A Riemann–Lebesgue Lemma for Jacobi expansions

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Dedicated to P. L. Butzer on the occasion of his 65-th birthday

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Abstract. A Lemma of Riemann–Lebesgue type for Fourier–Jacobi coefficients is derived. Via integral representations of Dirichlet–Mehler type for Jacobi polynomials its proof directly reduces to the classical Riemann–Lebesgue Lemma for Fourier coefficients. Other proofs are sketched. Analogous results are also derived for Laguerre expansions and for Jacobi transforms.

Key words. Fourier–Jacobi coefficients, Riemann–Lebesgue Lemma, integrals of Dirichlet–Mehler type, Laguerre expansions, Jacobi transforms

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The classical Riemann–Lebesgue Lemma states (see [2, p. 168], [12, (4.4), p. 45]):

If $f \in L^1(-\pi,\pi)$, *then*

$$\lim_{|k|\to\infty} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} d\theta = 0.$$

Also, by using the identities $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$ *and* $e^{i\theta} = \cos\theta + i\sin\theta$, *if* $a, b, c_0, c_1 \in \mathbf{R}$ *and* $f \in L^1(a,b)$, *then*

$$\lim_{|k|\to\infty} \int_a^b f(\theta)\cos\left((k+c_0)\theta + c_1\right) d\theta = 0. \tag{1}$$

Here we will first consider the extension of this result to Fourier–Jacobi coefficients. For this purpose we need to introduce the following notation. Fix $\alpha, \beta > -1$ and let $L_{(\alpha,\beta)}$ denote the space of measurable functions on $[0,\pi]$ with finite norm

$$\|f\|_{L_{(\alpha,\beta)}} = \int_0^{\pi} \left|f(\theta)\right| \Big(\sin\frac{\theta}{2}\Big)^{2\alpha+1} \Big(\cos\frac{\theta}{2}\Big)^{2\beta+1} d\theta.$$

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Define normalized Jacobi polynomials by $R_k^{(\alpha,\beta)}(x) = P_k^{(\alpha,\beta)}(x)/P_k^{(\alpha,\beta)}(1)$, where $P_k^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree k and order (α,β), see [11]. For $f \in L_{(\alpha,\beta)}$, its k-th Fourier–Jacobi coefficient $\hat{f}_{(\alpha,\beta)}(k)$ is defined by

$$\hat{f}_{(\alpha,\beta)}(k) = \int_0^\pi f(\theta)\, R_k^{(\alpha,\beta)}(\cos\theta)\Big(\sin\frac{\theta}{2}\Big)^{2\alpha+1}\Big(\cos\frac{\theta}{2}\Big)^{2\beta+1} d\theta\,. \tag{2}$$

Then f has an expansion of the form

$$f(\theta) \sim \sum_{k=0}^{\infty} \hat{f}_{(\alpha,\beta)}(k)\, h_k^{(\alpha,\beta)} R_k^{(\alpha,\beta)}(\cos\theta)\,,$$

where the normalizing factors $h_k^{(\alpha,\beta)}$ are given by

$$\begin{aligned} h_k^{(\alpha,\beta)} &= \Big(\|(R_k^{(\alpha,\beta)})^2\|_{L_{(\alpha,\beta)}}\Big)^{-1} \\ &= \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)\Gamma(k+\alpha+1)}{\Gamma(k+\beta+1)\Gamma(k+1)\Gamma(\alpha+1)\Gamma(\alpha+1)} \approx (k+1)^{2\alpha+1} \end{aligned}$$

and the $\approx$ sign means that there are positive constants C, C' such that $C' h_k^{(\alpha,\beta)} \le (k+1)^{2\alpha+1} \le C h_k^{(\alpha,\beta)}$ holds.
In a recent paper [8] on ultraspherical multipiers we used the $\alpha = \beta$ case of the observation that if $(\alpha,\beta) \in S$ with

$$S := \{(\alpha,\beta) : \alpha \ge \beta > -1, \alpha \ge -1/2\},$$

then, by [11, Theorem 7.32.1],

$$\max_{-1\le x\le 1} |R_k^{(\alpha,\beta)}(x)| \le 1, \qquad k \in \mathbf{N_0},$$

and hence

$$|\hat{f}_{(\alpha,\beta)}(k)| \le \|f\|_{L_{(\alpha,\beta)}}, \qquad k \in \mathbf{N_0}.$$

Since $R_k^{(-1/2,-1/2)}(\cos\theta) = \cos k\theta$ and thus, by (1), $\lim_{k\to\infty} \hat{f}_{(-1/2,-1/2)}(k) = 0$ for $f \in L_{(-1/2,-1/2)}$, this led us to consider the problem of determining all (α,β) with $\alpha,\beta > -1$ for which

$$\lim_{k\to\infty} \hat{f}_{(\alpha,\beta)}(k) = 0 \tag{3}$$

whenever $f \in L_{(\alpha,\beta)}$, and to derive the following extension of the Riemann–Lebesgue Lemma to Fourier–Jacobi coefficients.

Lemma. *Let $\alpha,\beta > -1$. Then (3) holds for each $f \in L_{(\alpha,\beta)}$ if and only if $(\alpha,\beta) \in S$.*

Proof. First consider the case $\beta > \alpha > -1$. Formula 16.4 (1) in [4, p. 284] (corrected by inserting a missing $n!$ factor into the denominator of the right hand side) shows for $\alpha > -1,\ \beta + \rho > -1$ that

$$|\int_{-1}^{1} R_k^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta+\rho}dx| \approx (k+1)^{-2\rho-\alpha-\beta-2}. \tag{4}$$

Since $-2\rho - \alpha - \beta - 2 = \beta - \alpha - 2(\beta + \rho + 1)$, it follows that if $\beta > \alpha > -1$ and $0 < \beta + \rho + 1 < (\beta - \alpha)/2$, then the right hand side of (4) tends to ∞ as $k \to \infty$.

Now let $-1 < \beta \le \alpha < -1/2$. Introduce the linear functional $T_k : L_{(\alpha,\beta)} \to \mathbf{C},\ T_k f := \hat{f}_{(\alpha,\beta)}(k)$. By [11, Theorem 7.32.1], $|P_k^{(\alpha,\beta)}(x)|$ attains its maximum at a point x' (one of the two maximum points nearest $x_0 = (\beta-\alpha)/(\alpha+\beta+1)$) and $|P_k^{(\alpha,\beta)}(x')| \sim (k+1)^{-1/2}$. By the continuity of the Jacobi polynomial $P_k^{(\alpha,\beta)}(x)$ there exists a $\delta_k > 0$ such that $2|P_k^{(\alpha,\beta)}(x)| \ge |P_k^{(\alpha,\beta)}(x')|$ for all x with $|x - x'| \le \delta_k$. Now choose $f_k \in L_{(\alpha,\beta)}$ with $\mathrm{supp} f_k \subset \{x : |x - x'| \le \delta_k\}$, $\mathrm{sgn}\, f_k(x) = \mathrm{sgn}\, P_k^{(\alpha,\beta)}(x)$ if $|x - x'| \le \delta_k,\ \|f_k\|_{L_{(\alpha,\beta)}} = 1$. Then

$$|T_k f| = |\hat{f}_{\alpha,\beta}(k)| \ge \frac{C|P_k^{(\alpha,\beta)}(x')|}{2|P_k^{(\alpha,\beta)}(1)|}\, \|f_k\|_{L_{(\alpha,\beta)}} \ge C'(k+1)^{-\alpha-1/2},$$

i.e., $\|T_k\| \ge C'(k+1)^{-\alpha-1/2}$. Hence, by the uniform boundedness principle there exists an $f^* \in L_{(\alpha,\beta)}$ with $\lim_{k\to\infty} |T_k f^*| = \infty$. Summarizing, (3) cannot hold for each $f \in L_{(\alpha,\beta)}$ when $(\alpha,\ \beta) \notin S$.

Now let $\alpha \ge -1/2,\ \alpha > \beta > -1$. By the definition of the Fourier–Jacobi coefficients,

$$\hat{f}_{(\alpha,\beta)}(k) = \Big(\int_0^{\pi/2} + \int_{\pi/2}^{\pi}\Big) f(\theta) R_k^{(\alpha,\beta)}(\cos\theta)\Big(\sin\frac{\theta}{2}\Big)^{2\alpha+1}\Big(\cos\frac{\theta}{2}\Big)^{2\beta+1} d\theta =: I_k + J_k.$$

Then J_k tends to 0 for $k \to \infty$ if $\alpha > -1/2$ and $\alpha > \beta > -1$ since, by [11, paragraph below Theorem 7.32.1],

$$\max_{\pi/2\le\theta\le\pi} |R_k^{(\alpha,\beta)}(\cos\theta)| \le C(k+1)^{\max\{\beta,-1/2\}-\alpha}.$$

If $\alpha = \beta$, then J_k is of the same type as I_k. Thus we can restrict ourselves to a discussion of I_k.

To estimate I_k we will use the formula of Dirichlet–Mehler type in Gasper [6, (6)]

$$\begin{aligned} R_k^{(\alpha,\beta)}(\cos\theta) &= \frac{2^{(\alpha+\beta+1)/2}\Gamma(\alpha+1)}{\Gamma(1/2)\Gamma(\alpha+1/2)}\,(1-\cos\theta)^{-\alpha} \\ &\quad \times \int_0^{\theta} \cos\,(k+(\alpha+\beta+1)/2)\phi\, \frac{(\cos\phi-\cos\theta)^{\alpha-1/2}}{(1+\cos\phi)^{(\alpha+\beta)/2}} \\ &\quad \times {}_2F_1\Big[\frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta}{2}; \alpha+\frac{1}{2}; \frac{\cos\phi-\cos\theta}{1+\cos\phi}\Big] d\phi, \end{aligned} \tag{5}$$

which is valid for $\alpha > -1/2,\ 0 < \theta < \pi$. Inserting this integral representation into I_k and interchanging the order of integration we find that

$$I_k = C \int_0^{\pi/2} g(\phi) \cos\left(k + (\alpha+\beta+1)/2\right)\phi \, d\phi$$

with

$$g(\phi) = \Big(\cos\frac{\phi}{2}\Big)^{-\alpha-\beta} \int_\phi^{\pi/2} f(\theta)(\cos\phi - \cos\theta)^{\alpha-1/2}\Big(\sin\frac{\theta}{2}\Big)\Big(\cos\frac{\theta}{2}\Big)^{2\beta+1}$$
$$\times\, {}_2F_1\Big[\frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta}{2}; \alpha+\frac{1}{2}; \frac{\cos\phi - \cos\theta}{1+\cos\phi}\Big]\, d\theta.$$

Now notice that $0 < \cos\theta < \cos\phi < 1$ and $(\cos\phi - \cos\theta)/(1+\cos\phi) < 1/2$ when $0 < \phi < \theta < \pi/2$. Thus the ${}_2F_1$ function in the above integrand is uniformly bounded and

$$\int_0^{\pi/2} |g(\phi)|\, d\phi \le C \int_0^{\pi/2}\int_\phi^{\pi/2} |f(\theta)|(\cos\phi - \cos\theta)^{\alpha-1/2}\Big(\sin\frac{\theta}{2}\Big)\Big(\cos\frac{\theta}{2}\Big)^{2\beta+1} d\theta\, d\phi$$
$$\le C \int_0^{\pi/2} |f(\theta)|\, h(\theta)\Big(\sin\frac{\theta}{2}\Big)^{2\alpha+1}\Big(\cos\frac{\theta}{2}\Big)^{2\beta+1} d\theta$$

with

$$h(\theta) = (\sin\theta)^{-2\alpha}\int_0^\theta (\cos\phi - \cos\theta)^{\alpha-1/2}\, d\phi.$$

Since

$$\begin{aligned}
h(\theta) &\le C(\sin\theta)^{-2\alpha}\int_0^\theta \Big(\sin\frac{\theta+\phi}{2}\,\sin\frac{\theta-\phi}{2}\Big)^{\alpha-1/2} d\phi \\
&\le C(\sin\theta)^{-\alpha-1/2}\Big[\int_0^{\theta/2} + \int_{\theta/2}^{\theta}\Big]\Big(\sin\frac{\theta-\phi}{2}\Big)^{\alpha-1/2} d\phi \\
&\le C(\sin\theta)^{-1}\int_0^{\theta/2} d\phi \,+\, C(\sin\theta)^{-\alpha-1/2}\int_{\theta/2}^{\theta}(\theta-\phi)^{\alpha-1/2} d\phi \le C
\end{aligned}$$

when $\alpha > -1/2$ and $0 < \theta \le \pi/2$, it follows that $g(\phi)$ is integrable on $[0, \pi/2]$ and hence $I_k \to 0$ as $k \to \infty$ by (1).

It remains to consider the case $\alpha = -1/2, -1 < \beta < -1/2$. To handle this case we first observe that by letting α decrease to $-1/2$ in (5) and proceeding as in the derivation of formula (6.13) in [5] we obtain that

$$\begin{aligned}
R_k^{(-1/2,\beta)}(\cos\theta) &= \Big(\cos\frac{\theta}{2}\Big)^{-\beta-1/2}\cos\left(k+\beta/2+1/4\right)\theta \\
&\quad + \frac{1}{4}\Big(\beta^2 - \frac{1}{4}\Big)\Big(\sin\frac{\theta}{2}\Big)\int_0^\theta \Big(\cos\frac{\phi}{2}\Big)^{-\beta-3/2}\cos\left(k+\beta/2+1/4\right)\phi \\
&\qquad \times\, {}_2F_1\Big[\beta/2+5/4, \beta/2+3/4; 2; \frac{\cos\phi-\cos\theta}{1+\cos\phi}\Big]\, d\phi
\end{aligned} \tag{6}$$

for $0 < \theta < \pi$. Since the series ${}_2F_1[\beta/2+5/4, \beta/2+3/4; 2; x]$ converges at $x = 1$ when $\beta < 0$, it converges uniformly on $[0, 1]$, and thus, observing that

$$0 < \frac{\cos\phi - \cos\theta}{1+\cos\phi} < 1\,, \qquad 0 < \phi < \theta < \pi,$$

it is clear that the ${}_2F_1$ series in the above integral is uniformly bounded when $\beta < 0$. Hence, for $-1 < \beta < -1/2$, the use of (6) in (2) gives

$$\hat{f}_{(-1/2,\beta)}(k) = c_1 \int_0^\pi f(\theta)\Big(\cos\frac{\theta}{2}\Big)^{\beta+1/2} \cos{(k+\beta/2+1/4)\theta}\, d\theta$$
$$+c_2 \int_0^\pi f(\theta)\Big(\sin\frac{\theta}{2}\Big)\Big(\cos\frac{\theta}{2}\Big)^{2\beta+1}\Big[\int_0^\theta \Big(\cos\frac{\phi}{2}\Big)^{-\beta-3/2} \cos{(k+\beta/2+1/4)\phi}$$
$$\times\, {}_2F_1\Big[\beta/2+5/4, \beta/2+3/4; 2; \frac{\cos\phi-\cos\theta}{1+\cos\phi}\Big] d\phi\Big]\, d\theta = M_k + N_k,$$

say. Since $\|f\|_{L_{(-1/2,\beta/2-1/4)}} \le \|f\|_{L_{(-1/2,\beta)}}$ for $\beta \le -1/2$, $M_k \to 0$ as $k \to \infty$ by the Riemann–Lebesgue Lemma (1).
Concerning N_k, after an interchange of integration one arrives at

$$N_k = c_2 \int_0^\pi \cos{(k+\beta/2+1/4)\phi}\, \Big[\Big(\cos\frac{\phi}{2}\Big)^{-\beta-3/2} \int_\phi^\pi f(\theta)\Big(\sin\frac{\theta}{2}\Big)\Big(\cos\frac{\theta}{2}\Big)^{2\beta+1}$$
$$\times\, {}_2F_1\Big[\beta/2+5/4, \beta/2+3/4; 2; \frac{\cos\phi-\cos\theta}{1+\cos\phi}\Big] d\theta\Big]\, d\phi$$

and the assertion again follows by (1) once we have shown that $\int_0^\pi |[\ldots]|\, d\phi < \infty$. But this is immediate, since the occurring ${}_2F_1$ function is uniformly bounded,

$$\Big|\int_\phi^\pi \ldots\, d\theta\Big| \le C \int_0^\pi |f(\theta)|\Big(\cos\frac{\theta}{2}\Big)^{2\beta+1} d\theta < \infty, \qquad 0 < \phi < \pi,$$

and

$$\int_0^\pi \Big(\cos\frac{\phi}{2}\Big)^{-\beta-3/2} d\phi < \infty\,, \qquad \beta < -1/2.$$

Thus the Lemma is established.

Remarks. 1) Since $P_k^{(\alpha,\beta)}(-x) = (-1)^k P_k^{(\beta,\alpha)}(x)$ and $P_k^{(\alpha,\beta)}(1) = \binom{k+\alpha}{k} \approx (k+1)^\alpha$, the Lemma implies that if $f \in L_{(\alpha,\beta)}$ and $\max\{\alpha,\beta\} \ge -1/2$, then

$$\int_0^\pi f(\theta) P_k^{(\alpha,\beta)}(\cos\theta)\Big(\sin\frac{\theta}{2}\Big)^{2\alpha+1}\Big(\cos\frac{\theta}{2}\Big)^{2\beta+1} d\theta\; = o(k^{\max\{\alpha,\beta\}}), \qquad k \to \infty.$$

2) Notice that the above proof via the Dirichlet-Mehler type integral is an elementary one; it reduces the problem straight to the classical Riemann-Lebesgue Lemma for

Fourier coefficients and does not use any density properties of subspaces of $L_{(\alpha,\beta)}$. By making use of such properties we can give the following simpler proofs. It is well known that the subspaces of cosine polynomials, simple functions, and of step functions are dense in $L_{(\alpha,\beta)}$. Thus, if $f \in L_{(\alpha,\beta)}$ and $\varepsilon > 0$, then we can write $f = g + h$, where $\|h\|_{L_{(\alpha,\beta)}} < \varepsilon$ and g is a cosine polynomial, a simple function, or a step function. Now let $(\alpha,\beta) \in S$. Since

$$|\hat{h}_{(\alpha,\beta)}(k)| \leq \|h\|_{L_{(\alpha,\beta)}} < \varepsilon, \qquad k \in \mathbf{N_0},$$

because $|R_k^{(\alpha,\beta)}(\cos\theta)| \leq 1$ for $(\alpha,\beta) \in S$, to prove that $\hat{f}_{(\alpha,\beta)}(k) \to 0$ as $k \to \infty$ it suffices to show that $\hat{g}_{(\alpha,\beta)}(k) \to 0$ as $k \to \infty$. This is obvious when g is a cosine polynomial since $\hat{g}_{(\alpha,\beta)}(k) = 0$ for all sufficiently large k. If g is a simple function then, being bounded, it is square integrable with respect to the weight function $\left(\sin\frac{\theta}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta}{2}\right)^{2\beta+1}$ and the Parseval formula gives $\sum(k+1)^{2\alpha+1}|\hat{g}_{(\alpha,\beta)}(k)|^2 < \infty$, which implies that $\hat{g}_{(\alpha,\beta)}(k) \to 0$ as $k \to \infty$ when $(\alpha,\beta) \in S$. If g is a step function then it is a finite linear combination of characteristic functions $\chi(\theta)$ of subintervals of $(0,\pi)$, so that it remains to show for such χ that $\hat{\chi}_{(\alpha,\beta)}(k) \to 0$ as $k \to \infty$; but this easily follows by using the integral [3, 10.8 (38)] and the asymptotic expansion [11, (8.21.10)].

In the case of Laguerre expansions one does not have a Dirichlet–Mehler type formula at one's disposal. However, the preceding three arguments apply. To sketch this we introduce the Lebesgue space

$$L_{w(\alpha)} = \{f : \ \|f\|_{L_{w(\alpha)}} = \int_0^\infty |f(x)|e^{-x/2}x^\alpha \, dx < \infty\}, \qquad \alpha > -1,$$

and the normalized Laguerre polynomials $R_k^\alpha(x)$

$$R_k^\alpha(x) = L_k^\alpha(x)/L_k^\alpha(0), \qquad L_k^\alpha(0) = A_k^\alpha = \binom{k+\alpha}{k} = \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1)},$$

where $L_k^\alpha(x), k \in \mathbf{N}_0$, is the classical Laguerre polynomial of degree k and order α (see Szegö [11, p. 100]). Associate to f its formal Laguerre series

$$f(x) \sim (\Gamma(\alpha+1))^{-1} \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L_k^\alpha(x),$$

where the Fourier Laguerre coefficients of f are defined by

$$\hat{f}_\alpha(k) = \int_0^\infty f(x) R_k^\alpha(x) x^\alpha e^{-x} \, dx$$

when the integrals exist (for a more detailed description see, e.g., [7]). If $\alpha \geq 0$ then, by [3, 10.18 (14)],

$$|e^{-x/2}R_k^\alpha(x)| \leq 1, \qquad x \geq 0, \quad k \in \mathbf{N_0}.$$

Since polynomials and simple functions are dense in $L_{w(\alpha)}$, by proceeding as above it follows again that a Riemann-Lebesgue Lemma for Fourier-Laguerre coefficients holds:

$$\lim_{k\to\infty} \hat{f}_\alpha(k) = 0, \quad f \in L_{w(\alpha)}\,, \quad \alpha \geq 0\,.$$

This can also be proved by using the density of step functions in $L_{w(\alpha)}$ and the observation that, by [3, 10.12 (28)] and [11, (8.22.1)],

$$\int_0^a R_k^\alpha(x) e^{-x} x^\alpha \, dx = \frac{1}{\alpha+1}\, e^{-a} a^{\alpha+1} R_{k-1}^{\alpha+1}(a) \to 0, \qquad k \to \infty,$$

for each $a > 0$.

3) One could ask: Does a Riemann–Lebesgue Lemma also hold for the system $\{\sqrt{h_k^{(\alpha,\beta)}} R_k^{(\alpha,\beta)}(\cos\theta)\}$, which is orthonormal with respect to the weight function $(\sin\frac{\theta}{2})^{2\alpha+1}(\cos\frac{\theta}{2})^{2\beta+1}$? That this cannot be true for $\alpha > -1/2$ can be seen by an argument analogous to that at the beginning of the proof of the Lemma: introduce a corresponding linear functional, estimate its norm from below by considering a neighborhood of $x = 1$, and apply the uniform boundedness principle. This also shows (replace $(k+1)^{\alpha+1/2}$ by some $(k+1)^\varepsilon$, $\varepsilon > 0$) that a "better" result than that given in the Lemma, better in the sense that for general $f \in L_{(\alpha,\beta)}$ there is a certain rate of decrease of the Fourier–Jacobi coefficients $\hat{f}_{(\alpha,\beta)}(k)$, cannot hold.

4) For Fourier coefficients of a function $f \in L^1(-\pi,\pi)$ it is well known that they decrease faster for smoother functions (see, e.g., [12, (4.3), p. 45]). This phenomenon also occurs for Jacobi expansions. Let us illustrate this by considering a special case of Besov spaces investigated by Runst and Sickel [10] (for $\alpha \geq \beta \geq -1/2$): Let $\delta > 0$. We say that $f \in B^\delta_{1,\infty,\alpha,\beta}$ if

$$\|f\|_{B^\delta_{1,\infty,\alpha,\beta}} := \sup_{j\in\mathbf{N}_0} 2^{j\delta} \Big\| \sum_{k=0}^\infty \hat{f}_{(\alpha,\beta)}(k)\, \varphi_j(k)\, h_k^{(\alpha,\beta)} R_k^{(\alpha,\beta)}(\cos\theta) \Big\|_{L_{(\alpha,\beta)}} < \infty\,,$$

where $\varphi(x) \in C^\infty(\mathbf{R})$ has compact support in $(2^{j-1}, 2^{j+2})$ and is identicallly 1 for $2^j \leq x \leq 2^{j+1}$. Then for $n \in [2^j, 2^{j+1}]$ one obtains

$$|\hat{f}_{(\alpha,\beta)}(n)| \leq \Big\| \sum_{k=0}^\infty \hat{f}_{(\alpha,\beta)}(k)\, \varphi_j(k)\, h_k^{(\alpha,\beta)} R_k^{(\alpha,\beta)}(\cos\theta) \Big\|_{L_{(\alpha,\beta)}} \leq C n^{-\delta} \|f\|_{B^\delta_{1,\infty,\alpha,\beta}}$$

uniformly in j. Bavinck [1] introduced Lipschitz spaces based on the generalized Jacobi translation operator (see [5]). These coincide with the above Besov spaces (see [1, p. 374] and [10, Remark 15 and Theorem 5] and observe that the domain of the infinitesimal generator considered by Bavinck is just the domain of the square of the infinitesimal generator considered by Runst and Sickel).

5) In the same spirit we can extend the Riemann–Lebesgue Lemma for cosine transforms on the half-axis to Jacobi transforms. For $\alpha > -1$ and $\beta \in \mathbf{R}$ we denote the underlying space of measurable functions on $\mathbf{R}_+$ by

$$L_{(\alpha,\beta)}(\mathbf{R}_+) = \{f \,:\, \|f\|_{L_{(\alpha,\beta)}(\mathbf{R}_+)} := \int_0^\infty |f(t)|\Big(\sinh t\Big)^{2\alpha+1}\Big(\cosh t\Big)^{2\beta+1} dt < \infty\}\,.$$

Then the Jacobi transform of an $L_{(\alpha,\beta)}(\mathbf{R}_+)$ function f is defined by

$$\mathcal{J}^{(\alpha,\beta)}[f](\tau) = \frac{2^{2(\alpha+\beta+1)+1/2}}{\Gamma(\alpha+1)} \int_0^\infty f(t)\, \varphi_\tau^{(\alpha,\beta)}(t) \Big(\sinh t\Big)^{2\alpha+1}\Big(\cosh t\Big)^{2\beta+1} dt,$$

whenever the integral converges, where $\varphi_\tau^{(\alpha,\beta)}(t)$ is the Jacobi function defined by

$$\varphi_\tau^{(\alpha,\beta)}(t) = {}_2F_1\Big[\frac{1}{2}(\rho+i\tau), \frac{1}{2}(\rho-i\tau); \alpha+1; -(\sinh t)^2\Big]$$

with $\rho = \alpha+\beta+1$; see, e.g., Koornwinder [9]. Note that $\mathcal{J}^{(-1/2,-1/2)}[f]$ is just the cosine transform of f, see [9, (3.4)]. This time, one can reduce the problem to the classical Riemann-Lebesgue Lemma for the cosine transform: for $\alpha > -1/2$ Koornwinder [9, (2.21)] has shown the following integral representation of Dirichlet–Mehler type

$$\varphi_\tau^{(\alpha,\beta)}(t) = 2^{-\alpha+3/2} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1/2)\Gamma(1/2)} \frac{1}{(\sinh t)^{2\alpha}(\cosh t)^{\alpha+\beta}}$$

$$\times \int_0^t \cos\tau s\; (\cosh 2t - \cosh 2s)^{\alpha-1/2} {}_2F_1\Big[\alpha+\beta, \alpha-\beta; \alpha+\frac{1}{2}; \frac{\cosh t - \cosh s}{2\cosh t}\Big]\, ds\,.$$

From this integral it follows (cf. [9, p. 150]) that

$$|\varphi_\tau^{(\alpha,\beta)}(t)| \le C(1+t)e^{-(\alpha+\beta+1)t}\,, \quad t,\tau \in \mathbf{R}_+\,, \quad \alpha > -1/2. \tag{7}$$

Hence the Jacobi transform of a function $f \in L_{(\alpha,\beta)}(\mathbf{R}_+)$ exists as a uniformly bounded function of $\tau \in \mathbf{R}_+$ if $\alpha > -1/2$ and $\alpha+\beta > -1$. Then, by proceeding as in proof of the Lemma, the Riemann-Lebesgue Lemma for cosine transforms now implies that $\mathcal{J}^{(\alpha,\beta)}[f](\tau)$ vanishes at infinity when $\alpha > -1/2$ and $\alpha+\beta > -1$. This result can also be proved by using the density in $L_{(\alpha,\beta)}(\mathbf{R}_+)$ of finite linear combinations of characteristic functions of bounded intervals, [9, (2.10)], (7), and the method described in Remark 2. This result and the inequality in (7) can be extended to $\alpha = -1/2$, $\alpha+\beta > -1$ by using the $\alpha \searrow -1/2$ limit case of the above integral representation:

$$\varphi_\tau^{(-1/2,\beta)}(t) = (\cosh t)^{-\beta-1/2} \cos\tau t$$

$$+ (\frac{1}{4} - \beta^2)(\sinh t)(\cosh t)^{-\beta-1/2} \int_0^t \cos\tau s\; (\cosh t + \cosh s)^{-1}$$

$$\times {}_2F_1\Big[\frac{1}{2}+\beta, \frac{1}{2}-\beta; 2; \frac{\cosh t - \cosh s}{2\cosh t}\Big]\, ds\,.$$

We intend to consider the general complex parameter case and related problems in another paper.

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