$$

and $2a_1=a+b$, $2a_{s+1}=c+d$.

Tratnik showed that these polynomials contain multivariable Jacobi, Meixner-Pollaczek, Laguerre, continuous Charlier, and Hermite polynomials as limit cases, and he used a permutation of the parameters and variables in (1.2) and (1.4) to show that the polynomials

(1.7)
$$\begin{aligned}\tilde{W}_{\mathbf{n}}(\mathbf{x}) &= \tilde{W}_{\mathbf{n}}(\mathbf{x};a,b,c,d,a_2,a_3,\ldots,a_s)\\ &= w_{n_1}(x_1;c+\alpha_{2,s}+N_{2,s},d+\alpha_{2,s}+N_{2,s},a,b)\\ &\times\prod_{k=2}^{s} w_{n_k}(x_k;c+\alpha_{k+1,s}+N_{k+1,s},d+\alpha_{k+1,s}+N_{k+1,s},a_k+ix_{k-1},a_k-ix_{k-1})\end{aligned}$$

also form a complete system of multivariable polynomials of total degree N_s in the variables $y_k=x_k^2$, $k=1,\ldots,s$, that is orthogonal with respect to the weight function $\rho(\mathbf{x})$ in (1.5), and with the normalization constant

$$(1.8)\qquad \begin{aligned}\tilde{\lambda}_{\mathbf{n}} &= (4\pi)^s\left[\prod_{k=1}^{s} n_k!(N_{k,s}+N_{k+1,s}+2\alpha_{k,s+1}-1)_{n_k}\right.\\ &\times\left.\frac{\Gamma(N_{k,s}+N_{k+1,s}+2\alpha_{k+1,s+1})\Gamma(n_k+2a_k)}{\Gamma(2N_{k,s}+2\alpha_{k,s+1})}\right]\\ &\times\Gamma(a+c+\alpha_{2,s}+N_s)\Gamma(a+d+\alpha_{2,s}+N_s)\\ &\times\Gamma(b+c+\alpha_{2,s}+N_s)\Gamma(b+d+\alpha_{2,s}+N_s).\end{aligned}$$

The Askey-Wilson polynomials defined as in [1] and [3] by

$$\begin{aligned}(1.9)\qquad p_n(x|q) &= p_n(x;a,b,c,d|q)\\ &= a^{-n}(ab,ac,ad;q)_n\ {}_4\phi_3\left[\begin{matrix}q^{-n},abcdq^{n-1},ae^{i\theta},ae^{-i\theta}\\ ab,ac,ad\end{matrix};q,q\right],\end{aligned}$$

where $x=\cos\theta$, are a q-analogue of the Wilson polynomials (for the definition of the q-shifted factorials and the basic hypergeometric series ${}_4\phi_3$ see [3]). These polynomials satisfy the orthogonality relation

$$(1.10)\qquad \int_{-1}^{1} p_n(x|q)p_m(x|q)\rho(x|q)dx = \lambda_n(q)\delta_{n,m}$$

with $\max(|q|,|a|,|b|,|c|,|d|)<1$,

$$\begin{aligned}(1.11)\qquad \rho(x|q) &= \rho(x;a,b,c,d|q)\\ &= \frac{(e^{2i\theta},e^{-2i\theta};q)_\infty(1-x^2)^{-1/2}}{(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},ce^{i\theta},ce^{-i\theta},de^{i\theta},de^{-i\theta};q)_\infty}\end{aligned}$$

and

$$\begin{aligned}(1.12)\qquad \lambda_n(q) &= \lambda_n(a,b,c,d|q)\\ &= \frac{2\pi(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}\\ &\quad\times\frac{(q,ab,ac,ad,bc,bd,cd;q)_n(1-abcdq^{-1})}{(abcdq^{-1};q)_n(1-abcdq^{2n-1})}.\end{aligned}$$

In this paper we extend Tratnik's systems of multivariable Wilson polynomials to systems of multivariable Askey-Wilson polynomials and consider their special cases. Some q-extensions of Tratnik's [9] multivariable biorthogonal generalization of the Wilson polynomials are considered in this Proceedings [4]. q-Extensions of Tratnik's [11] system of multivariable orthogonal Racah polynomials and their special cases will be considered in a subsequent paper.

2. Multivariable Askey-Wilson polynomials. In terms of the Askey-Wilson polynomials a q-analogue of the multivariable Wilson polynomials can be defined by

$$\begin{aligned}(2.1)\qquad P_{\mathbf{n}}(\mathbf{x}|q) &= P_{\mathbf{n}}(\mathbf{x};a,b,c,d,a_2,a_3,\ldots,a_s|q)\\ &= \left[\prod_{k=1}^{s-1} p_{n_k}(x_k;aA_{2,k}q^{N_{k-1}},bA_{2,k}q^{N_{k-1}},a_{k+1}e^{i\theta_{k+1}},a_{k+1}e^{-i\theta_{k+1}}|q)\right]\\ &\quad\times p_{n_s}(x_s;aA_{2,s}q^{N_{s-1}},bA_{2,s}q^{N_{s-1}},c,d\,|q)\end{aligned}$$

where $x_k = \cos\theta_k,\ A_{j,k} = \prod_{i=j}^{k} a_i,\ A_{k+1,k} = 1,\ A_k = A_{1,k},\ 1 \le j \le k \le s$. Our main aim in this section is to show that these polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} \cdots \int_{-1}^{1} P_{\mathbf{n}}(\mathbf{x}|q) P_{\mathbf{m}}(\mathbf{x}|q) \rho(\mathbf{x}|q) dx_1 \cdots dx_s = \lambda_{\mathbf{n}}(q)\, \delta_{\mathbf{n},\mathbf{m}} \tag{2.2}$$

with $\max(|q|, |a|, |b|, |c|, |d|, |a_2|, \ldots, |a_s|) < 1$,

(2.3)

$$\begin{aligned}
\rho(\mathbf{x}|q) &= \rho(\mathbf{x}; a, b, c, d, a_2, a_3, \ldots, a_s|q) \\
&= (ae^{i\theta_1}, ae^{-i\theta_1}, be^{i\theta_1}, be^{-i\theta_1}; q)_\infty^{-1} \\
&\quad \times \left[\prod_{k=1}^{s-1} \frac{(e^{2i\theta_k}, e^{-2i\theta_k}; q)_\infty (1 - x_k^2)^{-1/2}}{(a_{k+1}e^{i\theta_{k+1}+i\theta_k}, a_{k+1}e^{i\theta_{k+1}-i\theta_k}, a_{k+1}e^{i\theta_k - i\theta_{k+1}}, a_{k+1}e^{-i\theta_{k+1}-i\theta_k}; q)_\infty} \right] \\
&\quad \times \frac{(e^{2i\theta_s}, e^{-2i\theta_s}; q)_\infty (1 - x_s^2)^{-1/2}}{(ce^{i\theta_s}, ce^{-i\theta_s}, de^{i\theta_s}, de^{-i\theta_s}; q)_\infty},
\end{aligned}$$

(2.4)

$$\begin{aligned}
\lambda_{\mathbf{n}}(q) &= \lambda_{\mathbf{n}}(a, b, c, d, a_2, a_3, \ldots, a_s|q) \\
&= (2\pi)^s \left[\prod_{k=1}^{s} \frac{(q, A_{k+1}^2 q^{N_k + N_{k-1} - 1}; q)_{n_k} (A_{k+1}^2 q^{2N_k}; q)_\infty}{(q, A_k^2 q^{N_k + N_{k-1}}, a_{k+1}^2 q^{n_k}; q)_\infty} \right] \\
&\quad \times (acA_{2,s}q^{N_s}, adA_{2,s}q^{N_s}, bcA_{2,s}q^{N_s}, bdA_{2,s}q^{N_s}; q)_\infty^{-1},
\end{aligned}$$

where $a_1^2 = ab$ and $a_{s+1}^2 = cd$. The two-dimensional case was considered by Koelink and Van der Jeugt [6], but they did not give the value of the norm. First observe that by (1.10)—(1.12) the integration over x_1 in (2.2) can be evaluated to obtain that

$$\begin{aligned}
&\int_{-1}^{1} p_{n_1}(x_1; a, b, a_2e^{i\theta_2}, a_2e^{-i\theta_2}|q) p_{m_1}(x_1; a, b, a_2e^{i\theta_2}, a_2e^{-i\theta_2}|q) \\
&\quad \times \rho(x_1; a, b, a_2e^{i\theta_2}, a_2e^{-i\theta_2}|q) dx_1 \\
&= \delta_{n_1,m_1} \frac{2\pi (q, aba_2^2 q^{n_1 - 1}; q)_{n_1} (aba_2^2 q^{2n_1}; q)_\infty}{(q, abq^{n_1}, a_2^2 q^{n_1}; q)_\infty} \\
&\quad \times (aa_2q^{n_1}e^{i\theta_2}, aa_2q^{n_1}e^{-i\theta_2}, ba_2q^{n_1}e^{i\theta_2}, ba_2q^{n_1}e^{-i\theta_2}; q)_\infty^{-1}.
\end{aligned} \tag{2.5}$$

After doing the integrations over $x_1, x_2, \ldots, x_j$ for a few j one is led to conjecture that

(2.6)

$$\begin{aligned}
&\int_{-1}^{1} \cdots \int_{-1}^{1} P_{\mathbf{n}}^{(j)}(\mathbf{x}|q) P_{\mathbf{m}}^{(j)}(\mathbf{x}|q) \rho^{(j)}(\mathbf{x}|q) dx_1 \cdots dx_j \\
&= (2\pi)^j \left[\prod_{k=1}^{j} \delta_{n_k,m_k} \frac{(q, A_{k+1}^2 q^{N_k + N_{k-1} - 1}; q)_{n_k} (A_{k+1}^2 q^{2N_k}; q)_\infty}{(q, A_k^2 q^{N_k + N_{k-1}}, a_{k+1}^2 q^{n_k}; q)_\infty} \right] \\
&\quad \times \left(aA_{2,j+1}q^{N_j}e^{i\theta_{j+1}}, aA_{2,j+1}q^{N_j}e^{-i\theta_{j+1}}, bA_{2,j+1}q^{N_j}e^{i\theta_{j+1}}, bA_{2,j+1}q^{N_j}e^{-i\theta_{j+1}}; q \right)_\infty^{-1},
\end{aligned}$$

where

$$P_{\mathbf{n}}^{(j)}(\mathbf{x}|q) = \prod_{k=1}^{j} p_{n_k}(x_k; aA_{2,k}q^{N_{k-1}}, bA_{2,k}q^{N_{k-1}}, a_{k+1}e^{i\theta_{k+1}}, a_{k+1}e^{-i\theta_{k+1}}|q),$$

$$\rho^{(j)}(\mathbf{x}|q) = (ae^{i\theta_1}, ae^{-i\theta_1}, be^{i\theta_1}, be^{-i\theta_1}; q)_\infty^{-1} \times \prod_{k=1}^{j} \frac{(e^{2i\theta_k}, e^{-2i\theta_k}; q)_\infty (1-x_k^2)^{-1/2}}{(a_{k+1}e^{i\theta_{k+1}+i\theta_k}, a_{k+1}e^{i\theta_{k+1}-i\theta_k}, a_{k+1}e^{i\theta_k-i\theta_{k+1}}, a_{k+1}e^{-i\theta_{k+1}-i\theta_k}; q)_\infty}$$

for $j = 1, 2, \ldots, s-1$. To prove this by induction on j, suppose that $j < s-1$, multiply (2.6) by the x_{j+1}-dependent parts of the weight function and orthogonal polynomials, and then integrate with respect to x_{j+1} to get

(2.7)

$$\begin{aligned}
&(2\pi)^j \left[\prod_{k=1}^{j} \delta_{n_k,m_k} \frac{(q, A_{k+1}^2 q^{N_k+N_{k-1}-1}; q)_{n_k} (A_{k+1}^2 q^{2N_k}; q)_\infty}{(q, A_k^2 q^{N_k+N_{k-1}}, a_{k+1}^2 q^{n_k}; q)_\infty}\right] \\
&\times \int_{-1}^{1} p_{n_{j+1}}(x_{j+1}; aA_{2,j+1}q^{N_j}, bA_{2,j+1}q^{N_j}, a_{j+2}e^{i\theta_{j+2}}, a_{j+2}e^{-i\theta_{j+2}}|q) \\
&\times p_{m_{j+1}}(x_{j+1}; aA_{2,j+1}q^{N_j}, bA_{2,j+1}q^{N_j}, a_{j+2}e^{i\theta_{j+2}}, a_{j+2}e^{-i\theta_{j+2}}|q) \\
&\times \rho(x_{j+1}; aA_{2,j+1}q^{N_j}, bA_{2,j+1}q^{N_j}, a_{j+2}e^{i\theta_{j+2}}, a_{j+2}e^{-i\theta_{j+2}}|q) dx_{j+1} \\
&= (2\pi)^{j+1} \left[\prod_{k=1}^{j+1} \delta_{n_k,m_k} \frac{(q, A_{k+1}^2 q^{N_k+N_{k-1}-1}; q)_{n_k} (A_{k+1}^2 q^{2N_k}; q)_\infty}{(q, A_k^2 q^{N_k+N_{k-1}}, a_{k+1}^2 q^{n_k}; q)_\infty}\right] \\
&\times \left(aA_{2,j+2}q^{N_{j+1}}e^{i\theta_{j+2}}, aA_{2,j+2}q^{N_{j+1}}e^{-i\theta_{j+2}}, bA_{2,j+2}q^{N_{j+1}}e^{i\theta_{j+2}}, bA_{2,j+2}q^{N_{j+1}}e^{-i\theta_{j+2}}; q\right)_\infty^{-1},
\end{aligned}$$

which is the $j \to j+1$ case of (2.6), completing the induction proof.

Now set $j = s-1$ in (2.6) and use it and (2.5) to find that

(2.8)
$$\begin{aligned}
&\int_{-1}^{1} \cdots \int_{-1}^{1} P_{\mathbf{n}}(\mathbf{x}|q) P_{\mathbf{m}}(\mathbf{x}|q) \rho(\mathbf{x}|q) dx_1 \cdots dx_s \\
&= (2\pi)^{s-1} \left[\prod_{k=1}^{s-1} \delta_{n_k,m_k} \frac{(q, A_{k+1}^2 q^{N_k+N_{k-1}-1}; q)_{n_k} (A_{k+1}^2 q^{2N_k}; q)_\infty}{(q, A_k^2 q^{N_k+N_{k-1}}, a_{k+1}^2 q^{n_k}; q)_\infty}\right] \\
&\times \int_{-1}^{1} p_{n_s}(x_s; aA_{2,s}q^{N_{s-1}}, bA_{2,s}q^{N_{s-1}}, c, d|q) \\
&\times p_{m_s}(x_s; aA_{2,s}q^{N_{s-1}}, bA_{2,s}q^{N_{s-1}}, c, d|q) \\
&\times \frac{(e^{2i\theta_s}, e^{-2i\theta_s}; q)_\infty (1-x_s^2)^{-1/2}}{(ce^{i\theta_s}, ce^{-i\theta_s}, de^{i\theta_s}, de^{-i\theta_s}; q)_\infty} \\
&\times (aA_{2,s}q^{N_{s-1}}e^{i\theta_s}, aA_{2,s}q^{N_{s-1}}e^{-i\theta_s}, bA_{2,s}q^{N_{s-1}}e^{i\theta_s}, bA_{2,s}q^{N_{s-1}}e^{-i\theta_s}; q)_\infty^{-1} dx_s \\
&= \lambda_{\mathbf{n}}(q)\, \delta_{\mathbf{n},\mathbf{m}},
\end{aligned}$$

where $\lambda_{\mathbf{n}}(q)$ is given by (2.4). This completes the proof of (2.2).

Note that the integration region and weight function in (2.2) and (2.3) are invariant under the permutation of variables and parameters

$$
\begin{aligned}
(2.9) \qquad & a \leftrightarrow c,\ b \leftrightarrow d,\ a_{k+1} \leftrightarrow a_{s-k+1}, \quad k = 1, 2, \dots, s-1, \\
& \theta_k \leftrightarrow \theta_{s-k+1}, \quad k = 1, 2, \dots, s.
\end{aligned}
$$

Hence, when these permutations are applied to (2.2) and (2.3) the transformed polynomials also form an orthogonal system with the same weight function. Since the polynomials $P_{\mathbf{n}}(\mathbf{x}|q)$ in (2.1) are not invariant under (2.9), we obtain a second system of multivariable orthogonal Askey-Wilson polynomials, which is a q-analogue of Tratnik's second system (1.7) of multivariable Wilson polynomials. After doing the permutation $n_k \leftrightarrow n_{s-k+1}$, $k = 1, \dots, s$, the transformed polynomials and the normalization constant are given by

$$
\begin{aligned}
(2.10) \quad \tilde{P}_{\mathbf{n}}(\mathbf{x}|q) &= \tilde{P}_{\mathbf{n}}(\mathbf{x}; a, b, c, d, a_2, a_3, \dots, a_s|q) \\
&= p_{n_1}(x_1; cA_{2,s}q^{N_{2,s}}, dA_{2,s}q^{N_{2,s}}, a, b|q) \\
&\times \left[\prod_{k=2}^{s} p_{n_k}(x_k; cA_{k+1,s}q^{N_{k+1,s}}, dA_{k+1,s}q^{N_{k+1,s}}, ae^{i\theta_{k-1}}, ae^{-i\theta_{k-1}}|q) \right],
\end{aligned}
$$

$$
\begin{aligned}
(2.11) \quad \tilde{\lambda}_{\mathbf{n}}(q) &= \tilde{\lambda}_{\mathbf{n}}(a, b, c, d, a_2, a_3, \dots, a_s|q) \\
&= (2\pi)^s \left[\prod_{k=1}^{s} \frac{(q, A_{k,s+1}^2 q^{N_{k,s}+N_{k+1,s}-1}; q)_{n_k} (A_{k,s+1}^2 q^{2N_{k,s}}; q)_\infty}{(q, A_{k+1,s+1}^2 q^{N_{k,s}+N_{k+1,s}}, a_k^2 q^{n_k}; q)_\infty} \right] \\
&\times (acA_{2,s}q^{N_s}, adA_{2,s}q^{N_s}, bcA_{2,s}q^{N_s}, bdA_{2,s}q^{N_s}; q)_\infty^{-1},
\end{aligned}
$$

with $a_1^2 = ab$, $a_{s+1}^2 = cd$, and $\max(|q|, |a|, |b|, |c|, |d|, |a_2|, |a_3|, \dots, |a_s|) < 1$. These polynomials are of total degree N_s in the variables $x_1, \dots, x_s$ and they form a complete set.

A five-parameter system of multivariable Askey-Wilson polynomials which is associated with a root system of type BC was introduced by Koornwinder [7] and studied with four of the parameters generally complex in Stokman [8].

3. Special Cases of (2.2). First observe that the continuous dual q-Hahn polynomial defined by

$$
(3.1) \qquad d_n(x; a, b, c|q) = a^{-n}(ab, ac; q)_n \ {}_3\phi_2 \left[\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac \end{matrix} ; q, q \right]
$$

is obtained by taking $d = 0$ in (1.9) and $x = \cos\theta$. Since $d_n(x; a, b, c|q)$ is symmetric in its parameters by [3, (3.2.3)], we may define the multivariable dual q-Hahn polynomials by

$$
\begin{aligned}
(3.2) \qquad D_{\mathbf{n}}(\mathbf{x}|q) &= D_{\mathbf{n}}(\mathbf{x}; a, b, c, a_2, a_3, \cdots, a_s|q) \\
&= \left[\prod_{k=1}^{s-1} d_{n_k}(x_k; a_{k+1}e^{i\theta_{k+1}}, a_{k+1}e^{-i\theta_{k+1}}, aA_{2,k}q^{N_{k-1}}|q) \right] \\
&\times d_{n_s}(x_s; b, c, aA_{2,s}q^{N_{s-1}}|q),
\end{aligned}
$$

with $x_k = \cos\theta_k$ for $k = 1, 2, \ldots, s$. It follows from the $b = 0$ case of (2.2)—(2.4) that the orthogonality relation for these polynomials is

$$\int_{-1}^{1} \cdots \int_{-1}^{1} D_{\mathbf{n}}(\mathbf{x}|q) D_{\mathbf{m}}(\mathbf{x}|q) \rho(\mathbf{x}|q) dx_1 \cdots dx_s = \lambda_{\mathbf{n}}(q)\, \delta_{\mathbf{n},\mathbf{m}} \tag{3.3}$$

with

(3.4)

$$\begin{aligned} \rho(\mathbf{x}|q) &= \rho(\mathbf{x}; a, b, c, a_2, a_3, \ldots, a_s|q) \\ &= (ae^{i\theta_1}, ae^{-i\theta_1}; q)_\infty^{-1} \\ &\quad \times \left[\prod_{k=1}^{s-1} \frac{(e^{2i\theta_k}, e^{-2i\theta_k}; q)_\infty (1 - x_k^2)^{-1/2}}{(a_{k+1}e^{i\theta_{k+1}+i\theta_k}, a_{k+1}e^{i\theta_{k+1}-i\theta_k}, a_{k+1}e^{i\theta_k - i\theta_{k+1}}, a_{k+1}e^{-i\theta_{k+1}-i\theta_k}; q)_\infty} \right] \\ &\quad \times \frac{(e^{2i\theta_s}, e^{-2i\theta_s}; q)_\infty (1 - x_s^2)^{-1/2}}{(be^{i\theta_s}, be^{-i\theta_s}, ce^{i\theta_s}, ce^{-i\theta_s}; q)_\infty}, \end{aligned}$$

(3.5)

$$\begin{aligned} \lambda_{\mathbf{n}}(q) &= \lambda_{\mathbf{n}}(a, b, c, a_2, a_3, \ldots, a_s|q) \\ &= (2\pi)^s \left[\prod_{k=1}^{s} (q^{n_k+1}, a_{k+1}^2 q^{n_k}; q)_\infty^{-1} \right] (abA_{2,s}q^{N_s}, acA_{2,s}q^{N_s}; q)_\infty^{-1}, \end{aligned}$$

where $a_{s+1}^2 = bc$ and $\max(|q|, |a|, |b|, |c|, |a_2|, |a_3|, \ldots, |a_s|) < 1$.

By taking the limit $a \to 0$ in (3.2)—(3.5) we can now deduce that the multivariable Al-Salam-Chihara polynomials defined by

$$\begin{aligned} S_{\mathbf{n}}(\mathbf{x}|q) &= S_{\mathbf{n}}(\mathbf{x}; b, c, a_2, a_3, \ldots, a_s|q) \\ &= \left[\prod_{k=1}^{s-1} p_{n_k}(x_k; a_{k+1}e^{i\theta_{k+1}}, a_{k+1}e^{-i\theta_{k+1}}|q) \right] \\ &\quad \times p_{n_s}(x_s; b, c|q). \end{aligned} \tag{3.6}$$

satisfy the orthogonality relation

$$\int_{-1}^{1} \cdots \int_{-1}^{1} S_{\mathbf{n}}(\mathbf{x}|q) S_{\mathbf{m}}(\mathbf{x}|q) \rho(\mathbf{x}|q) dx_1 \cdots dx_s = \lambda_{\mathbf{n}}(q)\, \delta_{\mathbf{n},\mathbf{m}} \tag{3.7}$$

with

(3.8)

$$\begin{aligned} \rho(\mathbf{x}|q) &= \left[\prod_{k=1}^{s-1} \frac{(e^{2i\theta_k}, e^{-2i\theta_k}; q)_\infty (1 - x_k^2)^{-1/2}}{(a_{k+1}e^{i\theta_{k+1}+i\theta_k}, a_{k+1}e^{i\theta_{k+1}-i\theta_k}, a_{k+1}e^{i\theta_k - i\theta_{k+1}}, a_{k+1}e^{-i\theta_{k+1}-i\theta_k}; q)_\infty} \right] \\ &\quad \times \frac{(e^{2i\theta_s}, e^{-2i\theta_s}; q)_\infty (1 - x_s^2)^{-1/2}}{(be^{i\theta_s}, be^{-i\theta_s}, ce^{i\theta_s}, ce^{-i\theta_s}; q)_\infty}, \end{aligned}$$

(3.9)

$$\lambda_{\mathbf{n}}(q) = (2\pi)^s \prod_{k=1}^{s} (q^{n_k+1}, a_{k+1}^2 q^{n_k}; q)_\infty^{-1},$$

where $a_{s+1}^2 = bc$, $\max(|q|, |b|, |c|, |a_2|, |a_3|, \ldots, |a_s|) < 1$, and the Al-Salam-Chihara polynomial $p_n(x; b, c|q)$ is defined by

$$p_n(x; b, c|q) = b^{-n}(bc; q)_n\ {}_3\phi_2 \left[\begin{matrix} q^{-n}, be^{i\theta}, be^{-i\theta} \\ bc, 0 \end{matrix} ; q, q \right], \tag{3.10}$$

see [5, (3.8.1)].

Setting

$$a = q^{(2\alpha+1)/4},\ b = q^{(2\alpha+3)/4},\ c = -q^{(2\beta+1)/4},\ d = -q^{2\beta+3)/4} \tag{3.11}$$

in (2.1) and (2.2) gives a multivariable orthogonal extension of the continuous q-Jacobi polynomials $P_n^{(\alpha,\beta)}(x|q)$ defined in [3, (7.5.24)], while setting

$$a = q^{1/2},\ b = q^{\alpha+1/2},\ c = -q^{\beta+1/2},\ d = -q^{1/2} \tag{3.12}$$

in (2.1) and (2.2) gives a multivariable orthogonal extension of the $P_n^{(\alpha,\beta)}(x; q)$ polynomials defined in (7.5.25). Also, via [3, (7.5.33)] and [3, (7.5.34) with $q \to q^{1/2}$] the $\alpha = \beta = \lambda - 1/2$ substitution gives a multivariable orthogonal extension of the continuous q-ultrashperical polynomials $C_n(x; q^\lambda|q)$. By letting $\lambda \to \infty$ when we use (3.12), i.e. set $a = -d = q^{1/2}$ and $b = c = 0$, we get a multivariable orthogonal extension of the continuous q-Hermite polynomials defined in [3, Ex. 1.28].

A multivariable orthogonal extension of the continuous q-Hahn polynomials defined by

$$\begin{aligned} &p_n(\cos(\theta + \phi); a, b|q) \\ &= (a^2, ab, abe^{2i\phi}; q)_n (ae^{i\phi})^{-n}\ {}_4\phi_3 \left[\begin{matrix} q^{-n}, a^2b^2q^{n-1}, ae^{2i\phi+i\theta}, ae^{-i\theta} \\ a^2, ab, abe^{2i\phi} \end{matrix} ; q, q \right], \end{aligned} \tag{3.13}$$

see [3, (7.5.43)], is obtained from (2.1)—(2.4) by replacing a, b, c, d, θ_k and $x_k = \cos\theta_k$ by $a_1e^{i\phi}$, $a_1e^{-i\phi}$, $a_{s+1}e^{i\phi}$, $a_{s+1}e^{-i\phi}$, $\theta_k + \phi$ and $\cos(\theta_k + \phi)$, respectively.

It is clear that similar special cases of the second system of multivariable orthogonal Askey-Wilson polynomials can be obtained by appropriate specialization of the parameters in (2.10) and (2.11). Additional systems of multivariable orthogonal polynomials will be considered elsewhere.

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