May 29, 1973

Dear Tom and Dick,

(Dick should arrive in Amsterdam around June 22)

I've discovered some interesting connections between certain sums, integrals, and projection formulas which lead to the identity

$$(1) \qquad \sum_{k=0}^{n} P_k^{(\alpha,0)}(x) = \sum_{j=0}^{[\frac{n}{2}]} \frac{(\frac{1}{2})_j \left(\frac{\alpha+2}{2}\right)_{n-j} \left(\frac{\alpha+3}{2}\right)_{n-2j} (n-2j)!}{j!\left(\frac{\alpha+3}{2}\right)_{n-j} \left(\frac{\alpha+1}{2}\right)_{n-2j} (\alpha+1)_{n-2j}} \cdot \left\{ C_{n-2j}^{\frac{\alpha+1}{2}}\left[\left(\frac{1+x}{2}\right)^{\frac{1}{2}}\right] \right\}^2 .$$

From this identity it immediately follows that

$$(2) \qquad \sum_{k=0}^{n} P_k^{(\alpha,0)}(x) \geq 0, \qquad x \geq -1, \quad \alpha \geq -2,$$

and hence, by Bateman's integral,

$$(3) \qquad \sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\beta,\alpha)}(1)} \geq 0, \qquad x \geq -1,$$

for $\beta \geq 0$, $\alpha+\beta \geq -2$. ((3) does not hold if $\alpha+\beta < -2$)

To prove (1) I used the following string of identities

$$\sum_{k=0}^{n} P_k^{(\alpha,0)}(x) \overset{(a)}{=} \frac{(\alpha+2)_n}{n!}\, {}_3F_2\left[\begin{matrix}-n,\ n+\alpha+2,\ \frac{\alpha+1}{2};\ \frac{1-x}{2}\\ \frac{\alpha+3}{2},\ \alpha+1\end{matrix}\right]$$

(b) (Temporarily assume $\alpha > -1$)

$$= \frac{\Gamma(\alpha+1)\left(\frac{1-x}{2}\right)^{-\alpha}}{\left[\Gamma\left(\frac{\alpha+1}{2}\right)\right]^2} \int_0^{\frac{1-x}{2}} \left(\frac{1-x}{2}-t\right)^{\frac{\alpha-1}{2}} t^{\frac{\alpha-1}{2}} C_n^{\frac{\alpha+2}{2}}(1-2t)\,dt$$

(c)

$$= \sum_{j=0}^{[\frac{n}{2}]} \frac{\Gamma(\alpha+1)\left(\frac{1-x}{2}\right)^{-\alpha}\left(\frac{1}{2}\right)_j \left(\frac{\alpha+2}{2}\right)_{n-j}\left(\frac{\alpha+3}{2}\right)_{n-2j}}{\left[\Gamma\left(\frac{\alpha+1}{2}\right)\right]^2 j!\left(\frac{\alpha+3}{2}\right)_{n-j}\left(\frac{\alpha+1}{2}\right)_{n-2j}} \cdot$$

$$\cdot \int_0^{\frac{1-x}{2}} \left(\frac{1-x}{2}-t\right)^{\frac{\alpha-1}{2}} t^{\frac{\alpha-1}{2}} C_{n-2j}^{\frac{\alpha+1}{2}}(1-2t)\,dt$$

(d)

$$= \sum_{j=0}^{[\frac{n}{2}]} \frac{\left(\frac{1}{2}\right)_j \left(\frac{\alpha+2}{2}\right)_{n-j}(\alpha+1)_{n-2j}\left(\frac{\alpha+3}{2}\right)_{n-2j}}{j!\left(\frac{\alpha+3}{2}\right)_{n-j}(n-2j)!\left(\frac{\alpha+1}{2}\right)_{n-2j}}\, {}_3F_2\left[\begin{matrix}2j-n,\ n-2j+\alpha+1,\ \frac{\alpha+1}{2};\ \frac{1-x}{2}\\ \frac{\alpha+2}{2},\ \alpha+1\end{matrix}\right]$$

(e)

$$= \sum_{j=0}^{[\frac{n}{2}]} \frac{\left(\frac{1}{2}\right)_j \left(\frac{\alpha+2}{2}\right)_{n-j}(\alpha+1)_{n-2j}\left(\frac{\alpha+3}{2}\right)_{n-2j}}{j!\left(\frac{\alpha+3}{2}\right)_{n-j}(n-2j)!\left(\frac{\alpha+1}{2}\right)_{n-2j}} \left\{F\left[\begin{matrix}\frac{2j-n}{2},\ \frac{n-2j+\alpha+1}{2};\ \frac{1-x}{2}\\ \frac{\alpha+2}{2}\end{matrix}\right]\right\}^2$$

(f)

$$= \sum_{j=0}^{[\frac{n}{2}]} \frac{\left(\frac{1}{2}\right)_j \left(\frac{\alpha+2}{2}\right)_{n-j}\left(\frac{\alpha+3}{2}\right)_{n-2j}(n-2j)!}{j!\left(\frac{\alpha+3}{2}\right)_{n-j}\left(\frac{\alpha+1}{2}\right)_{n-2j}(\alpha+1)_{n-2j}} \left\{C_{n-2j}^{\frac{\alpha+1}{2}}\left[\left(\frac{1+x}{2}\right)^{\frac{1}{2}}\right]\right\}^2.$$

Then (1) follows by analytic continuation.

Identity (a) follows by using the ${}_2F_1[\ ;\frac{1-x}{2}]$ representation for $P_n^{(\alpha,0)}(x)$.

" (b) (here $\alpha>-1$) follows from the integral rep. [Bateman Tables of Integral Transforms Vol 2, p 191, (40)]

$$\frac{(2\lambda)_n\,\Gamma(\nu)}{n!\,\Gamma(\mu+\nu)}\,y^{\mu+\nu-1}\,{}_3F_2\left[\begin{matrix}-n,\ n+2\lambda,\ \nu;\ y\\ \lambda+\frac{1}{2},\ \mu+\nu\end{matrix}\right]$$

(4)

$$=\frac{1}{\Gamma(\mu)}\int_0^y (y-x)^{\mu-1}\,x^{\nu-1}\,C_n^{\lambda}(1-2x)\,dx,$$

$$(\mu>0;\ \nu>0)$$

by taking $\lambda=\frac{\alpha+2}{2}$ and $\nu=\mu=\frac{\alpha+1}{2}$.

(c) is a consequence of the (Gegenbauer) projection formula

(5) $$C_n^{\mu}(x)=\sum_{j=0}^{[\frac{n}{2}]}\frac{(\mu-\lambda)_j\,(\mu)_{n-j}\,(\lambda+1)_{n-2j}}{j!\,\cancel{(n-j)!}\,(\lambda+1)_{n-j}\,(\lambda)_{n-2j}}\,C_{n-2j}^{\lambda}(x)$$ mailed correction May 30, 78

with $\mu=\frac{\alpha+2}{2}$, $\lambda=\frac{\alpha+1}{2}$. (d) is another application of (4). (e) follows from Clausen's formula [Ibid. Vol 2, p185] while (f) is from a quadratic transformation (Gauss').

The nonnegativity of the ${}_3F_2$ in line ⓓ suggest that we consider the general expansion

$$ {}_3F_2\left[\begin{matrix}-n,\ n+2\lambda,\ \lambda;\\ \lambda+\frac{1}{2},\ 2\lambda\end{matrix}\ \frac{1-x}{2}\right] = \sum_{k=0}^{n} \frac{(\alpha+\beta+1)_k\,(\beta+1)_k\,n!}{(\alpha+\beta+1)_{2k}\,k!\,(n-k)!}\times $$

$$ \times\frac{(n+2\lambda)_k\,(\lambda)_k}{(\lambda+\frac{1}{2})_k\,(2\lambda)_k}\ {}_4F_3\left[\begin{matrix}k-n,\ n+2\lambda+k,\ \lambda+k,\ \alpha+1+k;\\ \lambda+\frac{1}{2}+k,\ 2\lambda+k,\ \alpha+\beta+2+2k\end{matrix}\right]\ \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\beta,\alpha)}(1)}. $$

(should leads to alternate pf of (1) when $\beta=-\frac{1}{2}$ and Szegö, (4.22.2) is used (and gives an interesting identity))

This ${}_4F_3$ simplifies when $\lambda = \frac{\alpha+1}{2}$, $\lambda=\alpha+\frac{1}{2}$ or $\lambda=0$

Also it is (luckily?) Saalschützian when $\beta=-\frac{1}{2}$; so might be of use for the case $\beta=-\frac{1}{2}$, $\alpha\geq\frac{1}{2}$ of (3) (for ~~[illegible]~~ $\lambda=\alpha+\frac{1}{2}$, $\alpha=\frac{1}{2}$, $\beta=-\frac{1}{2}$ it gives (3)). There are several other nonnegative sums and integrals (e.g. compare lines ⓒ & ⓕ) which follow from the above observations; but, even using fractional integrals & sums, I haven't been able ~~[illegible]~~ so far to obtain any additional cases of (3).

Have a nice summer.

Sincerely,

George

June 12, 1973

Dear George,

Your proof of $\sum_{k=0}^{n} P_k^{(\alpha,0)}(x) \geq 0$, $x \geq -1$, $\alpha \geq -2$ is very pretty. It is clearly the right proof. I'm not too optimistic that the case $(\alpha, -\frac{1}{2})$ can be done by the same method, but who knows? Not me. I propose the following for publishing all of this work. I will rewrite P.J.P.S. II, and cut out the parts that no longer are true, and give some of the consequences of your result. For example,

$$\sum_{k=0}^{n} \frac{(\lambda+1)_{n-k}}{(n-k)!} \frac{(\lambda+1)_k}{k!} \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\beta,\alpha)}(1)} \geq 0, \quad -1 \leq x, \quad 0 \leq \lambda \leq \beta, \quad \alpha \geq -2-\beta+\lambda,$$

but not mention anything about $\alpha \to \infty$. You write P.J.P.S. III and include your proof for $(\alpha, 0)$ and all of the consequences when $\alpha \to \infty$. Then P.J.P.S. IV is the joint paper proving the result for $\alpha = 5/2$, $\beta = -1/2$. These should appear together, and I suggest the American Journal of Mathematics. It is a good general journal, and these results are important enough so they should be in a high quality general journal, rather than a specialized journal.

Mail to Budapest takes a long time, so you can write me in Amsterdam,

Nieuwe Prinsengracht 37

Amsterdam C, Holland.

I'll be there until the 5th of July (for mail, we leave the 6th). I've proven one theorem while here.

Let $\sum_{n=0}^{\infty} a_n$ have positive (C,γ) means. Then the (C,α) means of $\sum_{n=0}^{\infty} a_n r^n$ are positive for $0 \leq r \leq (\alpha+1)/(\gamma+1)$, $-1 < \alpha < \gamma$, and this is sharp. It isn't a deep theorem, but the proof is cute. First it is reduced to a problem about ${}_2F_1$'s, and then an induction gives the theorem. It is a good example of how hypergeometric functions come into problems, and how contiguous relations can be used. I only need one of the Gauss relations, i.e. a result of 1820 or so, while you needed the more recent result of Clausen (1828). Can you think of any other field where deep research was done 150 years ago that is almost completely unknown to the general mathematical community?

Legendre (1752–1833)
Euler (1707–1783) — Euler's integral formula.
Jacobi (1804–1851)

The old traditions of Fejér are much weaker here than I had hoped they would be, but people at least appreciate the results, even if they don't quite understand them.

The next result to try for is $\sum_{k=0}^{n} \frac{k!}{(2k)!} H_{2k}(x) \geq 0$. I think it

$\alpha \to \infty$, $\beta = -\frac{1}{2}$ limiting case

will have a number of interesting consequences, and it really shouldn't be that hard to prove.

Sincerely,
Dick