Lecture Notes For An Introductory Minicourse on q-Series

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These lecture notes were written for a minicourse that was designed to introduce students and researchers to *q-series,* which are also called *basic hypergeometric series* because of the parameter q that is used as a base in series that are "*over, above or beyond*" the *geometric series*. We start by considering q-extensions (also called q-analogues) of the binomial theorem, the exponential and gamma functions, and of the beta function and beta integral, and then progress on to the derivations of rather general summation, transformation, and expansion formulas, integral representations, and applications. Our main emphasis is on methods that can be used to **derive** formulas, rather than to just *verify* previously derived formulas. Since the best way to *learn* mathematics is to *do* mathematics, in order to enhance the learning process and enable the reader to practice with the discussed methods and formulas, we have provided several carefully selected exercises at the end of each section. We strongly encourage you to obtain a deeper understanding of q-series by looking at these exercises and doing *at least three* of them in each section. Solutions to several of the exercises are given in the Gasper and Rahman [1990a] "Basic Hypergeometric Series" book (which we will refer to as BHS) along with additional exercises and material on this subject, and references to applications to affine root systems (Macdonald identities), Lie algebras and groups, number theory, orthogonal polynomials, physics (such as representations of quantum groups and Baxter's work on the hard hexagon model), statistics, etc. In particular, for applications to orthogonal polynomials we recommend the Askey and Wilson [1985] A.M.S. Memoirs. For applications to number theory, physics and related fields, we recommend the Andrews [1986] and Berndt [1993] lecture notes and the Fine [1988] book. To add a historical perspective, we have followed the method in BHS of referring to papers and books in the References at the end of these notes by placing the year of publication in square brackets immediately after the author's name.

1 The q-binomial theorem and related formulas

1.1 The binomial theorem. One of the most elementary summation formulas for power series is the sum of the geometric series

$$\sum_{n=0}^{\infty} z^n = (1-z)^{-1}, \tag{1.1.1}$$

where z is a real or complex number and $|z| < 1$. By applying Taylor's theorem to the function $f(z) = (1-z)^{-a}$, which is an analytic function of z for $|z| < 1$, and observing

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by mathematical induction that $f^{(n)}(z) = (a)_n(1-z)^{-a-n}|_{z=0} = (a)_n$, where $(a)_n$ is the *shifted factorial* defined by

$$(a)_0 = 1,\ (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n = 1, 2, \ldots, \tag{1.1.2}$$

one can extend (1.1.1) to

$$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = (1-z)^{-a}, \qquad |z| < 1. \tag{1.1.3}$$

This formula is usually called the *binomial theorem* because, when $a = -m$ is a negative integer and $z = -x/y$, it reduces to the binomial theorem for the m-th power of the binomial $x + y$:

$$(x+y)^m = \sum_{n=0}^{m} \binom{m}{n} x^n y^{m-n}, \quad m = 0, 1, 2, \ldots\ . \tag{1.1.4}$$

1.2 The q-binomial theorem. Let $0 < q < 1$. Since, by l'Hôpital's rule,

$$\lim_{q \to 1^-} \frac{1-q^a}{1-q} = a \tag{1.2.1}$$

and hence

$$\lim_{q \to 1^-} \frac{(1-q^a)(1-q^{a+1})\cdots(1-q^{a+n-1})}{(1-q)(1-q^2)\cdots(1-q^n)} = \frac{(a)_n}{n!}, \tag{1.2.2}$$

it is natural to consider what happens when the coefficient of each z^n in (1.1.3) is replaced by the ratio displayed on the left side of (1.2.2) or, more generally, by

$$\frac{(a;q)_n}{(q;q)_n},$$

where $(a;q)_n$ is the *q-shifted factorial* defined by

$$(a;q)_n = \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n = 1, 2, \ldots\ . \end{cases} \tag{1.2.3}.$$

Hence, let us set

$$f(a,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n, \tag{1.2.4}$$

where a, q, z are real or complex numbers such that $|z| < 1$ and, unless stated otherwise, it is assumed that $|q| < 1$. The case when $|q| > 1$ will be considered later. Note that, by the ratio test, since $|q| < 1$ the series in (1.2.4) converges for $|z| < 1$ to a function, which we have denoted by $f(a,z)$. One way to find a formula for $f(a,z)$ analogous to that for the sum of the series in (1.1.3) is to first observe that, since $1-a=(1-aq^n)-a(1-q^n)$,

$$\begin{aligned} f(a,z) &= 1+\sum_{n=1}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n \\ &= 1+\sum_{n=1}^{\infty}\frac{(aq;q)_{n-1}}{(q;q)_n}[(1-aq^n)-a(1-q^n)]z^n \\ &= 1+\sum_{n=1}^{\infty}\frac{(aq;q)_n}{(q;q)_n}z^n - a\sum_{n=1}^{\infty}\frac{(aq;q)_{n-1}}{(q;q)_{n-1}}z^n \\ &= f(aq,z)-azf(aq,z)=(1-az)f(aq,z). \end{aligned} \tag{1.2.5}$$

By iterating this functional equation $n-1$ times we get that

$$f(a,z)=(az;q)_n\ f(aq^n,z), \qquad n=1,2,\ldots,$$

which on letting $n\to\infty$ and using $q^n\to 0$ gives

$$f(a,z)=(az;q)_\infty\ f(0,z) \tag{1.2.6}$$

with $(a;q)_\infty$ defined by

$$(a;q)_\infty=\lim_{n\to\infty}(a;q)_n=\prod_{k=0}^{\infty}(1-aq^k), \qquad |q|<1. \tag{1.2.7}$$

Since the above infinite product diverges when $a\neq 0$ and $|q|\geq 1$, whenever $(a;q)_\infty$ appears in a formula, it is usually assumed that $|q|<1$. Now set $a=q$ in (1.2.6) to obtain

$$f(0,z)=\frac{f(q,z)}{(qz;q)_\infty}=\frac{(1-z)^{-1}}{(qz;q)_\infty}=\frac{1}{(z;q)_\infty},$$

which, combined with (1.2.6) and (1.2.4), shows that

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n=\frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z|<1,\ |q|<1. \tag{1.2.8}$$

This summation formula was derived by Cauchy [1843] and Heine [1847]. It is called the *q-binomial theorem*, because it is a q-analogue of the binomial theorem in the sense that

$$\lim_{q\to 1^-}\sum_{n=0}^{\infty}\frac{(q^a;q)_n}{(q;q)_n}z^n=\sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^n=(1-z)^{-a}, \qquad |z|<1. \tag{1.2.9}$$

Notice that (1.2.8) and (1.2.9) yield

$$\lim_{q\to 1^-}\frac{(q^az;q)_\infty}{(z;q)_\infty}=(1-z)^{-a}, \quad |z|<1, \quad a \text{ real}, \tag{1.2.10}$$

which, by analytic continuation, holds for z in the complex plane cut along the positive real axis from 1 to ∞, with $(1-z)^{-a}$ positive when z is real and less than 1. In the special case $a = q^{-m}, m = 0, 1, 2, \ldots$, (1.2.8) gives

$$\sum_{n=0}^{m} \frac{(q^{-m};q)_n}{(q;q)_n} z^n = (zq^{-m};q)_m, \tag{1.2.11}$$

where, by analytic continuation, z can be any complex number.

Heine's proof of the q-binomial theorem, which is presented in Heine [1878], Bailey [1935, p. 66], Slater [1966, p. 92], and in §1.3 of BHS along with some motivation from Askey [1980], consists of using series manipulations to derive two difference equations that are then used to derive the functional equation

$$(1-z)f(a,z) = (1-az)f(a,qz). \tag{1.2.12}$$

Iterating this equation $n-1$ times and letting $n \to \infty$ gives

$$\begin{aligned} f(a,z) &= \frac{(az;q)_n}{(z;q)_n} f(a,q^n z) \\ &= \frac{(az;q)_\infty}{(z;q)_\infty} f(a,0) = \frac{(az;q)_\infty}{(z;q)_\infty} \end{aligned} \tag{1.2.13}$$

and completes the proof. Since it is easy to use series manipulations to show that the series in (1.2.4) satisfies the functional equation (1.2.12), once (1.2.12) has been *discovered* it can be used to give a short *verification* type proof of the q-binomial theorem.

Another derivation of the q-binomial theorem can be given by calculating the coefficients $c_n = g_a^{(n)}(0)/n!, n = 0, 1, 2, \ldots$, in the Taylor series expansion of the function

$$g_a(z) = \frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} c_n z^n, \tag{1.2.14}$$

which is an analytic function of z when $|z| < 1$ and $|q| < 1$. Clearly $c_0 = g_a(0) = 1$. One may show that $c_1 = g_a'(0) = (1-a)/(1-q)$ by taking the logarithmic derivative of $\frac{(az;q)_\infty}{(z;q)_\infty}$ and then setting $z = 0$. But, unfortunately, the succeeding higher order derivatives of $g_a(z)$ become more and more difficult to calculate, and so one is forced to abandon this approach and to search for another way to calculate all of the c_n coefficients. One simple method is to notice that from the definition of $g_a(z)$ as the quotient of two infinite products, it immediately follows that $g_a(z)$ satisfies the functional equation

$$(1-z)\; g_a(z) = (1-az)\; g_a(qz), \tag{1.2.15}$$

which is of course the same as the functional equation (1.2.12) satisfied by $f(a,z)$. In a *verification* type proof of the q-binomial theorem, (1.2.15) provides *substantial* motivation for showing, as in Heine's proof, that the sum of the q-binomial series $f(a,z)$ satisfies the functional equation (1.2.12).

To calculate the c_n coefficients, we first use (1.2.15) to find that

$$\sum_{n=0}^{\infty} c_n\, z^n - \sum_{n=0}^{\infty} c_n\, z^{n+1} = \sum_{n=0}^{\infty} c_n\, q^n\ z^n - a\sum_{n=0}^{\infty} c_n\, q^n\, z^{n+1},$$

or, equivalently,

$$1+\sum_{n=1}^{\infty}(c_n - c_{n-1})\, z^n = 1+\sum_{n=1}^{\infty}(c_n q^n - ac_{n-1}\, q^{n-1})\, z^n,$$

which implies that

$$c_n - c_{n-1} = c_n q^n - ac_{n-1}\, q^{n-1}$$

and hence

$$c_n = \frac{1-aq^{n-1}}{1-q^n}\, c_{n-1} \tag{1.2.16}$$

for $n = 1, 2, \ldots$. Iterating the recurrence relation (1.2.16) $n-1$ times gives

$$c_n = \frac{(a;q)_n}{(q;q)_n} c_0 = \frac{(a;q)_n}{(q;q)_n}, \qquad n = 0, 1, 2, \ldots, \tag{1.2.17}$$

which concludes our third derivation of the q-binomial theorem (1.2.8). For a combinatorial proof using a bijection between two classes of partitions, see Andrews [1969].

1.3 Related formulas. One immediate consequence the q-binomial theorem is the product formula

$$\sum_{j=0}^{\infty} \frac{(a;q)_j}{(q;q)_j}\, z^j \ \sum_{k=0}^{\infty} \frac{(b;q)_k}{(q;q)_k}\ (az)^k = \sum_{n=0}^{\infty} \frac{(ab;q)_n}{(q;q)_n}\, z^n, \qquad |z|<1,\ |q|<1, \tag{1.3.1}$$

which is a q-analogue of $(1-z)^{-a}(1-z)^{-b} = (1-z)^{-a-b}$. By setting $j = n-k$ in the product on the left side of (1.3.1) and comparing the coefficients of z^n on both sides of the equation, we get

$$\frac{(ab;q)_n}{(q;q)_n} = \sum_{k=0}^{n} \frac{(a;q)_{n-k}(b;q)_k}{(q;q)_{n-k}(q;q)_k}\, a^k, \tag{1.3.2}$$

which gives a q-analogue of (1.1.4) in the form

$$(ab;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (a;q)_{n-k}(b;q)_k\, a^k, \tag{1.3.3}$$

where the *q-binomial coefficient* is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \qquad k = 0, 1, \ldots, n. \tag{1.3.4}$$

If we let

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n}, \qquad |z| < 1, \tag{1.3.5}$$

then the case $a = 0$ of (1.2.8) gives

$$e_q(z) = \frac{1}{(z;q)_\infty}, \qquad |z| < 1, \ |q| < 1. \tag{1.3.6}$$

The function $e_q(z)$ is a q-analogue of the exponential function e^z, since

$$\lim_{q \to 1^-} e_q(z(1-q)) = e^z. \tag{1.3.7}$$

Another q-analogue of e^z can be obtained from (1.2.8) by replacing z with $-z/a$ and then letting $a \to \infty$ to find that the *q-exponential function* defined by

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} \, z^n, \qquad |z| < \infty, \ |q| < 1, \tag{1.3.8}$$

equals $(-z;q)_\infty$ for all complex values of z and satisfies the limit relation

$$\lim_{q \to 1^-} E_q(z(1-q)) = e^z, \qquad |z| < \infty. \tag{1.3.9}$$

Hence, $e_q(z) E_q(-z) = 1$ when $|z| < 1$ and $|q| < 1$.

In Exercise 1.1 below you will be asked to verify the *inversion identity*

$$(a;q)_n = (a^{-1}; q^{-1})_n (-a)^n q^{\binom{n}{2}} \tag{1.3.10}$$

for $n = 0, 1, 2, \ldots$. This identity enables us to convert a q-series formula containing sums of quotients of products of q-shifted factorials in base q to a similar formula with base q^{-1}. In particular, it follows from (1.2.8) and (1.3.10) that if $|q| > 1$, then the q-binomial theorem takes the form

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} \, z^n = \frac{(z/q; q^{-1})_\infty}{(az/q; q^{-1})_\infty}, \quad |az/q| < 1, \ |q| > 1. \tag{1.3.11}$$

Exercises 1

1.1 Verify the inversion identity (1.3.10) and show for nonnegative integers k and n that

$$(i) \qquad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

$$(ii) \qquad (a;q)_{n+k} = (a;q)_n(aq^n;q)_k,$$

$$(iii) \qquad (aq^n;q)_k = \frac{(a;q)_k(aq^k;q)_n}{(a;q)_n},$$

$$(iv) \qquad (aq^{-n};q)_k = \frac{(a;q)_k(qa^{-1};q)_n}{(a^{-1}q^{1-k};q)_n}\, q^{-nk},$$

$$(v) \qquad \frac{1-aq^{2n}}{1-a} = \frac{(qa^{\frac{1}{2}};q)_n(-qa^{\frac{1}{2}};q)_n}{(a^{\frac{1}{2}};q)_n(-a^{\frac{1}{2}};q)_n}.$$

1.2 Use series manipulations to verify that the function $f(a,z)$ defined in (1.2.4) satisfies the functional equation (1.2.12).

1.3 Show that if we define $e_q(z)$ for $|q|>1$ by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n}, \qquad |z|<\infty,\ |q|>1,$$

then it follows from (1.3.8) and (1.3.10) that $e_{q^{-1}}(z) = E_q(-qz)$ when $|z|<\infty$ and $|q|<1$.

1.4 Derive (1.3.11) by using one (or more) of the methods used in §1.2 to derive (1.2.8).

1.5 Investigate what happens if, as in the third derivation of the q-binomial theorem presented in §1.2, you try to calculate the c_n coefficients in (1.2.14) by starting with the functional equation $g_a(z) = (1-az)\, g_{aq}(z)$, which corresponds to the functional equation derived in (1.2.5), instead of with the more complicated functional equation (1.2.15).

1.6 Extend the definition of the q-binomial coefficient (1.3.4) to

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_q = \frac{(q^{\beta+1};q)_\infty(q^{\alpha-\beta+1};q)_\infty}{(q;q)_\infty(q^{\alpha+1};q)_\infty}$$

for complex α and β when $|q|<1$. Show that

$$(i) \qquad \begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \frac{(q^{-\alpha};q)_k}{(q;q)_k}\,(-q^\alpha)^k\, q^{-\binom{k}{2}},$$

$$(ii) \qquad \begin{bmatrix} k+\alpha \\ k \end{bmatrix}_q = \frac{(q^{\alpha+1};q)_k}{(q;q)_k},$$

$$(iii) \qquad \begin{bmatrix} \alpha+1 \\ k \end{bmatrix}_q = \begin{bmatrix} \alpha \\ k \end{bmatrix}_q q^k + \begin{bmatrix} \alpha \\ k-1 \end{bmatrix}_q = \begin{bmatrix} \alpha \\ k \end{bmatrix}_q + \begin{bmatrix} \alpha \\ k-1 \end{bmatrix}_q q^{\alpha+1-k},$$

where k and n are nonnegative integers.

1.7 Use Ex. 1.6 (iii) and induction to prove that if x and y are indeterminates such that $xy = qyx, q$ commutes with x and y, and the associative law holds, then

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k x^{n-k} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} x^k y^{n-k}.$$

See, e.g., Koornwinder[1989].

2 Ramanujan's ${}_1\psi_1$ summation formula, Jacobi's triple product identity and theta functions

2.1 Ramanujan's ${}_1\psi_1$ summation formula. Since

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \qquad |q|<1,\ n=0,1,2,\ldots, \tag{2.1.1}$$

we may extend the definition (1.2.3) of $(a;q)_n$ to

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \qquad |q|<1, \tag{2.1.2}$$

for any complex number α, where the principal value of q^α is (usually) taken when $q \neq 0$. In particular, if $0<|q|<1$ and $n=0,1,2,\ldots$, then

$$(a;q)_{-n} = \frac{(a;q)_\infty}{(aq^{-n};q)_\infty} = \frac{1}{(aq^{-n};q)_n} = \frac{(-q/a)^n}{(q/a;q)_n}\, q^{\binom{n}{2}} \tag{2.1.3}$$

and

$$\frac{1}{(q^n;q)_k} = (q^{n+k};q)_{-k} = 0, \qquad n \geq 1,\ k=-n,-n-1,\ldots\ . \tag{2.1.4}$$

Let $0<|q|<1, 0<|z|<1$, and $n=1,2,\ldots$. Then the q-binomial theorem (1.2.8) gives

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{k=-\infty}^{\infty} \frac{(a;q)_k}{(q;q)_k}\, z^k = \sum_{j=-\infty}^{\infty} \frac{(a;q)_{j+n-1}}{(q;q)_{j+n-1}}\, z^{j+n-1}$$

$$= \sum_{j=-\infty}^{\infty} \frac{(a;q)_{n-1}(aq^{n-1};q)_j}{(q;q)_{n-1}(q^n;q)_j}\, z^{j+n-1} = \frac{(a;q)_{n-1}}{(q;q)_{n-1}}\, z^{n-1} \sum_{j=-\infty}^{\infty} \frac{(aq^{n-1};q)_j}{(q^n;q)_j}\, z^j \tag{2.1.5}$$

by a shift in the index of summation. After replacing a by aq^{1-n} and then setting $q^n = b$ it follows from the left and right sides of (2.1.5) that

$$\sum_{j=-\infty}^{\infty} \frac{(a;q)_j}{(b;q)_j}\, z^j = \frac{(q, b/a, az, q/az;q)_\infty}{(b, q/a, z, b/az;q)_\infty} \tag{2.1.6}$$

for $b = q^n$ when $0 < |q| < 1, 0 < |z| < 1$, and $n = 1, 2, \ldots$, where we used the compact notation

$$(a_1, a_2, \ldots, a_k; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_k; q)_\infty. \tag{2.1.7}$$

By applying (2.1.3) to the terms of the series in (2.1.6) with negative j, we get

$$\sum_{j=-\infty}^{\infty} \frac{(a; q)_j}{(b; q)_j} z^j = \sum_{j=0}^{\infty} \frac{(a; q)_j}{(b; q)_j} z^j + \sum_{j=1}^{\infty} \frac{(q/b; q)_j}{(q/a; q)_j} \left(\frac{b}{az}\right)^j, \tag{2.1.8}$$

from which it is clear that the bilateral series on the left side of (2.1.8) converges in the annulus $|b/a| < |z| < 1$ when $|q| < 1$ and $a \neq 0$. Since both sides of (2.1.8) are analytic functions of b when $|b| < \min(1, |az|)$ and $|z| < 1$, and since (2.1.6) holds for $b = q^n$ when $0 < |q| < 1$, $n = 1, 2, \ldots$, it follows by analytic continuation that we have derived the summation formula

$$\sum_{j=-\infty}^{\infty} \frac{(a; q)_j}{(b; q)_j} z^j = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}, \qquad |b/a| < |z| < 1,\ |q| < 1, \tag{2.1.9}$$

which is an extension of the q-binomial theorem. This formula was stated without proof in Ramanujan's second notebook (see Hardy [1940] and Berndt [1993]). The first published proofs were given by Hahn [1949] and M. Jackson [1950]. The above proof is essentially the reverse order of Ismail's [1977] proof, which first reduces the proof of (2.1.9) to the case when $b = q^n$, where n is a positive integer, and then proves this case by using a shift in the index of summation to obtain a series that is summable by the q-binomial theorem. In the next section, when we introduce the ${}_r\psi_s$ notation for general bilateral basic hypergeometric series, it will be seen that (2.1.9) gives the sum of a ${}_1\psi_1$ series. Hence, (2.1.9) is sometimes called Ramanujan's ${}_1\psi_1$ summation formula.

In BHS a proof due to Andrews and Askey [1978] is given, which first considers the sum of the series in (2.1.9) as a function of b, say $f(b)$, and then shows that this function satisfies the functional equation

$$f(b) = \frac{1 - b/a}{(1 - b)(1 - b/az)} f(bq). \tag{2.1.10}$$

Iterating (2.1.10) $n - 1$ times gives

$$f(b) = \frac{(b/a; q)_n}{(b, b/az; q)_n} f(bq^n). \tag{2.1.11}$$

Since $f(b)$ is analytic for $|b| < \min(1, |az|)$, we may let $n \to \infty$ in (2.1.11) to obtain

$$f(b) = \frac{(b/a; q)_\infty}{(b, b/az; q)_\infty} f(0). \tag{2.1.12}$$

To calculate $f(0)$, it suffices to set $b = q$ in (2.1.12) and observe that

$$f(0) = \frac{(q, q/az; q)_\infty}{(q/a; q)_\infty} f(q) = \frac{(q, q/az, az; q)_\infty}{(q/a, z; q)_\infty}, \tag{2.1.13}$$

since

$$f(q)=\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\,z^n=\frac{(az;q)_\infty}{(z;q)_\infty}$$

by the q-binomial theorem. Combining (2.1.12) and (2.1.13) gives (2.1.9).

Just as in the third derivation given in §1.2 for the q-binomial theorem, one can also derive (2.1.9) by using a functional equation and a recurrence relation to calculate the coefficients c_n in the Laurent expansion

$$\frac{(az,q/az;q)_\infty}{(z,b/az;q)_\infty}=\sum_{n=-\infty}^{\infty}c_n z^n. \tag{2.1.14}$$

See, e.g., Venkatachaliengar [1988], whose proof is reproduced in Berndt [1993] (also see Exercise 2.4).

2.2 Jacobi's triple product identity and theta functions. If we set $b=0$ in (2.1.9), replace q and z by q^2 and $-qz/a$, respectively, and then let $a\to\infty$, we obtain Jacobi's [1829] *triple product identity*

$$\sum_{k=-\infty}^{\infty}q^{k^2}z^k=\left(q^2,-qz,-q/z;q^2\right)_\infty. \tag{2.2.1}$$

In BHS this formula is derived by using Heine's [1847] summation formula for a ${}_2\phi_1(c/ab)$ series, which we will consider in §3.

Jacobi's triple product identity has many important applications. In particular, it can be used to express the *theta functions* (Whittaker and Watson [1965, Chapter 21])

$$\vartheta_1(x)=2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin(2n+1)x, \tag{2.2.2}$$

$$\vartheta_2(x)=2\sum_{n=0}^{\infty}q^{(n+1/2)^2}\cos(2n+1)x, \tag{2.2.3}$$

$$\vartheta_3(x)=1+2\sum_{n=1}^{\infty}q^{n^2}\cos 2nx, \tag{2.2.4}$$

$$\vartheta_4(x)=1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nx \tag{2.2.5}$$

in terms of infinite products. To derive the infinite product representations, replace q by q^2 in (2.2.1) and then set z equal to $qe^{2ix},-qe^{2ix},-e^{2ix},e^{2ix}$, respectively, to obtain

$$\vartheta_1(x)=2q^{1/4}\sin x\prod_{n=1}^{\infty}(1-q^{2n})(1-2q^{2n}\cos 2x+q^{4n}), \tag{2.2.6}$$

$$\vartheta_2(x) = 2q^{1/4}\cos x \prod_{n=1}^{\infty}(1-q^{2n})(1+2q^{2n}\cos 2x + q^{4n}), \tag{2.2.7}$$

$$\vartheta_3(x) = \prod_{n=1}^{\infty}(1-q^{2n})(1+2q^{2n-1}\cos 2x + q^{4n-2}), \tag{2.2.8}$$

and

$$\vartheta_4(x) = \prod_{n=1}^{\infty}(1-q^{2n})(1-2q^{2n-1}\cos 2x + q^{4n-2}). \tag{2.2.9}$$

For other applications of (2.2.1), including a proof of the *quintuple product identity*

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} z^{3n}(1+zq^n) = (q,-z,-q/z;q)_\infty (qz^2, q/z^2; q^2)_\infty, \qquad z \neq 0, \tag{2.2.10}$$

see Berndt [1993] and the Notes to Exercise 5.6 in BHS.

Exercises 2

2.1 Verify the identities (2.1.3) and (2.1.4), and show that

(i) $$(aq^k;q)_{n-k} = \frac{(a;q)_n}{(a;q)_k},$$

(ii) $$(aq^{2k};q)_{n-k} = \frac{(a;q)_n (aq^n;q)_k}{(a;q)_{2k}},$$

(iii) $$(aq^{-n};q)_k = \frac{(a;q)_k (qa^{-1};q)_n}{(a^{-1}q^{1-k};q)_n}\, q^{-nk},$$

(iv) $$(q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk},$$

(v) $$(a;q)_{n-k} = \frac{(a;q)_n}{(a^{-1}q^{1-n};q)_k}\,(-qa^{-1})^k q^{\binom{k}{2}-nk},$$

when k and n are integers and both sides of the identity are well-defined.

2.2 Use series manipulations to verify that the function $f(b)$ defined in §2.1 satisfies the functional equation (2.1.10).

2.3 Apply the inversion identity (1.3.10) to the q-shifted factorials in the left side of (2.1.9) to derive a bilateral extension of the $|q| > 1$ case of the q-binomial given in (1.3.11).

2.4 Show that the function

$$h(z) = \frac{(az, q/az;q)_\infty}{(z, b/az;q)_\infty}$$

satisfies the functional equation $(b-aqz)\,h(qz) = q(1-z)\,h(z)$ when $|b/aq| < |z| < 1$. Use this equation to derive a recurrence relation for the coefficients c_n in (2.1.14). As in the derivation of (1.2.17), use the recurrence relation to calculate these coefficients, and hence derive Ramanujan's ${}_1\psi_1$ summation formula (2.1.9). See Venkatachaliengar [1988] and Berndt [1993].

2.5 Investigate what happens if, as in Ex. 2.4, you try to generalize Ramanujan's ${}_1\psi_1$ summation formula by using a functional equation and a recurrence relation to calculate the coefficients in the Laurent expansion

$$\frac{(az, b/z; q)_\infty}{(z, c/z; q)_\infty} = \sum_{n=-\infty}^{\infty} d_n\, z^n, \qquad |c| < |z| < 1.$$

2.6 Use functional equations in a and b to prove that

$$\int_0^\infty \frac{(-tq^b, -q^{a+1}/t; q)_\infty}{(-t, -q/t; q)_\infty} \frac{dt}{t} = -\log q\, \frac{(q, q^{a+b}; q)_\infty}{(q^a, q^b; q)_\infty}$$

when $0 < q < 1$, Re $a > 0$ and Re $b > 0$. See Askey and Roy [1986], and Gasper [1987].

2.7 Extend the integral formula in Ex. 2.6 by evaluating the more general integral

$$\int_0^\infty t^{c-1} \frac{(-tq^b, -q^{a+1}/t; q)_\infty}{(-t, -q/t; q)_\infty}\, dt$$

when $0 < q < 1$, Re $(a+c) > 0$ and Re $(b-c) > 0$. See Ramanujan [1915], Askey and Roy [1986], and Gasper [1987].

3 Basic hypergeometric series, q-gamma and q-beta functions, q-integrals, and some important summation and transformation formulas

3.1 Basic hypergeometric series. So far, because of the relatively simple q-series considered, we have not needed to introduce a compact notation for q-series containing several parameters. Recall that the *Gauss* [1813] *hypergeometric series* is (formally) defined by

$$F(a,\ b;\ c;\ z) \equiv {}_2F_1(a,\ b;\ c;\ z) \equiv {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n}\, z^n, \tag{3.1.1}$$

where it is assumed that $c \neq 0, -1, -2, ...,$ so that no zero factors appear in the denominators of the terms of the series. Gauss' series converges absolutely for $|z| < 1$, and for $|z| = 1$ when $\mathrm{Re}(c - a - b) > 0$. Heine [1846, 1847, 1878] introduced the series

$$\phi(\alpha, \beta, \gamma, q, z) = {}_2\phi_1(q^\alpha, q^\beta; q^\gamma; q, z) \tag{3.1.2}$$

with

$${}_2\phi_1(a, b; c; q, z) \equiv {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n}\, z^n, \tag{3.1.3}$$

where it is assumed that $\gamma \neq -m$ and $c \neq q^{-m}$ for $m = 0, 1, \ldots$. *Heine's series* converges absolutely for $|z| < 1$ when $|q| < 1$, and it is a q-analogue of Gauss' series because, by using (1.2.1) and taking a formal termwise limit,

$$\lim_{q \to 1^-} {}_2\phi_1(q^\alpha, q^\beta; q^\gamma; q, z) = {}_2F_1(\alpha, \beta; \gamma; z). \tag{3.1.4}$$

In view of the base q, Heine's series is also called the *basic hypergeometric series* or *q-hypergeometric series.* We prefer to use the ${}_2\phi_1(a, b; c; q, z)$ notation instead of Heine's $\phi(\alpha, \beta, \gamma, q, z)$ notation, because when $0 < |q| < 1$ the α, β, or $\gamma \to \infty$ limit cases of Heine's series correspond to setting a, b, or c, respectively, equal to zero in (3.1.3).

The *(generalized) hypergeometric series* with r numerator parameters $a_1, \ldots, a_r$ and s denominator parameters $b_1, \ldots, b_s$ is defined (formally) by

$$\begin{aligned} {}_rF_s(a_1, a_2, \ldots, a_r;\ b_1, \ldots, b_s; z) &\equiv {}_rF_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; z\right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{n! (b_1)_n \cdots (b_s)_n}\, z^n, \end{aligned} \tag{3.1.5}$$

and an ${}_r\phi_s$ *basic hypergeometric series* is defined by

$$\begin{aligned} {}_r\phi_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_s; q, z) &\equiv {}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z\right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n \end{aligned} \tag{3.1.6}$$

where $\binom{n}{2} = n(n-1)/2$ and, analogous to (2.1.7), we employed the compact notation $(a_1, a_2, \ldots, a_r; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n$. As in (3.1.4), ${}_r\phi_s$ is a q-analogue of ${}_rF_s$ and we have (formally)

$$\lim_{q \to 1^-} {}_r\phi_s\left[\begin{matrix} q^{a_1}, q^{a_2}, \ldots, q^{a_r} \\ q_1^b, \ldots, q^{b_s} \end{matrix}; q, (q-1)^{1+s-r} z\right] = {}_rF_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; z\right]. \tag{3.1.7}$$

In terms of these notations, the binomial theorem (1.1.3) and the q-binomial theorem (1.2.8) may be written in the forms

$$ {}_2F_1(a, c; c; z) = {}_1F_0(a; —; z) = (1-z)^{-a}, \quad |z| < 1, \tag{3.1.8}$$

and

$$ {}_2\phi_1(a, c; c; q, z) = {}_1\phi_0(a; —; q, z) = \frac{(az; q)_\infty}{(z; q)_\infty}, \quad |z| < 1,\ |q| < 1, \tag{3.1.9}$$

where a dash is used to indicate the absence of either numerator (when $r = 0$) or denominator (when $s = 0$) parameters. Many other important special cases are considered in BHS, its references, and in what follows.

It is assumed in (3.1.5) and (3.1.6) that the parameters $b_1, \ldots, b_s$ are such that the denominator factors in the terms of the series are never zero, and in (3.1.6) it is assumed that $q \neq 0$ when $r > s+1$. From

$$(-m)_n = \left(q^{-m}; q\right)_n = 0, \quad n = m+1, m+2, \ldots, \tag{3.1.10}$$

we see that an ${}_rF_s$ series terminates if one of its numerator parameters is zero or a negative integer, and an ${}_r\phi_s$ series terminates if one of its numerator parameters is of the form q^{-m} with $m = 0, 1, 2, \ldots$, and $q \neq 0$. Unless stated otherwise, when working with nonterminating basic hypergeometric series, we will (for simplicity) assume that $|q| < 1$ and that the parameters and variables are such that the series converge absolutely. As in our derivation of (1.3.11), if $|q| > 1$ then we can use (1.3.10) to perform an inversion with respect to the base to transform the series (3.1.6) into a series in base q^{-1}, with $|q^{-1}| < 1$ (see Exercise 3.2).

The ratio test shows that an ${}_rF_s$ series converges absolutely for all z if $r \leq s$, and for $|z| < 1$ if $r = s+1$. By Raabe's test this series also converges absolutely for $|z| = 1$ if $r = s+1$ and Re $[b_1 + \cdots + b_s - (a_1 + \cdots + a_r)] > 0$. If $r > s+1$ and $z \neq 0$ or $r = s+1$ and $|z| > 1$, then this series diverges, unless it terminates. Similarly, if $0 < |q| < 1$, then the ${}_r\phi_s$ series converges absolutely for all z if $r \leq s$ and for $|z| < 1$ if $r = s+1$. It converges absolutely if $|q| > 1$ and $|z| < |b_1 b_2 \cdots b_s| / |a_1 a_2 \cdots a_r|$. It diverges for $z \neq 0$ if $0 < |q| < 1$ and $r > s+1$, and if $|q| > 1$ and $|z| > |b_1 b_2 \cdots b_s| / |a_1 a_2 \cdots a_r|$, unless it terminates. The ${}_rF_s$ and ${}_r\phi_s$ notations are also used for the sums of these series inside the circle of convergence and for their analytic continuations (called *hypergeometric functions* and *basic hypergeometric functions*, respectively) outside the circle of convergence.

The series (3.1.6) has the property that if we replace z by z/a_r and let $a_r \to \infty$ (called a *confluence process*), then we obtain a series that is of the form (3.1.6) with r replaced by $r-1$. This is not the case for the ${}_r\phi_s$ series defined without the factors $[(-1)^n q^{\binom{n}{2}}]^{1+s-r}$ in Bailey [1935] and Slater [1966]. It is because we need a notation that includes such limit cases as special cases that we have chosen to use the definition (3.1.6). Also, there is no loss in generality because the Bailey and Slater series can be obtained from the $r = s+1$ case of (3.1.6) by choosing s sufficiently large and setting some of the parameters equal to zero.

Notice that if we denote the terms of the series (3.1.5) and (3.1.6) which contain z^n by u_n and v_n, respectively, then $\frac{u_{n+1}}{u_n}$ is a rational function of n and $\frac{v_{n+1}}{v_n}$ is a rational function of q^n. Conversely, any rational function of n can be written in the form of $\frac{u_{n+1}}{u_n}$, and any rational function of q^n can be written in the form of $\frac{v_{n+1}}{v_n}$. Hence, we have the characterization that if $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$ are series with $u_0 = v_0 = 1$ such that u_{n+1}/u_n is a rational function of n and v_{n+1}/v_n is a rational function of q^n, then these series are of the forms (3.1.5) and (3.1.6), respectively. This characterization provides another reason why we defined ${}_r\phi_s$ in (3.1.6) with the $[(-1)^n q^{\binom{n}{2}}]^{1+s-r}$ factor.

Generalizing the bilateral q-series that we considered in §2.1, we define the *bilateral* *basic hypergeometric series* in base q with r numerator and s denominator parameters by

$$\begin{aligned} {}_r\psi_s(z) \equiv {}_r\psi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right] \\ = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} (-1)^{(s-r)n} q^{(s-r)\binom{n}{2}} z^n, \end{aligned} \tag{3.1.11}$$

where it is assumed that q, z and the parameters are such that each term of the series is well-defined (i.e., the denominator factors are never zero, $q \neq 0$ if $s < r$, and $z \neq 0$ if negative powers of z occur). A bilateral basic hypergeometric series may be characterized as being a series $\sum_{n=-\infty}^{\infty} v_n$ such that $v_0 = 1$ and v_{n+1}/v_n is a rational function of q^n.

Employing the ${}_1\psi_1$ notation, Ramanujan's summation formula (2.1.9) may be stated in the form

$${}_1\psi_1(a; b; q, z) = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}, \qquad |b/a| < |z| < 1,\ |q| < 1. \tag{3.1.12}$$

As in our derivation of (2.1.8), to determine when ${}_r\psi_s$ series converge we first apply (1.3.10) to the terms with negative n to obtain the decomposition

$$\begin{aligned} {}_r\psi_s(z) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} (-1)^{(s-r)n} q^{(s-r)\binom{n}{2}} z^n \\ + \sum_{n=1}^{\infty} \frac{(q/b_1, q/b_2, \ldots, q/b_s; q)_n}{(q/a_1, q/a_2, \ldots, q/a_r; q)_n} \left(\frac{b_1 \cdots b_s}{a_1 \cdots a_r z}\right)^n. \end{aligned} \tag{3.1.13}$$

Set $R = |(b_1 \cdots b_s)/(a_1 \cdots a_r)|$. From (3.1.13) it is obvious that if $r > s$ and $0 < |q| < 1$, then the first series on the right side of (3.1.13) diverges for $z \neq 0$; if $r > s$ and $|q| > 1$, then the first series converges for $|z| < R$ and the second series converges for all $z \neq 0$. When $r < s$ and $|q| < 1$ the first series converges for all z, but the second series converges only when $|z| > R$. If $r < s$ and $|q| > 1$, the second series diverges for all $z \neq 0$. If $r = s$ and $|q| < 1$, the first series converges when $|z| < 1$ and the second when $|z| > R$; while if $|q| > 1$, then the second series converges when $|z| > 1$ and the first when $|z| < R$. In particular, if $r = s$ and $|q| < 1$, which is the most important case, the region of convergence of the series ${}_r\psi_r(z)$ is the annulus $|(b_1 \cdots b_r)/(a_1 \cdots a_r)| < |z| < 1$.

3.2 q-gamma and q-beta functions. Analogous to Gauss' infinite product representation for the gamma function

$$\Gamma(z) = z^{-1} \prod_{k=1}^{\infty} [(1 + 1/k)^z (1 + z/k)^{-1}], \tag{3.2.1}$$

the *q-gamma function* $\Gamma_q(z)$ is defined as in Thomae [1869] and Jackson [1904] by

$$\Gamma_q(z) = \frac{(q; q)_\infty}{(q^z; q)_\infty} (1 - q)^{1-z},\ 0 < q < 1. \tag{3.2.2}$$

When $z = n + 1$ is a positive integer, this definition reduces to

$$\Gamma_q(n+1) = 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}), \tag{3.2.3}$$

which tends to $n!$ as $q \to 1^-$. Thus $\Gamma_q(n+1) \to \Gamma(n+1) = n!$ as $q \to 1^-$. One can extend the definition of $\Gamma_q(z)$ to $|q| < 1$ by using the principal values of q^z and $(1-q)^{1-z}$ in (3.2.2). In BHS it is shown that

$$\lim_{q \to 1^-} \Gamma_q(z) = \Gamma(z) \tag{3.2.4}$$

and that $\Gamma_q(z)$ satisfies the functional equation

$$f(z+1) = \frac{1-q^z}{1-q} f(z), \quad f(1) = 1, \tag{3.2.5}$$

which is a q-analogue of the well-known functional equation for the gamma function

$$\Gamma(z+1) = z\,\Gamma(z), \quad \Gamma(1) = 1. \tag{3.2.6}$$

Analogous to the definition of the the *beta function*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{3.2.7}$$

the *q-beta function* is defined by

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}, \tag{3.2.8}$$

which tends to $B(x,y)$ as $q \to 1^-$. From (3.2.2) and (1.2.8),

$$\begin{aligned} B_q(x,y) &= (1-q)\frac{(q, q^{x+y}; q)_\infty}{(q^x, q^y; q)_\infty} \\ &= (1-q)\frac{(q;q)_\infty}{(q^y;q)_\infty} \sum_{n=0}^{\infty} \frac{(q^y;q)_n}{(q;q)_n} q^{nx} \\ &= (1-q)\sum_{n=0}^{\infty} \frac{(q^{n+1};q)_\infty}{(q^{n+y};q)_\infty} q^{nx}, \quad \text{Re}\, x,\ \text{Re}\, y > 0. \end{aligned} \tag{3.2.9}$$

This expansion will be utilized in §3.3 to derive a q-integral representation for $B_q(x,y)$.

3.3 q-integrals. Thomae [1869, 1870] defined the *q-integral* on the interval $[0,1]$ by

$$\int_0^1 f(t)\, d_qt = (1-q)\sum_{n=0}^{\infty} f(q^n)q^n. \tag{3.3.1}$$

The right side of (3.3.1) corresponds to using a Riemann sum with partition points $t_n = q^n, n = 0, 1, 2, \ldots$. Jackson [1910b] extended this to the interval $[a, b]$ via

$$\int_a^b f(t)\ d_qt = \int_0^b f(t)\ d_qt - \int_0^a f(t)\ d_qt, \tag{3.3.2}$$

where

$$\int_0^a f(t)\ d_qt = a(1-q)\sum_{n=0}^{\infty} f(aq^n)\, q^n. \tag{3.3.3}$$

He also defined an integral on $(0, \infty)$ by

$$\int_0^{\infty} f(t)\ d_qt = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n. \tag{3.3.4}$$

On the interval $(-\infty, \infty)$ the *bilateral q-integral* is defined by

$$\int_{-\infty}^{\infty} f(t)\ d_qt = (1-q)\sum_{n=-\infty}^{\infty} [f(q^n) + f(-q^n)]\, q^n. \tag{3.3.5}$$

When f is continuous on $[0, a]$, it can be shown that

$$\lim_{q\to 1}\int_0^a f(t)\ d_qt = \int_0^a f(t)\ dt \tag{3.3.6}$$

and that a similar limit holds for (3.3.4) and (3.3.5) for suitable functions. By employing the q-integral definition (3.3.1), the series expansion (3.2.9) for the q-beta function can be rewritten in the form

$$B_q(x, y) = \int_0^1 t^{x-1}\frac{(tq;q)_\infty}{(tq^y;q)_\infty}\ d_qt, \quad \operatorname{Re} x > 0, \quad y \neq 0, -1, -2, \ldots, \tag{3.3.7}$$

which, as $q \to 1^-$, tends to the beta function integral

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\ dt, \qquad \operatorname{Re} x,\ \operatorname{Re} y > 0. \tag{3.3.8}$$

We can derive a q-integral representation for ${}_2\phi_1(q^a, q^b; q^c; q, z)$ by using (3.3.7) to get

$$\frac{(q^b;q)_n}{(q^c;q)_n} = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)}\int_0^1 t^{b+n-1}\frac{(tq;q)_\infty}{(tq^{c-b};q)_\infty}\ d_qt \tag{3.3.9}$$

for $n \geq 0,\ \operatorname{Re} b > 0,\ c - b \neq 0, -1, -2, \ldots,$ which leads to

$${}_2\phi_1(q^a, q^b; q^c; q, z) = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)}\sum_{n=0}^{\infty}\frac{(q^a;q)_n}{(q;q)_n}z^n\int_0^1 t^{b+n-1}\frac{(tq;q)_\infty}{(tq^{c-b};q)_\infty}\ d_qt$$

$$= \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \int_0^1 t^{b-1} \frac{(tq;q)_\infty}{(tq^{c-b};q)_\infty} {}_1\phi_0(q^a;—;q,tz)\ d_qt \tag{3.3.10}$$

when $|z| < 1$, Re $b > 0$, and $c - b \neq 0, -1, -2, \dots$. Employing (3.1.9) to sum the ${}_1\phi_0$ series in (3.3.10) gives Thomae's [1869] q-integral

$${}_2\phi_1(q^a, q^b; q^c; q, z) = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \int_0^1 t^{b-1} \frac{(tzq^a, tq; q)_\infty}{(tz, tq^{c-b}; q)_\infty}\ d_qt, \tag{3.3.11}$$

where $|z| < 1, \mathrm{Re}\, b > 0$, and $c - b \neq 0, -1, -2, \dots$. This is a q-analogue of Euler's integral representation

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\ dt, \tag{3.3.12}$$

where $|\arg(1-z)| < \pi$ and Re $c >$ Re $b > 0$.

3.4 Heine's ${}_2\phi_1$ transformation and summation formulas. By employing Thomae's definition (3.3.1) of a q-integral to rewrite the integral in (3.3.11) as a series, we obtain Heine's [1847], [1878] ${}_2\phi_1$ transformation formula

$${}_2\phi_1(q^a, q^b; q^c; q, z) = \frac{(q^b, q^az; q)_\infty}{(q^c, z; q)_\infty}\ {}_2\phi_1(q^{c-b}, z; q^az; q, q^b), \tag{3.4.1}$$

or, equivalently, on replacing q^a, q^b, q^c by a, b, c, respectively,

$${}_2\phi_1(a, b; c; q, z) = \frac{(b, az; q)_\infty}{(c, z; q)_\infty}\ {}_2\phi_1(c/b, z; az; q, b), \tag{3.4.2}$$

provided $|z| < 1$ and $|b| < 1$. Thomae [1869] had derived his integral representation (3.3.11) by rewriting Heine's formula (3.4.2) in the form (3.3.11), the reverse of our derivation of (3.4.2). Formula (3.4.2) is derived in BHS by using the q-binomial theorem and series manipulations of double series.

One particularly attractive feature of (3.4.2) is that it can be iterated to produce the following chain of transformation formulas

$${}_2\phi_1(a, b; c; q, z) = \frac{(b, az; q)_\infty}{(c, z; q)_\infty}\ {}_2\phi_1(c/b, z; az; q, b) \tag{3.4.3}$$

$$= \frac{(c/b, bz; q)_\infty}{(c, z; q)_\infty}\ {}_2\phi_1(abz/c, b; bz; q, c/b) \tag{3.4.4}$$

$$= \frac{(abz/c; q)_\infty}{(z; q)_\infty}\ {}_2\phi_1(c/a, c/b; c; q, abz/c), \tag{3.4.5}$$

provided the series converge. In particular, (3.4.5) and the left side of (3.4.3) yield Heine's transformation formula

$${}_2\phi_1(a, b; c; q, z) = \frac{(abz/c; q)_\infty}{(z; q)_\infty}\ {}_2\phi_1(c/a, c/b; c; q, abz/c) \tag{3.4.6}$$

when $|z| < 1$ and $|abz/c| < 1$, which is a q-analogue of Euler's transformation formula for ${}_2F_1$ series

$$ {}_2F_1(a, b; c; z) = (1-z)^{c-a-b}\ {}_2F_1(c-a, c-b; c; z), \qquad |z| < 1. \tag{3.4.7} $$

Formulas (3.4.2) – (3.4.6) and (3.3.11) can be used to analytically continue ${}_2\phi_1(a, b; c; q, z)$ functions to regions outside the unit disk $|z| < 1$.

Another important application of (3.4.2) is that in the case $z = c/ab$ it reduces to

$$ {}_2\phi_1(a, b; c; q, c/ab) = \frac{(b, c/b; q)_\infty}{(c, c/ab; q)_\infty}\ {}_1\phi_0(c/ab; —; q, b), \qquad |c/ab| < 1,\ |b| < 1, \tag{3.4.8} $$

where the ${}_1\phi_0$ can be summed via the q-binomial theorem to yield Heine's [1847] summation formula for a ${}_2\phi_1$ series

$$ {}_2\phi_1(a, b; c; q, c/ab) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \tag{3.4.9} $$

which, by analytic continuation, holds when $|c/ab| < 1$. This formula is Heine's [1847] q-analogue of Gauss' [1813] famous summation formula

$$ F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}\ (c-a-b) > 0. \tag{3.4.10} $$

In §1.6 of BHS it is shown that the Jacobi triple product identity (2.2.1) can be easily derived from Heine's summation formula (3.4.9).

For the terminating case when $b = q^{-n}$, (3.4.9) reduces to

$$ {}_2\phi_1(a, q^{-n}; c; q, cq^n/a) = \frac{(c/a; q)_n}{(c; q)_n}, \tag{3.4.11} $$

which, by using the inversion identity (1.3.10) or by changing the order of summation, is equivalent to

$$ {}_2\phi_1(a, q^{-n}; c; q, q) = \frac{(c/a; q)_n}{(c; q)_n}\, a^n. \tag{3.4.12} $$

Formulas (3.4.11) and (3.4.12) are equivalent to (1.3.3), and they are q-analogues of the Chu–Vandermonde summation formula

$$ F(a, -n; c; 1) = \frac{(c-a)_n}{(c)_n}, \quad n = 0, 1, \ldots\ . \tag{3.4.13} $$

In §1.5 of BHS it is shown that (3.4.11) and (3.4.12) can be used to derive Sears' [1951b] transformation formula

$$ \begin{aligned} &{}_2\phi_1(q^{-n}, b; c; q, z) \\ &= \frac{(c/b; q)_n}{(c; q)_n}\left(\frac{bz}{q}\right)^n\ {}_3\phi_2(q^{-n}, q/z, c^{-1}q^{1-n}; bc^{-1}q^{1-n}, 0; q, q) \end{aligned} \tag{3.4.14} $$

and Jackson's [1910a] transformation formula

$$
\begin{aligned}
{}_2\phi_1(a,b;c;q,z) &= \frac{(az;q)_\infty}{(z;q)_\infty} \sum_{k=0}^{\infty} \frac{(a,c/b;q)_k}{(q,c,az;q)_k} (-bz)^k q^{\binom{k}{2}} \\
&= \frac{(az;q)_\infty}{(z;q)_\infty}\, {}_2\phi_2(a,c/b;c,az;q,bz), \qquad (3.4.15)
\end{aligned}
$$

which is a q-analogue of the Pfaff-Kummer transformation formula

$$
{}_2F_1(a,b;c;z) = (1-z)^{-a}\, {}_2F_1(a,c-b;c;z/(z-1)). \qquad (3.4.16)
$$

3.5 q-analogue of the Pfaff–Saalschütz summation formula. Observe that, since

$$
\frac{(abz/c;q)_\infty}{(z;q)_\infty} = \sum_{k=0}^{\infty} \frac{(ab/c;q)_k}{(q;q)_k}\, z^k
$$

by the q-binomial theorem, the right side of (3.4.6) equals

$$
\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(ab/c;q)_k (c/a,c/b;q)_m}{(q;q)_k (q,c;q)_m} \left(\frac{ab}{c}\right)^m z^{k+m},
$$

and hence, by equating the coefficients of z^n on both sides of (3.4.6),

$$
\sum_{j=0}^{n} \frac{(q^{-n},c/a,c/b;q)_j}{(q,c,cq^{1-n}/ab;q)_j}\, q^j = \frac{(a,b;q)_n}{(c,ab/c;q)_n}.
$$

After replacing a,b by $c/a, c/b$, respectively, this yields Jackson's [1910a] summation formula for a terminating ${}_3\phi_2$ series

$$
{}_3\phi_2(a,b,q^{-n};c,abc^{-1}q^{1-n};q,q) = \frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n}, \quad n=0,1,\ldots\;. \qquad (3.5.1)
$$

By replacing a,b,c in (3.5.1) by q^a, q^b, q^c, respectively, and letting $q \to 1$ one obtains the sum of a terminating ${}_3\phi_2$ series

$$
{}_3F_2(a,b,-n;c,1+a+b-c-n;1) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}, \quad n=0,1,\ldots, \qquad (3.5.2)
$$

which was discovered by Pfaff [1797] and rediscovered by Saalschütz [1890]. Since (3.5.2) is usually called the *Saalschütz formula* or the *Pfaff-Saalschütz formula*, Jackson's formula (3.5.1) is usually called the *q-Saalschütz* or the *q-Pfaff-Saalschütz formula*. It should be noted that letting $a \to \infty$ in (3.5.1) gives (3.4.11), while letting $a \to 0$ gives (3.4.12).

The Pfaff-Saalschütz ${}_3F_2(1)$ series is said to be a *balanced* series (or *Saalschützian*) because $z=1$ and the sum of its denominator parameters equals the sum of its numerator parameters plus 1. Analogously, the q-Pfaff-Saalschütz ${}_3\phi_2(q)$ series is also called *balanced* because $z=q$ and the product of its denominator parameters equals the product of its numerator parameters times q. In general, as defined in BHS, an ${}_{r+1}F_r$ series is called *k-balanced* if $b_1+b_2+\cdots+b_r=k+a_1+a_2+\cdots+a_{r+1}$ and $z=1$; a 1-balanced series is called *balanced* (or *Saalschützian*). Analogously, an ${}_{r+1}\phi_r$ series is called *k-balanced* if $b_1b_2\cdots b_r = q^k a_1a_2\cdots a_{r+1}$ and $z=q$, and a 1-balanced series is called *balanced* (or *Saalschützian*). In general an ${}_{r+1}F_r(z)$ series is called *balanced* if $b_1+b_2+\cdots+b_r=1+a_1+a_2+\cdots+a_{r+1}$ and $z=1$, and an ${}_{r+1}\phi_r(z)$ series is called *balanced* if $b_1b_2\cdots b_r=qa_1a_2\cdots a_{r+1}$ and $z=q$. We will encounter several balanced series in §4.

Exercises 3

3.1 Verify the convergence conditions stated for ${}_rF_s$, ${}_r\phi_s$, and ${}_r\psi_s$ series in §3.1.

3.2 Use (1.3.10) to derive the (formal) inversion formula

$$ {}_r\phi_s\left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,z\right] = \sum_{n=0}^{\infty} \frac{(a_1^{-1},\ldots,a_r^{-1};q^{-1})_n}{(q^{-1},b_1^{-1},\ldots,b_s^{-1};q^{-1})_n} \left(\frac{a_1\cdots a_r z}{b_1\cdots b_s q}\right)^n. $$

3.3 Show that if z is replaced by z/a_r in the series (3.1.6), then on letting $a_r\to\infty$ one obtains a series that is of the form (3.1.6) with r replaced by $r-1$. Show that this is not the case if the ${}_r\phi_s$ series in (3.1.6) is defined without the factors $[(-1)^n q^{\binom{n}{2}}]^{1+s-r}$.

3.4 Prove the limit relation (3.2.4) and the functional equation (3.2.5).

3.5 Derive the transformation formulas (3.4.14) and (3.4.15). Show that (3.4.15) is a q-analogue of (3.4.16).

3.6 Derive the generating function

$$ \sum_{n=0}^{\infty} \frac{H_n(x|q)}{(q;q)_n}\, t^n = \frac{1}{(te^{i\theta},te^{-i\theta};q)_\infty}, \quad |t|<1, $$

for the *continuous q-Hermite polynomials* defined by

$$ H_n(x|q) = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}\, e^{i(n-2k)\theta}, $$

where $x=\cos\theta$. See Rogers [1894] and Askey and Ismail [1983].

3.7 Derive the generating function

$$ \sum_{n=0}^{\infty} C_n(x;\beta|q)\, t^n = \frac{(\beta te^{i\theta},\beta te^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}, \quad |t|<1, $$

for the *continuous q-ultraspherical polynomials* defined by

$$C_n(x;\beta|q) = \sum_{k=0}^{n} \frac{(\beta;q)_k(\beta;q)_{n-k}}{(q;q)_k(q;q)_{n-k}}\, e^{i(n-2k)\theta},$$

where $x = \cos\theta$. See Rogers [1895] and Askey and Ismail [1983].

3.8 The *little q-Jacobi polynomials* are defined in Andrews and Askey [1977] by

$$p_n(x;a,b;q) = {}_2\phi_1(q^{-n}, abq^{n+1}; aq; q, qx).$$

Prove that they satisfy the orthogonality relation

$$\sum_{j=0}^{\infty} \frac{(bq;q)_j}{(q;q)_j}(aq)^j\, p_n(q^j;a,b;q)\, p_m(q^j;a,b;q)$$
$$= \begin{cases} 0, & \text{if } m \neq n, \\ \dfrac{(q,bq;q)_n(1-abq)(aq)^n}{(aq,abq;q)_n(1-abq^{2n+1})}\dfrac{(abq^2;q)_\infty}{(aq;q)_\infty}, & \text{if } m = n. \end{cases}$$

3.9 Show that the little q-Jacobi polynomials defined in Ex. 3.8 satisfy the *connection coefficient formula*

$$p_n(x;c,d;q) = \sum_{k=0}^{n} a_{k,n}\; p_k(x;a,b;q)$$

with

$$a_{k,n} = \frac{(q^{-n}, aq, cdq^{n+1};q)_k}{(q, cq, abq^{k+1};q)_k}\; {}_3\phi_2\left[\begin{matrix} q^{k-n}, cdq^{n+k+1}, aq^{k+1} \\ cq^{k+1}, abq^{2k+2} \end{matrix}; q, q\right].$$

4 Summation, transformation, and expansion formulas, integral representations, and applications

4.1 Finite differences, bibasic and very-well-poised series. Let $n, m = 0, \pm1, \pm2, \ldots$. Another method to derive summation formulas is to use the fact that if a finite difference operator Δ is defined for any sequence $\{u_k\}$ (of real or complex numbers), by $\Delta u_k = u_k - u_{k-1}$, then

$$\sum_{k=m}^{n} \Delta u_k = u_n - u_{m-1}, \tag{4.1.1}$$

where we employed the convention of defining

$$\sum_{k=m}^{n} a_k = \begin{cases} a_m + a_{m+1} + \cdots + a_n, & m \leq n, \\ 0, & m = n+1, \\ -(a_{n+1} + a_{n+2} + \cdots + a_{m-1}), & m \geq n+2, \end{cases} \tag{4.1.2}$$

for any sequence $\{a_k\}$. An excellent way to motivate this definition is to (formally) let

$$u_n = \sum_{k=-\infty}^{n} a_k$$

and then observe that, by cancelling the a_k's that appear in both series in the difference

$$u_n - u_{m-1} = \sum_{k=-\infty}^{n} a_k - \sum_{k=-\infty}^{m} a_k,$$

we obtain (4.1.1) with $\sum_{k=m}^{n} a_k$ defined by (4.1.2).

If, as in Gasper [1989a], we let

$$s_k = \frac{(ap, bp; p)_k (cq, aq/bc; q)_k}{(q, aq/b; q)_k (ap/c, bcp; p)_k}, \tag{4.1.3}$$

where q and p are independent bases, we obtain the factorization

$$\begin{aligned}
\Delta s_k &= s_k - s_{k-1} \\
&= \frac{(ap, bp; p)_{k-1} (cq, aq/bc; q)_{k-1}}{(q, aq/b; q)_k (ap/c, bcp; p)_k} \\
&\quad \cdot \big\{(1 - ap^k)(1 - bp^k)(1 - cq^k)(1 - aq^k/bc) \\
&\qquad -(1 - q^k)(1 - aq^k/b)(1 - ap^k/c)(1 - bcp^k)\big\} \\
&= \frac{(1 - ap^k q^k)(1 - bp^k/q^k)}{(1 - a)(1 - b)} \frac{(a, b; p)_k (c, a/bc; q)_k}{(q, aq/b; q)_k (ap/c, bcp; p)_k} q^k,
\end{aligned} \tag{4.1.4}$$

which is equivalent to the easily checked factorization

$$\begin{aligned}
&(1 - a)(1 - b)(1 - c)(1 - ad^2/bc) - (1 - d)(1 - ad/b)(1 - ad/c)(1 - bc/d) \\
&= (1 - ad)(1 - b/d)(1 - ad/bc)(d - c).
\end{aligned} \tag{4.1.5}$$

Since $s_k = 0$ when $k \leq -1$, (4.1.4) gives the *indefinite bibasic summation formula*

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{(1 - ap^k q^k)(1 - bp^k q^{-k})}{(1 - a)(1 - b)} \frac{(a, b; p)_k (c, a/bc; q)_k}{(q, aq/b; q)_k (ap/c, bcp; p)_k} q^k \\
&= \frac{(ap, bp; p)_n (cq, aq/bc; q)_n}{(q, aq/b; q)_n (ap/c, bcp; p)_n}
\end{aligned} \tag{4.1.6}$$

for $n = 0, 1, \ldots$. In particular, since $\left(q^{1-n}; q\right)_n = 0$ unless $n = 0$, when $p = q$ and $c = q^{-n}$, $n = 0, 1, \ldots$, (4.1.6) reduces to the ${}_6\phi_5$ summation formula

$${}_6\phi_5 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, aq^n/b, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, bq^{1-n}, aq^{n+1} \end{matrix} ; q, q \right] = \delta_{n,0} \,, \tag{4.1.7}$$

where

$$\delta_{m,n} = \begin{cases} 1, & m = n, \\ 0, & m \neq n, \end{cases}$$

is the Kronecker delta function. In §4.2 we will need to use the ${}_4\phi_3$ summation formula

$${}_4\phi_3 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq^{n+1} \end{matrix} ; q, q^n \right] = \delta_{n,0} \tag{4.1.8}$$

when $n = 0, 1, \ldots$, which is the $b \to 0$ and the $b \to \infty$ limit cases of (4.1.7). The above derivation of (4.1.8) is substantially simpler than that in §2.3 of BHS, which used the q-Pfaff–Saalschütz formula (3.5.1) and the Bailey [1941] and Daum [1942] summation formula

$${}_2\phi_1(a, b; aq/b; q, -q/b) = \frac{(-q; q)_\infty (aq, aq^2/b^2; q^2)_\infty}{(aq/b, -q/b; q)_\infty}. \tag{4.1.9}$$

Formula (4.1.9) is a q-analogue of Kummer's formula

$${}_2F_1(a, b; 1 + a - b; -1) = \frac{\Gamma(1 + a - b)\Gamma(1 + \frac{1}{2}a)}{\Gamma(1 + a)\Gamma(1 + \frac{1}{2}a - b)}. \tag{4.1.10}$$

The series in (4.1.7) and (4.1.8) are the form

$${}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix} ; q, z \right] \tag{4.1.11}$$

in which the the parameters satisfy the relations

$$qa_1 = a_2 b_1 = a_3 b_2 = \cdots = a_{r+1} b_r. \tag{4.1.12}$$

Such series are called *well-poised* series, and they are called *very-well-poised* if, in addition $a_2 = qa_1^{\frac{1}{2}}$, $a_3 = -qa_1^{\frac{1}{2}}$, which is the case for (4.1.7) and (4.1.8). Since very-well-poised series appear quite offen, to simplify some of the displays containing very-well-poised ${}_{r+1}\phi_r$ series, we will frequently replace

$${}_{r+1}\phi_r \left[\begin{matrix} a_1, qa_1^{\frac{1}{2}}, -qa_1^{\frac{1}{2}}, a_4, \ldots, a_{r+1} \\ a_1^{\frac{1}{2}}, -a_1^{\frac{1}{2}}, qa_1/a_4, \ldots, qa_1/a_{r+1} \end{matrix} ; q, z \right] \tag{4.1.13}$$

with the more compact notation

$${}_{r+1}W_r \left(a_1; a_4, a_5, \ldots, a_{r+1}; q, z \right). \tag{4.1.14}$$

Observe that the expression on the right side of (4.1.6) is *balanced* and *well-poised* since

$$(ap)(bp)(cq)(aq/bc) = q(aq/b)(ap/c)(bcp)$$

and

$$(ap)q = (bp)(aq/b) = (cq)(ap/c) = (aq/bc)(bcp).$$

Also, the part of the series on the left side of (4.1.6) containing the q-shifted factorials is *split-poised* in the sense that $aq = b(aq/b)$ and $c(ap/c) = (a/bc)(bcp) = ap$. It is the first property and previous special cases, such as Wm. Gosper's $b \to 0$ limit case of (4.1.6)

$$\sum_{k=0}^{n} \frac{1-ap^kq^k}{1-a} \frac{(a;p)_k(c;q)_k}{(q;q)_k(ap/c;p)_k} c^{-k} = \frac{(ap;p)_n(cq;q)_n}{(q;q)_n(ap/c;p)_n} c^{-n}, \tag{4.1.15}$$

that led the author to consider the sequence s_k in (4.1.3) and obtain the factorization (4.1.4). See the discussion in Gasper [1989a, p. 259] .

This and the importance of the applications of (4.1.6) considered in Gasper [1989a] led Gasper and Rahman [1990b] to generalize (4.1.3) to

$$s_k = \frac{(ap,bp;p)_k(cq,ad^2q/bc;q)_k}{(dq,adq/b;q)_k(adp/c,bcp/d;p)_k} \tag{4.1.16}$$

for $k = 0, \pm 1, \pm 2, \ldots$, and observe that

$$\begin{aligned}
\Delta t_k &= t_k - t_{k-1} \\
&= \frac{(ap,bp;p)_{k-1}(cq,ad^2q/bc;q)_{k-1}}{(dq,adq/b;q)_k(adp/c,bcp/d;p)_k} \\
&\quad \cdot \big\{(1-ap^k)(1-bp^k)(1-cq^k)(1-ad^2q^k/bc) \\
&\qquad -(1-dq^k)(1-adq^k/b)(1-adp^k/c)(1-bcp^k/d)\big\} \\
&= \frac{d(1-c/d)(1-ad/bc)(1-adp^kq^k)(1-bp^k/dq^k)}{(1-a)(1-b)(1-c)(1-ad^2/bc)} \\
&\quad \cdot \frac{(a,b;p)_k(c,ad^2/bc;q)_kq^k}{(dq,adq/b;q)_k(adp/c,bcp/d;p)_k}
\end{aligned} \tag{4.1.17}$$

by (4.1.5), which, in view of (4.1.1), gives the following generalization of (4.1.6)

$$\begin{aligned}
&\sum_{k=-m}^{n} \frac{(1-adp^kq^k)(1-bp^k/dq^k)}{(1-ad)(1-b/d)} \frac{(a,b;p)_k(c,ad^2/bc;q)_k}{(dq,adq/b;q)_k(adp/c,bcp/d;p)_k} q^k \\
&= \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)} \\
&\quad \cdot \left\{ \frac{(ap,bp;p)_n(cq,ad^2q/bc;q)_n}{(dq,adq/b;q)_n(adp/c,bcp/d;p)_n} - \frac{(c/ad,d/bc;p)_{m+1}(1/d,b/ad;q)_{m+1}}{(1/c,bc/ad^2;q)_{m+1}(1/a,1/b;p)_{m+1}} \right\},
\end{aligned} \tag{4.1.18}$$

where we used $-m$ as the lower limit of summation instead of m. This has the advantage that, by letting $m \to \infty$ in (4.1.18), we immediately see that if $|p| < 1$ and $|q| < 1$, then

(4.1.18) tends to the bibasic summation formula

$$\sum_{k=-\infty}^{n} \frac{(1-adp^kq^k)(1-bp^k/dq^k)}{(1-ad)(1-b/d)} \frac{(a,b;p)_k(c,ad^2/bc;q)_k}{(dq,adq/b;q)_k(adp/c,bcp/d;p)_k} q^k$$
$$= \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}$$
$$\cdot \left\{ \frac{(ap,bp;p)_n(cq,ad^2q/bc;q)_n}{(dq,adq/b;q)_n(adp/c,bcp/d;p)_n} - \frac{(c/ad,d/bc;p)_\infty(1/d,b/ad;q)_\infty}{(1/c,bc/ad^2;q)_\infty(1/a,1/b;p)_\infty} \right\} \tag{4.1.19}$$

for integer n, and with n replaced by ∞ . Some generalizations of (4.1.18) are given in Chu [1993] and Bhatnagar and Milne [1995].

Returning to (4.1.6), notice that when $c = q^{-n}$ it reduces to

$$\sum_{k=0}^{n} \frac{(1-ap^kq^k)(1-bp^kq^{-k})}{(1-a)(1-b)} \frac{(a,b;p)_k(q^{-n},aq^n/b;q)_k}{(q,aq/b;q)_k(apq^n,bpq^{-n};p)_k} q^k = \delta_{n,0} \tag{4.1.20}$$

when $n = 0, 1, \ldots$. This formula was derived independently by Bressoud [1988], Gasper [1989a], and Krattenthaler [1995]. If we replace n, a, b and k by $n-m, ap^mq^m, bp^mq^{-m}$ and $j-m$, respectively, (4.1.20) gives the rather general orthogonality relation

$$\sum_{j=m}^{n} a_{nj} b_{jm} = \delta_{n,m} \tag{4.1.21}$$

with

$$a_{nj} = \frac{(-1)^{n+j}(1-ap^jq^j)(1-bp^jq^{-j})(apq^n,bpq^{-n};p)_{n-1}}{(q;q)_{n-j}(apq^n,bpq^{-n};p)_j(bq^{1-2n}/a;q)_{n-j}},$$

$$b_{jm} = \frac{(ap^mq^m,bp^mq^{-m};p)_{j-m}}{(q,aq^{1+2m}/b;q)_{j-m}} \left(-\frac{a}{b}q^{1+2m}\right)^{j-m} q^{2\binom{j-m}{2}}.$$

Hence, the triangular matrix $A = (a_{nj})$ is inverse to the triangular matrix $B = (b_{jm})$, where j, m, n are nonnegative integers. Since inverse matrices commute, a calculation of the jk^{th} term of BA leads to the orthogonality relation

$$\sum_{n=0}^{j-k} \frac{(1-ap^kq^k)(1-bp^kq^{-k})(ap^{k+1}q^{k+n},bp^{k+1}q^{-k-n};p)_{j-k-1}}{(q;q)_n(q;q)_{j-k-n}(aq^{2k+n}/b;q)_{j-k-1}}$$
$$\cdot \left(1-\frac{a}{b}q^{2k+2n}\right)(-1)^n q^{n(j-k-1)+\binom{j-k-n}{2}} = \delta_{j,k} \ . \tag{4.1.22}$$

By replacing j, n, a, b by $n+k, k, ap^{-k-1}q^{-k}, bp^{-k-1}q^k$, respectively, we see that (4.1.22) is equivalent to the bibasic summation formula

$$\left(1-\frac{a}{p}\right)\left(1-\frac{b}{p}\right) \sum_{k=0}^{n} \frac{(aq^k,bq^{-k};p)_{n-1}(1-aq^{2k}/b)}{(q;q)_k(q;q)_{n-k}(aq^k/b;q)_{n+1}} (-1)^k q^{\binom{k}{2}} = \delta_{n,0} \tag{4.1.23}$$

when $n = 0, 1, \ldots$. Al-Salam and Verma [1984] derived the $b \to 0$ limit case of (4.1.23)

$$\left(1 - \frac{a}{p}\right) \sum_{k=0}^{n} \frac{(aq^k; p)_{n-1}}{(q;q)_k (q;q)_{n-k}} (-1)^k q^{\binom{n-k}{2}} = \delta_{n,0} \tag{4.1.24}$$

when $n = 0, 1, \ldots$, by using the fact that the n^{th} q-difference of a polynomial in q of degree less than n is equal to zero. For applications to q-analogues of Lagrange inversion, general expansion formulas, and to the positivity of certain sums (kernels), see Gessel and Stanton [1986], Exercises 4.2 – 4.4, and Gasper [1989a, 1989b].

4.2 Expansion, summation and transformation formulas. Let k and n be nonnegative integers. In order to derive rather general summation and transformation formulas for ${}_{r+1}\phi_r$ series, it is efficient to first derive a rather general expansion formula that follows by using, as in §2.2 of BHS, the q-Pfaff–Saalschütz formula (3.5.1) in the form

$$\begin{aligned} &{}_3\phi_2\left(q^{-k}, aq^k, aq/bc; aq/b, aq/c; q, q\right) \\ &= \frac{(c, q^{1-k}/b; q)_k}{(aq/b, cq^{-k}/a; q)_k} = \frac{(b, c; q)_k}{(aq/b, aq/c; q)_k} \left(\frac{aq}{bc}\right)^k \end{aligned} \tag{4.2.1}$$

to obtain, for any sequence $\{v_k\}$,

$$\begin{aligned} &\sum_{k=0}^{n} \frac{(b, c, q^{-n}; q)_k}{(q, aq/b, aq/c; q)_k} v_k \\ &= \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{(aq/bc, aq^k, q^{-k}; q)_j (q^{-n}; q)_k}{(q, aq/b, aq/c; q)_j (q;q)_k} q^j \left(\frac{bc}{aq}\right)^k v_k \\ &= \sum_{j=0}^{n} \sum_{i=0}^{n-j} \frac{\left(aq/bc, aq^{i+j}, q^{-i-j}; q\right)_j (q^{-n}; q)_{i+j}}{(q, aq/b, aq/c; q)_j (q;q)_{i+j}} q^j \left(\frac{bc}{aq}\right)^{i+j} v_{i+j} \\ &= \sum_{j=0}^{n} \frac{\left(aq/bc, aq^j, q^{-n}; q\right)_j}{(q, aq/b, aq/c; q)_j} (-1)^j q^{-\binom{j}{2}} \\ &\quad \cdot \sum_{i=0}^{n-j} \frac{\left(q^{j-n}, aq^{2j}; q\right)_i}{(q, aq^j; q)_i} q^{-ij} \left(\frac{bc}{aq}\right)^{i+j} v_{i+j}. \end{aligned} \tag{4.2.2}$$

Setting

$$v_k = \frac{(a, a_1, \ldots, a_r; q)_k}{(b_1, b_2, \ldots, b_{r+1}; q)_k} z^k \tag{4.2.3}$$

in (4.2.2) yields the desired expansion formula

$$\begin{aligned} &{}_{r+4}\phi_{r+3}\left[\begin{matrix} a, b, c, a_1, a_2, \ldots, a_r, q^{-n} \\ aq/b, aq/c, b_1, b_2, \ldots, b_r, b_{r+1} \end{matrix}; q, z\right] \\ &= \sum_{j=0}^{n} \frac{(aq/bc, a_1, a_2, \ldots, a_r, q^{-n}; q)_j}{(q, aq/b, aq/c, b_1, \ldots, b_r, b_{r+1}; q)_j} \left(-\frac{bcz}{aq}\right)^j q^{-\binom{j}{2}} (a;q)_{2j} \\ &\quad \cdot {}_{r+2}\phi_{r+1}\left[\begin{matrix} aq^{2j}, a_1 q^j, a_2 q^j, \ldots, a_r q^j, q^{j-n} \\ b_1 q^j, b_2 q^j, \ldots, b_r q^j, b_{r+1} q^j \end{matrix}; q, \frac{bcz}{aq^{j+1}}\right]. \end{aligned} \tag{4.2.4}$$

This formula enables us to reduce the problem of deriving a transformation formula for a ${}_{r+4}\phi_{r+3}$ series in terms of a single series to that of summing the ${}_{r+2}\phi_{r+1}$ series in (4.2.4) for some values of the parameters.

By setting $a_1 = qa^{\frac{1}{2}}$, $a_2 = -qa^{\frac{1}{2}}, b_1 = a^{\frac{1}{2}}$, $b_2 = -a^{\frac{1}{2}}, b_{r+1} = aq^{n+1}$ and $a_k = b_k$, for $k = 3, 4, \ldots, r$ in (4.2.4), we get

$$
\begin{aligned}
&{}_6\phi_5 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq^{n+1} \end{matrix} ; q, z \right] \\
&= \sum_{j=0}^{n} \frac{(aq/bc, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, q^{-n}; q)_j (a;q)_{2j}}{(q, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq^{n+1}; q)_j} \left(-\frac{bcz}{aq} \right)^j q^{-\binom{j}{2}} \\
&\quad \cdot {}_4\phi_3 \left[\begin{matrix} aq^{2j}, q^{j+1}a^{\frac{1}{2}}, -q^{j+1}a^{\frac{1}{2}}, q^{j-n} \\ q^j a^{\frac{1}{2}}, -q^j a^{\frac{1}{2}}, aq^{j+n+1} \end{matrix} ; q, \frac{bcz}{aq^{j+1}} \right]. \qquad (4.2.5)
\end{aligned}
$$

If we now set $z = aq^{n+1}/bc$, then we can use (4.1.8) to sum the above ${}_4\phi_3$ series; thus deriving the summation formula

$$
\begin{aligned}
&{}_6\phi_5 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq^{n+1} \end{matrix} ; q, \frac{aq^{n+1}}{bc} \right] \\
&= \frac{(aq/bc, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, q^{-n}; q)_n \ (a;q)_{2n}}{(q, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq^{n+1}; q)_n} (-1)^n q^{n(n+1)/2} \\
&= \frac{(aq, aq/bc; q)_n}{(aq/b, aq/c; q)_n}, \qquad (4.2.6)
\end{aligned}
$$

which sums a terminating very-well-poised ${}_6\phi_5$ series.

Similarly, from (4.2.4), we obtain

$$
\begin{aligned}
&{}_8\phi_7 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e, aq^{n+1} \end{matrix} ; q, \frac{a^2q^{2+n}}{bcde} \right] \\
&= \sum_{j=0}^{n} \frac{(aq/bc, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, d, e, q^{-n}; q)_j \ (a;q)_{2j}}{(q, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e, aq^{n+1}; q)_j} \left(-\frac{aq^{n+1}}{de} \right)^j q^{-\binom{j}{2}} \\
&\quad \cdot {}_6\phi_5 \left[\begin{matrix} aq^{2j}, q^{j+1}a^{\frac{1}{2}}, -q^{j+1}a^{\frac{1}{2}}, dq^j, eq^j, q^{j-n} \\ q^j a^{\frac{1}{2}}, -q^j a^{\frac{1}{2}}, aq^{j+1}/d, aq^{j+1}/e, aq^{j+n+1} \end{matrix} ; q, \frac{aq^{1+n-j}}{de} \right] \qquad (4.2.7)
\end{aligned}
$$

in which we can employ (4.2.6) to sum the ${}_6\phi_5$ series and derive Watson's [1929] transformation formula for a terminating very-well-poised ${}_8\phi_7$ series as a multiple of a terminating balanced ${}_4\phi_3$ series:

$$
\begin{aligned}
&{}_8\phi_7 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e, aq^{n+1} \end{matrix} ; q, \frac{a^2q^{2+n}}{bcde} \right] \\
&= \frac{(aq, aq/de; q)_n}{(aq/d, aq/e; q)_n} \ {}_4\phi_3 \left[\begin{matrix} q^{-n}, d, e, aq/bc \\ aq/b, aq/c, deq^{-n}/a \end{matrix} ; q, q \right]. \qquad (4.2.8)
\end{aligned}
$$

If $a^2q^{n+1} = bcde$, the ${}_4\phi_3$ series in (4.2.8) becomes a terminating balanced ${}_3\phi_2$ series, which can be summed by the q-Pfaff–Saalschütz formula to derive Jackson's [1921] summation formula for a terminating very-well-poised ${}_8\phi_7$ series

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b, c, d, e, q^{-n} \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d, aq/e, aq^{n+1}\end{matrix}; q, q\right] \\
&= \frac{(aq, aq/bc, aq/bd, aq/cd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n}, \qquad (4.2.9)
\end{aligned}
$$

where $a^2q^{n+1} = bcde$. This formula is a q-analogue of Dougall's [1907] ${}_7F_6$ summation formula

$$
\begin{aligned}
&{}_7F_6\left[\begin{matrix} a,\ 1+\frac12 a,\ b,\ c,\ d,\ e,\ -n \\ \frac12 a, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a+n\end{matrix}; 1\right] \\
&= \frac{(1+a)_n(1+a-b-c)_n(1+a-b-d)_n(1+a-c-d)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(1+a-b-c-d)_n}, \qquad (4.2.10)
\end{aligned}
$$

where the series is 2-balanced, i.e, $1 + 2a + n = b + c + d + e$. The reason that this series is 2-balanced instead of balanced is that the *appropriate* q-analogue of the term $(1+\frac12 a)_k/(\frac12 a)_k = (a+2k)/a$ in the ${}_7F_6$ series is not $(qa^{\frac12}; q)_k/(a^{\frac12}; q)_k = (1-a^{\frac12}q^k)/(1-a^{\frac12})$ but $(qa^{\frac12}, -qa^{\frac12}; q)_k/(a^{\frac12}, -a^{\frac12}; q)_k = (1 - aq^{2k})/(1 - a)$, and this introduces an additional q-factor in the ratio of the products of the numerator and denominator parameters. Krattenthaler [1995] removed some of the mystery in the factorization (4.1.5) by observing that it is equivalent to the $n = 1$ case of Jackson's ${}_8\phi_7$ sum (4.2.9).

Watson [1929] showed that the $b, c, d, e \to \infty$ limit case of his transformation formula (4.2.9) and Jacobi's triple product identity can be used to give a relatively simple proof of the famous Rogers-Ramanujan identities:

$$
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{(q^2, q^3, q^5; q^5)_\infty}{(q;q)_\infty}, \qquad (4.2.11)
$$

$$
\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{(q, q^4, q^5; q^5)_\infty}{(q;q)_\infty}, \qquad (4.2.12)
$$

where $|q| < 1$. See Hardy [1940] for an early history of these identities.

An important limit case of Jackson's summation formula (4.2.9) is the sum of a nonterminating very-well-poised ${}_6\phi_5$ series

$$
\begin{aligned}
&{}_6\phi_5\left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b, c, d \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d\end{matrix}; q, \frac{aq}{bcd}\right] \\
&= \frac{(aq, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty} \qquad (4.2.13)
\end{aligned}
$$

with $|aq/bcd| < 1$, which follows by letting $n \to \infty$ in (4.2.9). When $d = a^{\frac{1}{2}}$ this formula reduces to

$$
\begin{aligned}
&{}_4\phi_3 \left[\begin{matrix} a, -qa^{\frac{1}{2}}, b, c \\ -a^{\frac{1}{2}}, aq/b, aq/c \end{matrix} ; q, \frac{qa^{\frac{1}{2}}}{bc} \right] \\
&= \frac{(aq, aq/bc, qa^{\frac{1}{2}}/b, qa^{\frac{1}{2}}/c; q)_\infty}{(aq/b, aq/c, qa^{\frac{1}{2}}, qa^{\frac{1}{2}}/bc; q)_\infty},
\end{aligned} \tag{4.2.14}
$$

where $|qa^{\frac{1}{2}}/bc| < 1$, which is a q-analogue of Dixon's [1903] formula for the sum of a well-poised ${}_3F_2$ series

$$
\begin{aligned}
&{}_3F_2\left[a, b, c; 1+a-b, 1+a-c\,; 1\right] \\
&= \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)},
\end{aligned} \tag{4.2.15}
$$

where Re $(1+\frac{1}{2}a-b-c) > 0$.

4.3 Bailey's transformation formulas and some integral representations. In this section we conclude the some of the most important transformation formulas for very-well-poised series. By rewriting Jackson's formula (4.2.9) in the form

$$
\begin{aligned}
&{}_8\phi_7 \left[\begin{matrix} \lambda, q\lambda^{\frac{1}{2}}, -q\lambda^{\frac{1}{2}}, \lambda b/a, \lambda c/a, \lambda d/a, aq^m, q^{-m} \\ \lambda^{\frac{1}{2}}, -\lambda^{\frac{1}{2}}, aq/b, aq/c, aq/d, \lambda q^{1-m}/a, \lambda q^{m+1} \end{matrix} ; q, q \right] \\
&= \frac{(b, c, d, \lambda q; q)_m}{(aq/b, aq/c, aq/d, a/\lambda; q)_m}
\end{aligned} \tag{4.3.1}
$$

with $\lambda = qa^2/bcd$ and proceeding as in (4.2.2), one obtains a sum of a terminating very-well-poised ${}_8\phi_7$ series that can be summed with Jackson's formula, yielding Bailey's [1929] transformation formula between two terminating ${}_{10}W_9$ series

$$
\begin{aligned}
&{}_{10}W_9\left(a; b, c, d, e, f, \lambda aq^{n+1}/ef, q^{-n}; q, q\right) \\
&= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q)_n}{(aq/e, aq/f, \lambda q/ef, \lambda q; q)_n} \\
&\quad \cdot {}_{10}W_9\left(\lambda; \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda aq^{n+1}/ef, q^{-n}; q, q\right)
\end{aligned} \tag{4.3.2}
$$

with $\lambda = qa^2/bcd$, where for compactness we employed the ${}_{10}W_9$ notation for very-well-poised series.

Watson's transformation formula (4.2.8) follows from (4.3.2) by letting b, c, or $d \to \infty$. By taking the limit $n \to \infty$ of (4.3.2) we obtain a transformation formula for nonterminating very-well-poised ${}_8\phi_7$ series

$$
\begin{aligned}
&{}_8W_7\left(a; b, c, d, e, f; q, \lambda q/ef\right) \\
&= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q)_\infty}{(aq/e, aq/f, \lambda q, \lambda q/ef; q)_\infty} \\
&\quad \cdot {}_8W_7\left(\lambda; \lambda b/a, \lambda c/a, \lambda d/a, e, f; q, aq/ef\right)
\end{aligned} \tag{4.3.3}
$$

with $\lambda = qa^2/bcd$, where, for convergence, $\max(|aq/ef|,\ |\lambda q/ef) < 1$. Bailey iterated (4.3.2) to obtain

$$
\begin{aligned}
&{}_{10}W_9\left(a; b, c, d, e, f, a^3q^{n+2}/bcdef, q^{-n}; q, q\right) \\
&= \frac{(aq, aq/de, aq/df, aq/ef; q)_n}{(aq/d, aq/e, aq/f, aq/def; q)_n} \\
&\quad \cdot {}_{10}W_9\left(defq^{-n-1}/a; aq/bc, d, e, f, bdefq^{-n-1}/a^2, cdefq^{-n-1}/a^2, q^{-n}; q, q\right).
\end{aligned} \tag{4.3.4}
$$

It is clear that the ${}_{10}W_9$ on the left side of (4.3.4) tends to the ${}_8\phi_7$ series on the left side of (4.3.3) as $n \to \infty$. However, in trying to take the $n \to \infty$ limit of the right side we run into the problem that the terms near both ends of the series on the right side of (4.3.4) are large compared to those in the middle for large n, which prevents us from directly taking the term-by-term limit. Bailey overcame this difficulty by choosing n to be an odd integer $2m+1$, dividing the series on the right into two halves, each containing $m+1$ terms, reversing the order of the second series, and then taking the limit as $m \to \infty$ to derive the transformation formula

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b, c, d, e, f \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d, aq/e, aq/f \end{matrix}; q, \frac{a^2q^2}{bcdef}\right] \\
&= \frac{(aq, aq/de, aq/df, aq/ef; q)_\infty}{(aq/d, aq/e, aq/f, aq/def; q)_\infty}\, {}_4\phi_3\left[\begin{matrix} aq/bc, d, e, f \\ aq/b, aq/c, def/a \end{matrix}; q, q\right] \\
&\quad + \frac{(aq, aq/bc, d, e, f, a^2q^2/bdef,\ a^2q^2/cdef; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, aq/f, a^2q^2/bcdef, def/aq; q)_\infty} \\
&\quad \cdot {}_4\phi_3\left[\begin{matrix} aq/de, aq/df, aq/ef, a^2q^2/bcdef \\ a^2q^2/bdef, a^2q^2/cdef, aq^2/def \end{matrix}; q, q\right],
\end{aligned} \tag{4.3.5}
$$

provided $|a^2q^2/bcdef| < 1$ when the ${}_8\phi_7$ series on the left side does not terminate.

Al-Salam and Verma [1982] observed that (4.3.5) is equivalent to the q-integral formula

$$
\begin{aligned}
&\int_a^b \frac{(qt/a, qt/b, ct, dt; q)_\infty}{(et, ft, gt, ht; q)_\infty}\, d_qt \\
&= b(1-q)\frac{(q, bq/a, a/b, cd/eh, cd/fh, cd/gh, bc, bd; q)_\infty}{(ae, af, ag, be, bf, bg, bh, bcd/h; q)_\infty} \\
&\quad \cdot\ {}_8W_7\left(bcd/hq; be, bf, bg, c/h, d/h; q, ah\right),
\end{aligned} \tag{4.3.6}
$$

where $cd = abefgh$ and $|ah| < 1$. Setting $h = d$ in (4.3.6) and then replacing g by d gives Sears' [1951a] nonterminating extension of the q-Pfaff–Saalschütz formula

$$
\begin{aligned}
{}_3\phi_2\left[\begin{matrix} a, b, c \\ e, f \end{matrix}; q, q\right] &= \frac{(q/e, f/a, f/b, f/c; q)}{(aq/e, bq/e, cq/e, f; q)_\infty} \\
&- \frac{(q/e, a, b, c, qf/e; q)_\infty}{(e/q, aq/e, bq/e, cq/e, f; q)_\infty} \\
&\cdot\ {}_3\phi_2\left[\begin{matrix} aq/e, bq/e, cq/e \\ q^2/e, qf/e \end{matrix}; q, q\right]
\end{aligned} \tag{4.3.7}
$$

with $ef = abcq$ in the equivalent q-integral form

$$\int_a^b \frac{(qt/a, qt/b, ct; q)_\infty}{(dt, et, ft; q)_\infty}\, d_q t$$
$$= b(1-q)\frac{(q, bq/a, a/b, c/d, c/e, c/f; q)_\infty}{(ad, ae, af, bd, be, bf; q)_\infty}, \tag{4.3.8}$$

where $c = abdef$.

Rahman [1984] employed this q-integral to give a rather simple proof of the Askey and Wilson [1985] q-beta integral formula

$$\int_{-1}^1 \frac{h(x; 1, -1, q^{\frac12}, -q^{\frac12})}{h(x; a, b, c, d)} \frac{dx}{\sqrt{1-x^2}}$$
$$= \frac{2\pi(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}, \tag{4.3.9}$$

where

$$h(x; a_1, a_2, \ldots, a_m) \equiv h(x; a_1, a_2, \ldots, a_m; q) = h(x; a_1)h(x; a_2)\cdots h(x; a_m),$$

$$h(x; a) \equiv h(x; a; q) = \prod_{n=0}^\infty (1 - 2axq^n + a^2q^{2n})$$

and

$$\max(|a|, |b|, |c|, |d|, |q|) < 1.$$

This is the integral that Askey and Wilson [1985] used to derived their orthogonality relation for the polynomials

$$p_n(x) \equiv p_n(x; a, b, c, d|q)$$
$$= (ab, ac, ad; q)_n a^{-n}\ {}_4\phi_3\left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q\right], \tag{4.3.10}$$

which are now called the *Askey-Wilson polynomials.*

In §2.11 of BHS the q-integral representation (4.3.6) is applied to derive Bailey's [1936] 3-term transformation formula for ${}_8W_7$ series

$${}_8W_7(a, b, c, d, e, f; q, a^2q^2/bcdef)$$
$$= \frac{(aq, aq/de, aq/df, aq/ef, eq/c, fq/c, b/a, bef/a; q)_\infty}{(aq/d, aq/e, aq/f, aq/def, q/c, efq/c, be/a, bf/a; q)_\infty}$$
$$\cdot\ {}_8W_7(ef/c, aq/bc, aq/cd, ef/a, e, f; q, bd/a)$$
$$+ \frac{b}{a}\frac{(aq, bq/a, bq/c, bq/d, bq/e, bq/f, d, e, f; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, aq/f, bd/a, be/a, bf/a, def/a; q)_\infty}$$
$$\cdot \frac{(aq/bc, bdef/a^2, a^2q/bdef; q)_\infty}{(aq/def, q/c, b^2q/a; q)_\infty}$$
$$\cdot\ {}_8W_7(b^2/a, b, bc/a, bd/a, be/a, bf/a; q, a^2q^2/bcdef), \tag{4.3.11}$$

where $|bd/a| < 1$ and $|a^2q^2/bcdef| < 1$, and it is pointed out that the special case when $qa^2 = bcdef$ gives

$$\begin{aligned}
&{}_8W_7(a,b,c,d,e,f;q,q)\\
&\quad -\frac{b}{a}\frac{(aq,c,d,e,f,bq/a,bq/c,bq/d,bq/e,bq/f;q)_\infty}{(aq/b,aq/c,aq/d,aq/e,aq/f,bc/a,bd/a,be/a,bf/a,b^2q/a;q)_\infty}\\
&\quad \cdot {}_8W_7(b^2/a,b,bc/a,bd/a,be/a,bf/a;q,q)\\
&= \frac{(aq,b/a,aq/cd,aq/ce,aq/cf,aq/de,aq/df,aq/ef;q)_\infty}{(aq/c,aq/d,aq/e,aq/f,bc/a,bd/a,be/a,bf/a;q)_\infty},
\end{aligned} \tag{4.3.12}$$

where $qa^2 = bcdef$. This nonterminating extension of Jackson's summation formula (4.2.9), which is called Bailey's ${}_8\phi_7$ summation formula, can be written in the equivalent q-integral form

$$\begin{aligned}
&\int_a^b \frac{(qt/a,qt/b,t/a^{\frac12},-t/a^{\frac12},qt/c,qt/d,qt/e,qt/f;q)_\infty}{(t,bt/a,qt/a^{\frac12},-qt/a^{\frac12},ct/a,dt/a,et/a,ft/a;q)_\infty}\,d_qt\\
&= \frac{b(1-q)(q,a/b,bq/a,aq/cd,aq/ce,aq/cf,aq/de,aq/df,aq/ef;q)_\infty}{(b,c,d,e,f,bc/a,bd/a,be/a,bf/a;q)_\infty},
\end{aligned} \tag{4.3.13}$$

where $qa^2 = bcdef$.

Also, in §2.12 of BHS the q-integral (4.3.6) was used to give a short derivation of Bailey's [1947] 4-term transformation formula for ${}_{10}W_9$ series

$$\begin{aligned}
&{}_{10}W_9(a,b,c,d,e,f,g,h;q,q)\\
&\quad + \frac{(aq,b/a,c,d,e,f,g,h,bq/c,bq/d;q)_\infty}{(b^2q/a,a/b,aq/c,aq/d,aq/e,aq/f,aq/g,aq/h,bc/a,bd/a;q)_\infty}\\
&\quad \cdot \frac{(bq/e,bq/f,bq/g,bq/h;q)_\infty}{(be/a,bf/a,bg/a,bh/a;q)_\infty}\\
&\quad \cdot {}_{10}W_9(b^2/a,b,bc/a,bd/a,be/a,bf/a,bg/a,bh/a;q,q)\\
&= \frac{(aq,b/a,\lambda q/f,\lambda q/g,\lambda q/h,bf/\lambda,bg/\lambda,bh/\lambda;q)_\infty}{(\lambda q,b/\lambda,aq/f,aq/g,aq/h,bf/a,bg/a,bh/a;q)_\infty}\\
&\quad \cdot {}_{10}W_9(\lambda,b,\lambda c/a,\lambda d/a,\lambda e/a,f,g,h;q,q)\\
&\quad + \frac{(aq,b/a,f,g,h,bq/f,bq/g,bq/h,\lambda c/a,\lambda d/a;q)_\infty}{(b^2q/\lambda,\lambda/b,aq/c,aq/d,aq/e,aq/f,aq/g,aq/h,bc/a,bd/a;q)_\infty}\\
&\quad \cdot \frac{(\lambda e/a,abq/\lambda c,abq/\lambda d,abq/\lambda e;q)_\infty}{(be/a,bf/a,bg/a,bh/a;q)_\infty}\\
&\quad \cdot {}_{10}W_9(b^2/\lambda,b,bc/a,bd/a,be/a,bf/\lambda,bg/\lambda,bh/\lambda;q,q)
\end{aligned} \tag{4.3.14}$$

and it was observed that (4.3.14) can be written in terms of the q-integrals in the compact form

$$\int_a^b \frac{(qt/a,qt/b,ta^{-\frac12},-ta^{-\frac12},qt/c,qt/d,qt/e,qt/f,qt/g,qt/h;q)_\infty}{(t,bt/a,qta^{-\frac12},-qta^{-\frac12},ct/a,dt/a,et/a,ft/a,gt/a,ht/a;q)_\infty}\,d_qt$$

$$= \frac{a}{\lambda} \frac{(b/a, aq/b, \lambda c/a, \lambda d/a, \lambda e/a, bf/\lambda, bg/\lambda, bh/\lambda; q)_\infty}{(b/\lambda, \lambda q/b, c, d, e, bf/a, bg/a, bh/a; q)_\infty}$$

$$\cdot \int_\lambda^b \frac{(qt/\lambda, qt/b, t\lambda^{-\frac{1}{2}}, -t\lambda^{-\frac{1}{2}}, aqt/c\lambda, aqt/d\lambda, aqt/e\lambda, qt/f, qt/g, qt/h; q)_\infty}{(t, bt/\lambda, qt\lambda^{-\frac{1}{2}}, -qt\lambda^{-\frac{1}{2}}, ct/a, dt/a, et/a, ft/\lambda, gt/\lambda, ht/\lambda; q)_\infty} d_q t, \tag{4.3.15}$$

where $\lambda = qa^2/cde$ and $a^3q^2 = bcdefgh$.

Exercises 4

4.1 Prove that

$$\sum_{k=0}^{n} \frac{(1-adp^kq^k)(1-bp^k/dq^k)}{(1-ad)(1-b/d)} \frac{(a,b;p)_k(q^{-n}, ad^2q^n/b;q)_k}{(dq, adq/b;q)_k(adpq^n, bp/dq^n;p)_k} q^k$$
$$= \frac{(1-d)(1-ad/b)(1-adq^n)(1-dq^n/b)}{(1-ad)(1-d/b)(1-dq^n)(1-adq^n/b)}, \qquad n = 0, 1, \ldots .$$

4.2 Use (4.1.24) to show that Euler's transformation formula

$$\sum_{n=0}^{\infty} a_n b_n x^n = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!} f^{(k)}(x) \Delta^k a_0,$$

where

$$f(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$

and

$$\Delta^k a_0 = \sum_{j=0}^{k} (-1)^j \binom{k}{j} a_{k-j},$$

has the bibasic extension

$$\sum_{n=0}^{\infty} A_n B_n (xw)^n = \sum_{k=0}^{\infty} (apq^k;p)_{k-1} x^k \sum_{n=0}^{k} \frac{(1-ap^nq^n)w^n A_n}{(q;q)_{k-n}(apq^k;p)_n}$$
$$\cdot \sum_{j=0}^{\infty} \frac{(ap^kq^k;p)_j}{(q;q)_j} B_{j+k} (-x)^j q^{\binom{j}{2}}.$$

See Al-Salam and Verma [1984].

4.3 Use (4.1.6) to derive the Gasper [1989a] bibasic expansion formula

$$\sum_{n=0}^{\infty} A_n B_n \frac{(xw)^n}{(q;q)_n} = \sum_{n=0}^{\infty} \frac{(1-\gamma p^n q^n)(1-\sigma p^n q^n)}{(q;q)_n} (-x)^n q^{n+\binom{n}{2}}$$

$$\cdot \sum_{k=0}^{\infty} \frac{1-\gamma\sigma^{-1}q^{2n+2k}}{(q;q)_k(\gamma pq^{n+k},\sigma pq^{-n-k};p)_n} B_{n+k}x^k$$

$$\cdot \sum_{j=0}^{n} \frac{(q^{-n};q)_j(\gamma\sigma^{-1}q^{n+j+1};q)_{n+k-j-1}}{(q;q)_j}$$

$$\cdot (\gamma pq^j, \sigma pq^{-j};p)_{n-1} A_j C_{j,n+k-j} w^j q^{n(j-n-k)},$$

where $A_j, B_j, C_{j,k}$ are complex numbers such that the series converge absolutely and $C_{j,0}=1$, for $j=0,1,\ldots$.

4.4 Show that if $p=q$, then the $\sigma\to\infty$ limit case of Ex. 4.3 gives the expansion formula

$$_{r+t}\phi_{s+u}\left[\begin{matrix} a_R, c_T \\ b_S, d_U \end{matrix}; q, xw\right]$$

$$= \sum_{j=0}^{\infty} \frac{(c_T, e_K, \sigma, \gamma q^{j+1}/\sigma;q)_j}{(q,d_U,f_M,\gamma q^j;q)_j}\left(\frac{x}{\sigma}\right)^j [(-1)^j q^{\binom{j}{2}}]^{u+m-t-k}$$

$$\cdot\ {}_{t+k+4}\phi_{u+m+3}\left[\begin{matrix} \gamma q^{2j}/\sigma, q^{j+1}\sqrt{\gamma/\sigma}, -q^{j+1}\sqrt{\gamma/\sigma}, \sigma^{-1}, \\ q^j\sqrt{\gamma/\sigma}, -q^j\sqrt{\gamma/\sigma}, \gamma q^{2j+1}, d_U q^j, \end{matrix}\right.$$

$$\left.\begin{matrix} c_T q^j, e_K q^j \\ f_M q^j \end{matrix}; q, xq^{j(u+m-t-k)}\right]$$

$$\cdot\ {}_{r+m+2}\phi_{s+k+2}\left[\begin{matrix} q^{-j}, \gamma q^j, a_R, f_M \\ \gamma q^{j+1}/\sigma, q^{1-j}/\sigma, b_S, e_K \end{matrix}; q, wq\right],$$

where we employed the contracted notation of representing $a_1,\ldots,a_r$ by a_R, etc.

4.5 Derive formulas (4.3.2) and (4.3.4).

4.6 Show that the q-integral formulas (4.3.6) and (4.3.8) are equivalent to formulas (4.3.5) and (4.3.7), respectively.

4.7 Use the q-integral representation (4.3.6) to derive formulas (4.3.11) and (4.3.14).

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