USING INTEGRALS OF SQUARES OF CERTAIN REAL-VALUED SPECIAL FUNCTIONS TO PROVE THAT THE PÓLYA $\Xi^*(z)$ FUNCTION, THE FUNCTIONS $K_{iz}(a), a > 0,$ AND SOME OTHER ENTIRE FUNCTIONS HAVE ONLY REAL ZEROS

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Dedicated to Dan Waterman on the occasion of his 80th birthday

Abstract. Analogous to the use of sums of squares of certain real-valued special functions to prove the reality of the zeros of the Bessel functions $J_\alpha(z)$ when $\alpha \geq -1$, confluent hypergeometric functions ${}_0F_1(c;z)$ when $c > 0$ or $0 > c > -1$, Laguerre polynomials $L_n^\alpha(z)$ when $\alpha \geq -2$, Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ when $\alpha \geq -1$ and $\beta \geq -1$, and some other entire special functions considered in G. Gasper [*Using sums of squares to prove that certain entire functions have only real zeros,* in *Fourier Analysis: Analytic and Geometric Aspects,* W. O. Bray, P. S. Milojević and C. V. Stanojević, eds., Marcel Dekker, Inc., 1994, 171–186.], integrals of squares of certain real-valued special functions are used to prove the reality of the zeros of the Pólya $\Xi^*(z)$ function, the $K_{iz}(a)$ functions when $a > 0$, and some other entire functions.

1. Introduction

It is well-known [21] that the Riemann Hypothesis is equivalent to the statement that all of the zeros of the Riemann $\Xi(z)$ function are real. $\Xi(z)$ is an even entire function of z with the integral representations

$$\Xi(z) = \int_{-\infty}^{\infty} \Phi(u)\, e^{izu}\, du = 2\int_0^\infty \Phi(u)\cos(zu)\, du, \tag{1.1}$$

where

$$\Phi(u) = \sum_{n=1}^{\infty} (4n^4\pi^2 e^{\frac{9}{2}u} - 6n^2\pi e^{\frac{5}{2}u}) e^{-n^2\pi e^{2u}}. \tag{1.2}$$

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In a 1926 paper Pólya [18] observed that

$$\Phi(u) \sim 8\pi^2 \cosh(\frac{9}{2}u)\, e^{-2\pi\cosh(2u)} \quad \text{as } u \to \pm\infty \tag{1.3}$$

and, in view of this asymptotic equivalence to $\Phi(u)$, considered the problem of determining whether or not the entire function

$$\Xi^*(z) = 16\pi^2 \int_0^\infty \cosh(\frac{9}{2}u)\, e^{-2\pi\cosh(2u)} \cos(zu)\, du \tag{1.4}$$

has only real zeros. Here, as is now customary, the capital letter Ξ is used instead of the original lower case ξ. Pólya was able to prove that $\Xi^*(z)$ has only real zeros by using (in a different notation) a difference equation in z for the modified Bessel function of the third kind [7], [22]

$$K_z(a) = \int_0^\infty e^{-a\cosh u} \cosh(zu)\, du, \qquad a > 0, \tag{1.5}$$

to prove for each $a > 0$ that $K_{iz}(a)$ has only real zeros, and then applying the identity

$$\Xi^*(z) = 4\pi^2[K_{\frac{1}{2}iz-\frac{9}{4}}(2\pi) + K_{\frac{1}{2}iz+\frac{9}{4}}(2\pi)] \tag{1.6}$$

and the special case $G(z) = K_{iz/2}(2\pi)$, $c = \frac{9}{4}$, of the lemma (derived via an infinite product representation for $G(z)$):

> **Lemma.** *If $-\infty < c < \infty$ and $G(z)$ is an entire function of genus 0 or 1 that assumes real values for real z, has only real zeros and has at least one real zero, then the function*
>
> $$G(z - ic) + G(z + ic)$$
>
> *also has only real zeros.*

More generally, in a subsequent paper Pólya [19] pointed out that from the case $G(z) = K_{iz/2}(a)$ of this lemma it follows that each of the entire functions

$$\begin{aligned} F_{a,c}(z) &= K_{i(z-ic)}(a) + K_{i(z+ic)}(a) \\ &= 2\int_0^\infty \cosh(cu)\, e^{-a\cosh u} \cos(zu)\, du, \qquad a > 0,\ -\infty < c < \infty, \end{aligned} \tag{1.7}$$

has only real zeros. He also used a differential equation in the variable a to give a proof of the reality of the zeros of $K_{iz}(a), a > 0$, that was simpler than his previous proof.

Our main aim in this paper is to show how integrals of squares of certain real-valued special functions can be used to give new proofs of the reality of the zeros of the above $\Xi^*(z), K_{iz}(a)$, and $F_{a,c}(z)$ functions. This paper is a sequel to the author's 1994 paper [14] in which he showed how sums of squares of certain real-valued special functions could be used to prove the reality of the zeros of the Bessel functions $J_\alpha(z)$ when $\alpha \geq -1$, confluent hypergeometric functions ${}_0F_1(c; z)$ when $c > 0$ or $0 > c > -1$, Laguerre polynomials $L_n^\alpha(z)$ when $\alpha \geq -2$, Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ when $\alpha \geq -1$ and $\beta \geq -1$, and some other entire functions. Also see the applications of squares of real-valued special functions in [2], [3], [6], [8], [9], [10], [11], [12], [13], and [15].

2. Reality of the zeros of the functions $K_{iz}(a)$ when $a > 0$

Let $a > 0$ and $z = x + iy$, where x and y are real variables. First observe that, by the Meijer G-function representation for the product of two modified Bessel functions of the third kind in [7, Eq. 5.6(66)] and the definition of a Meijer G-function as a Mellin-Barnes integral in [7, Eq. 5.3(1)],

$$\begin{aligned}|K_{iz}(a)|^2 = K_{iz}(a)K_{i\bar{z}}(a) &= \frac{\sqrt{\pi}}{2}\, G_{24}^{40}\left[a^2 \middle| \begin{matrix} 0, \frac{1}{2} \\ ix, -ix, y, -y \end{matrix}\right] \\ &= \frac{\sqrt{\pi}}{4\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(ix-s)\Gamma(-ix-s)\Gamma(y-s)\Gamma(-y-s)}{\Gamma(-s)\Gamma(\frac{1}{2}-s)} a^{2s}\, ds \qquad (2.1)\end{aligned}$$

with $c < -|y|$, where the path of integration is along the upwardly oriented vertical line $\Re(s) = c$. Next note that, by Gauss' summation formula [15, (1.2.11)],

$$\sum_{k=0}^{\infty} \frac{(y)_k (y)_k}{k!\,(y-s)_k} = {}_2F_1\left[\begin{matrix} y, y \\ y-s \end{matrix}; 1\right] = \frac{\Gamma(y-s)\Gamma(-y-s)}{\Gamma(-s)\Gamma(-s)}, \qquad \Re(s) < -y,$$

where $(y)_k = \Gamma(y+k)/\Gamma(y)$ and the series is absolutely convergent. Hence,

$$|K_{iz}(a)|^2 = \frac{\sqrt{\pi}}{4\pi i}\int_{c-i\infty}^{c+i\infty} \sum_{k=0}^{\infty} \frac{((y)_k)^2 \Gamma(ix-s)\Gamma(-ix-s)\Gamma(y-s)\Gamma(-s)}{k!\,\Gamma(\frac{1}{2}-s)\Gamma(y+k-s)} a^{2s}\, ds$$

$$= [K_{ix}(a)]^2 + \frac{\sqrt{\pi}}{4\pi i}\sum_{k=1}^{\infty} \frac{((y)_k)^2}{k!}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(ix-s)\Gamma(-ix-s)\Gamma(y-s)\Gamma(-s)}{\Gamma(\frac{1}{2}-s)\Gamma(y+k-s)} a^{2s} ds$$

and, since $(y)_k = y(y+1)_{k-1}$ for $k \geq 0$,

$$|K_{iz}(a)|^2 = [K_{ix}(a)]^2 + y^2 L_a(x, y) \qquad (2.2)$$

with

$$L_a(x, y) = \frac{\sqrt{\pi}}{4\pi i}\sum_{k=0}^{\infty} \frac{((y+1)_k)^2}{(k+1)!}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(ix-s)\Gamma(-ix-s)\Gamma(y-s)\Gamma(-s)}{\Gamma(\frac{1}{2}-s)\Gamma(y+k-s)} a^{2s} ds. \qquad (2.3)$$

From (2.2) it follows that in order to prove that $K_{iz}(a)$ has only real zeros it suffices to prove that

$$L_a(x, y) > 0, \qquad -\infty < x, y < \infty. \qquad (2.4)$$

To prove (2.4) from (2.3) we observe that

$$\int_0^1 t^{y-s-1}(1-t)^{k-1}\, dt = \frac{\Gamma(k)\Gamma(y-s)}{\Gamma(y+k-s)}, \qquad k > 0,\ \Re(s) < y,$$

by the beta integral [15, (1.11.8)], and thus

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(ix-s)\Gamma(-ix-s)\Gamma(y-s)\Gamma(-s)}{\Gamma(\frac{1}{2}-s)\Gamma(y+k-s)}a^{2s}\,ds$$

$$=\frac{1}{\Gamma(k)}\int_0^1 t^{y-1}(1-t)^{k-1}\left[\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(ix-s)\Gamma(-ix-s)\Gamma(-s)}{2\pi i\,\Gamma(\frac{1}{2}-s)}\left(\frac{a^2}{t}\right)^s ds\right]dt$$

$$=\frac{2}{\sqrt{\pi}\Gamma(k)}\int_0^1 t^{y-1}(1-t)^{k-1}\left[K_{ix}\left(\frac{a}{\sqrt{t}}\right)\right]^2 dt$$

by Fubini's Theorem and (2.1), which gives

$$L_a(x,y)=\sum_{k=0}^{\infty}\frac{((y+1)_k)^2}{k!(k+1)!}\int_0^1 t^{y-1}(1-t)^{k-1}\left[K_{ix}\left(\frac{a}{\sqrt{t}}\right)\right]^2 dt>0 \tag{2.5}$$

for $-\infty<x,y<\infty$. Equations (2.2) and (2.5) can be combined to give the formula

$$|K_{iz}(a)|^2=[K_{ix}(a)]^2+y^2\int_0^1 t^{y-1}{}_2F_1\left[\begin{matrix}y+1,y+1\\2\end{matrix};1-t\right]\left[K_{ix}\left(\frac{a}{\sqrt{t}}\right)\right]^2 dt, \tag{2.6}$$

from which it follows that $K_{iz}(a)$ has only real zeros when $a>0$, since the integrand is clearly nonnegative.

The reality of the zeros of $K_{iz}(a), a>0$, can also be proved by taking the first and second partial derivatives of the formula (2.6) with respect to y to obtain the formulas

$$y\frac{\partial}{\partial y}|K_{iz}(a)|^2=\int_0^1 y\frac{\partial}{\partial y}f_t(y)\ \left[K_{ix}\left(\frac{a}{\sqrt{t}}\right)\right]^2\frac{dt}{t} \tag{2.7}$$

and

$$\frac{\partial^2}{\partial y^2}|K_{iz}(a)|^2=\int_0^1 \frac{\partial^2}{\partial y^2}f_t(y)\ \left[K_{ix}\left(\frac{a}{\sqrt{t}}\right)\right]^2\frac{dt}{t} \tag{2.8}$$

with

$$f_t(y)=y^2t^y{}_2F_1\left[\begin{matrix}y+1,y+1\\2\end{matrix};1-t\right]. \tag{2.9}$$

In [4] Askey and the author utilized the reality of the zeros of the continuous dual Hahn polynomials [1, p. 331] to show that $f_t(y)$ (and a generalization of it) is an (even) absolutely monotonic function (one whose power series coefficients are nonnegative) of y when $0<t<1$, which shows that

$$y\frac{\partial}{\partial y}f_t(y)\ge 0,\qquad -\infty<y<\infty,\quad 0<t<1, \tag{2.10}$$

and

$$\frac{\partial^2}{\partial y^2}f_t(y)\ge 0,\qquad -\infty<y<\infty,\quad 0<t<1. \tag{2.11}$$

and, hence, that the integrals in (2.7) and (2.8) are positive when $y\ne 0$. Thus, it follows from (2.7) that $|K_{iz}(a)|^2$ is an increasing (decreasing) even function of y when $y > 0$ ($y < 0$), and it follows from (2.8) that $|K_{iz}(a)|^2$ is a convex even non-constant function of y, each of which implies that for any x the function $|K_{iz}(a)|^2$ assumes its minimum value when $y = 0$ and proves that $K_{iz}(a)$ has only real zeros.

Corresponding to the above proofs via formulas (2.7) and (2.8), it should be noted that in 1913 Jensen [17] showed that each of the inequalities

$$y\frac{\partial}{\partial y}|F(x+iy)|^2 \geq 0, \quad -\infty < x, y < \infty, \tag{2.12}$$

and

$$\frac{\partial^2}{\partial y^2}|F(x+iy)|^2 \geq 0, \quad -\infty < x, y < \infty, \tag{2.13}$$

is necessary and sufficient for a real entire function $F(z) \not\equiv 0$ of genus 0 or 1 to have only real zeros (see [5, Chapter 2] and the necessary and sufficient conditions in [20]).

3. Reality of the zeros of the functions $\Xi^*(z)$ and $F_{a,c}(z)$

Because of $\Xi^*(z) = 4\pi^2 F_{2\pi,9/4}(z/2)$, it suffices to show how formula (2.6) can be used to prove that $F_{a,c}(z)$ has only real zeros when $a, c > 0$. Fix $a, c > 0$ and suppose that $z_0 = x_0 + iy_0$ is a zero of $F_{a,c}(z)$. Then $K_{i(x_0+i(y_0+c))}(a) = -K_{i(x_0+i(y_0-c))}(a)$ by (1.7) and hence

$$\begin{aligned} 0 &= |K_{i(x_0+i(y_0+c))}(a)|^2 - |K_{i(x_0+i(y_0-c))}(a)|^2 \\ &= \int_0^1 [f_t(y_0+c) - f_t(y_0-c)] \left[K_{ix_0}\left(\frac{a}{\sqrt{t}}\right)\right]^2 \frac{dt}{t} \end{aligned} \tag{3.1}$$

by (2.6) and (2.9). Since $f_t(y)$ is an even convex non-constant function of y when $0 < t < 1$,

$$f_t(y_0+c) - f_t(y_0-c) \begin{cases} > 0 & \text{if } y_0 > 0, \\ = 0 & \text{if } y_0 = 0, \\ < 0 & \text{if } y_0 < 0, \end{cases}$$

and it follows from (3.1) that $y_0 = 0$ and, thus, the function $F_{a,c}(z)$ has only real zeros when $a, c > 0$.

One can also try to give another proof of the reality of the zeros of $F_{a,c}(z)$ via a formula for the function

$$\begin{aligned} |F_{a,c}(z)|^2 &= F_{a,c}(z)F_{a,c}(\bar{z}) \\ &= K_{ix-y-c}(a)K_{ix+y-c}(a) + K_{ix-y-c}(a)K_{ix+y+c}(a) \\ &\quad + K_{ix-y+c}(a)K_{ix+y-c}(a) + K_{ix-y+c}(a)K_{ix+y+c}(a) \end{aligned} \tag{3.2}$$

that contains integrals of nonnegative functions as in (2.6). By using the right-hand side of (3.2), [7, Eq. 5.6(66)], [7, Eq. 5.3(1)], Gauss' summation formula and the beta integral, it can be shown that

$$|F_{a,c}(z)|^2 = [F_{a,c}(x)]^2 + \int_0^1 f_t(y)[F_{a/\sqrt{t},\,c}(x)]^2 \frac{dt}{t} + \int_0^1 g_{t,c}(y)|K_{i(x+ic)}(a/\sqrt{t})|^2 \frac{dt}{t} \tag{3.3}$$

with

$$g_{t,c}(y) = y(y+2c)t^{y+c}{}_2F_1\left[\begin{matrix} y+1, y+2c+1 \\ 2 \end{matrix}; 1-t\right]$$
$$+y(y-2c)t^{y-c}{}_2F_1\left[\begin{matrix} y+1, y-2c+1 \\ 2 \end{matrix}; 1-t\right] - 2y^2t^y{}_2F_1\left[\begin{matrix} y+1, y+1 \\ 2 \end{matrix}; 1-t\right]. \tag{3.4}$$

The function $g_{t,c}(y)$ is clearly an even function of c such that $g_{t,0}(y) = 0$ and $g_{t,c}(0) = 0$. It can be shown that $g_{t,c}(y)$ is also an even function of y by applying the Euler transformation formula [15, (1.4.2)] to the hypergeometric functions in (3.4). Extensive analysis of $g_{t,c}(y)$ via Mathematica and generating functions of the form [4, (5)] strongly suggest that $g_{t,c}(y)$ is nonnegative, convex, and an absolutely monotonic function of y when $0 < t < 1$ and $-\infty < c < \infty$. If the nonnegativity, convexity, or absolute monotonicity of $g_{t,c}(y)$ could be proved, then (3.3) and its partial derivatives with respect to y would give additional proofs of the reality of the zeros of $F_{a,c}(z)$.

Pólya [19] derived a theorem concerning the zeros of Fourier transforms and universal factors, and then used it to prove that his [18, p. 317]

$$\Xi^{**}(z) = 8\pi \int_0^\infty \left[2\pi\cosh(\frac{9}{2}u) - 3\cosh(\frac{5}{2}u)\right] e^{-2\pi\cosh(2u)} \cos(zu)\, du, \tag{3.5}$$

function and the more general functions

$$\Xi_{A,B,a,b,c}(z) = \int_0^\infty [A\cosh(au) - B\cosh(bu)]e^{-c\cosh(u)} \cos(zu)\, du \tag{3.6}$$

with $A > B > 0, a > b > 0, c > 0$, have only real zeros. Analogous to the formulas in (2.6) and (3.3), it might be possible to derive formulas containing squares of real-valued functions that give new proofs of the reality of the zeros of the entire functions in (3.5) and (3.6), and even, perhaps, prove that some of the entire functions in Hejhal [16, (0.3)] have only real zeros.

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