Ultraspherical multipliers revisited

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Dedicated to László Leindler on the occasion of his 60th birthday

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Abstract. Sufficient ultraspherical multiplier criteria are refined in such a way that they are comparable with necessary multiplier conditions. Also new necessary conditions for Jacobi multipliers are deduced which, in particular, imply known Cohen type inequalities. Muckenhoupt's transplantation theorem is used in an essential way.

Key words. Ultraspherical polynomials, multipliers, necessary conditions, sufficient conditions, Cohen type inequalities, fractional differences

AMS(MOS) subject classifications. 33C45, 42A45, 42C10

1 Introduction

Quite sharp sufficient conditions for ultraspherical multipliers are contained in papers by Muckenhoupt and Stein [15], Bonami and Clerc [4], Connett and Schwartz [5], Gasper and Trebels [9] and Muckenhoupt [14]. In [11] we gave comparable necessary conditions for Jacobi multipliers with parameters $(\alpha, -1/2)$ in the "natural" weight case (see [14, p. 2] and below). It is the goal of this paper to develop necessary conditions for ultraspherical (Jacobi) multipliers and to weaken the sufficient ones in such a way that they are comparable with the necessary ones. This is done by decomposing the relevant functions into even and odd parts; thus, by the quadratic transformations in [18, (4.1.5)], reducing the problem of controlling the multiplier sequence $\{m_k\}$ to a discussion of the subsequence $\{m_{2k}\}$ in the Jacobi case $(\alpha, -1/2)$ with natural weight and of $\{m_{2k+1}\}$ for the parameters $(\alpha, 1/2)$ (with an additional weight). An essential tool is Muckenhoupt's [14] transplantation theorem.

To become more precise let us introduce some notation. In view of the above it is reasonable to work within the framework of Jacobi expansions — the conversion to the standard notation for ultraspherical polynomials given in [18, (4.7.1)] does not change the involved multiplier spaces.

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Fix $\alpha \geq \beta \geq -1/2$ and let $L^p_{(a,b)}\,, 1 \leq p < \infty\,$, denote the space of measurable functions on $[0, \pi]$ with finite norm

$$\|f\|_{L^p_{(a,b)}} = \Big(\int_0^\pi \Big| f(\theta) \Big|^p \Big(\sin \frac{\theta}{2} \Big)^{2a+1} \Big(\cos \frac{\theta}{2} \Big)^{2b+1} d\theta \Big)^{1/p}.$$

If $a = b$ we use the abbreviation $L^p_a = L^p_{(a,a)}$. The "natural" weight case for expansions in Jacobi polynomials (when there is a nice convolution structure) is the case when $a = \alpha,\ b = \beta$. Define the normalized Jacobi polynomials by $R_k^{(\alpha,\beta)}(x) = P_k^{(\alpha,\beta)}(x)/P_k^{(\alpha,\beta)}(1)$, where $P_k^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree k and order (α, β), see [18]. For $f \in L^1_{(\alpha,\beta)}$, its k-th Fourier–Jacobi coefficient $\hat{f}_{(\alpha,\beta)}(k)$ is defined by

$$\hat{f}_{(\alpha,\beta)}(k) = \int_0^\pi f(\theta) R_k^{(\alpha,\beta)}(\cos\theta) \Big(\sin \frac{\theta}{2} \Big)^{2\alpha+1} \Big(\cos \frac{\theta}{2} \Big)^{2\beta+1} d\theta\,.$$

Then f has an expansion of the form

$$f(\theta) \sim \sum_{k=0}^{\infty} \hat{f}_{(\alpha,\beta)}(k) h_k^{(\alpha,\beta)} R_k^{(\alpha,\beta)}(\cos\theta)\,,$$

where the normalizing factors $h_k^{(\alpha,\beta)}$ are given by $h_k^{(\alpha,\beta)} = \|R_k^{(\alpha,\beta)}(\cos\theta)\|^{-2}_{L^2_{(\alpha,\beta)}} \approx (k+1)^{2\alpha+1}$ (here the $\approx$ sign means that there are positive constants C, C' such that $C' h_k^{(\alpha,\beta)} \leq (k+1)^{2\alpha+1} \leq C h_k^{(\alpha,\beta)}$ holds).

A sequence $m = \{m_k\}_{k=0}^\infty \in l^\infty$ is called a multiplier on $L^p_{(a,b)}$ with respect to an expansion into Jacobi polynomials of order (α, β), notation $m \in M^p_{(\alpha,\beta);(a,b)}\,$, if for each $f \in L^p_{(a,b)}$ there exists a function $T_m f \in L^p_{(a,b)}$ with

$$T_m f(\theta) \sim \sum_{k=0}^{\infty} m_k \hat{f}_{(\alpha,\beta)}(k) h_k^{(\alpha,\beta)} R_k^{(\alpha,\beta)}(\cos\theta)\,, \quad \|T_m f\|_{L^p_{(a,b)}} \leq C \|f\|_{L^p_{(a,b)}}\,. \tag{1}$$

The smallest constant C independent of f for which this holds is called the multiplier norm of m and is denoted by $\|m\|_{M^p_{(\alpha,\beta);(a,b)}}$. If $\alpha = \beta$ and $a = b$, we write $M^p_{\alpha;a}$.

Now decompose a function $f \in L^p_\alpha$ into its even part f_e and its odd part f_o with respect to the line $\theta = \pi/2$:

$$f_e(\theta) = \{f(\theta) + f(\pi - \theta)\}/2\,, \quad f_o = f - f_e\,.$$

Obviously, this decomposition is unique and there holds for the Fourier–Jacobi coefficients (observe that $R_k^{(\alpha,\alpha)}(x)$ is even when k is even and odd when k is odd)

$$(f_e)\hat{}_{(\alpha,\alpha)}(k) = \begin{cases} \hat{f}_{(\alpha,\alpha)}(k) & , k \text{ even} \\ 0 & , k \text{ odd} \end{cases}\,, \quad (f_o)\hat{}_{(\alpha,\alpha)}(k) = \begin{cases} 0 & , k \text{ even} \\ \hat{f}_{(\alpha,\alpha)}(k) & , k \text{ odd.} \end{cases} \tag{2}$$

Furthermore,

$$\|f\|_{L^p_\alpha} \le \|f_e\|_{L^p_\alpha} + \|f_o\|_{L^p_\alpha} \le 2\|f\|_{L^p_\alpha}\,, \quad 1 \le p < \infty\,. \tag{3}$$

In particular, the uniqueness theorem shows that

$$f_e(\theta) \sim \sum_{k=0}^{\infty} \hat{f}_{(\alpha,\alpha)}(2k) h_{2k}^{(\alpha,\alpha)} R_{2k}^{(\alpha,\alpha)}(\cos\theta)\,, \tag{4}$$

$$f_o(\theta) \sim \sum_{k=0}^{\infty} \hat{f}_{(\alpha,\alpha)}(2k+1) h_{2k+1}^{(\alpha,\alpha)} R_{2k+1}^{(\alpha,\alpha)}(\cos\theta)\,. \tag{5}$$

Given a sequence $m = \{m_k\}$ it is clear by the above that its $M^p_{\alpha;\alpha}$–multiplier norm is equivalent to the multiplier norm of m restricted to the subspace of even L^p_α–functions (with respect to the line $\theta = \pi/2$) plus the multiplier norm of m restricted to the subspace of odd L^p_α–functions, i.e.,

$$\|m\|_{M^p_{\alpha;\alpha}} \approx \|m\|_{M^p_{\alpha;\alpha}|_{\mathrm{even}}} + \|m\|_{M^p_{\alpha;\alpha}|_{\mathrm{odd}}}\,. \tag{6}$$

We can now state our first theorem.

Theorem 1.1 *Assume $\alpha \ge -1/2$ and define subsequences m_e and m_o of a given sequence m by $(m_e)_k = m_{2k},\ (m_o)_k = m_{2k+1},\ k \in \mathbf{N}_0$.*

a) If $1 \le p < \infty$, then there holds

$$\|m\|_{M^p_{\alpha;\alpha}|_{\mathrm{even}}} \approx \|m_e\|_{M^p_{(\alpha,-1/2);(\alpha,-1/2)}}\,,$$

$$\|m\|_{M^p_{\alpha;\alpha}|_{\mathrm{odd}}} \approx \|m_o\|_{M^p_{(\alpha,1/2);(\alpha,(p-1)/2)}}\,,$$

whenever one side in each of the equivalences is finite.

b) If $1 < p < \infty$, then

$$\|m\|_{M^p_{\alpha;\alpha}} \approx \|m_e\|_{M^p_{(\alpha,-1/2);(\alpha,-1/2)}} + \|m_o\|_{M^p_{(\alpha,-1/2);(\alpha,-1/2)}}\,.$$

We will combine Theorem 1.1 with known sufficient criteria and necessary ones. To this end define the fractional difference operator of order $\mu,\ \mu \in \mathbf{R}$, with increment $\kappa \in \mathbf{N}$ by

$$\Delta^\mu_\kappa m_k = \sum_{j=0}^{\infty} A_j^{-\mu-1} m_{k+\kappa j}, \quad A_j^\mu = \frac{\Gamma(j+\mu+1)}{\Gamma(j+1)\Gamma(\mu+1)}\,,$$

whenever the series converges; when $\kappa = 1$ we write $\Delta^\mu = \Delta^\mu_1$. An application of the multiplier criteria from [9, Theorem 4], [11, (3.8)] as well as of Askey's [1]

Marcinkiewicz multiplier theorem for Jacobi expansions to the sequences m_e and m_o and the observation that

$$\|m\|_\infty^q + \sup_{N\in\mathbf{N}_0} \sum_{k=N}^{2N} |(k+1)^\mu \Delta_2^\mu m_k|^q \frac{1}{k+1} \approx$$

$$\|m\|_\infty^q + \sup_{N\in\mathbf{N}_0} \sum_{k=N}^{2N} |(k+1)^\mu \Delta^\mu (m_e)_k|^q \frac{1}{k+1} + \sup_{N\in\mathbf{N}_0} \sum_{k=N}^{2N} |(k+1)^\mu \Delta^\mu (m_o)_k|^q \frac{1}{k+1}$$

immediately lead to

Corollary 1.2 *Let $\alpha \geq -1/2$, $1 < p < \infty$, and let $\{m_k\} \in l^\infty$ be a given sequence.*

a) If m satisfies for $\mu > \max\{(2\alpha+2)|1/p-1/2|, 1/2\}$ the condition

$$\|m\|_\infty + \sup_{N\in\mathbf{N}_0} \Big(\sum_{k=N}^{2N} |(k+1)^\mu \Delta_2^\mu m_k|^2 \frac{1}{k+1} \Big)^{1/2} \leq D < \infty\,,$$

then $m \in M^p_{\alpha;\alpha}$ and $\|m\|_{M^p_{\alpha;\alpha}} \leq C\,D$.

b) If $m \in M^p_{\alpha;\alpha}$, then

$$\|m\|_\infty + \sup_{N\in\mathbf{N}_0} \Big(\sum_{k=N}^{2N} |(k+1)^\nu \Delta_2^\nu m_k|^{p'} \frac{1}{k+1} \Big)^{1/p'} \leq C \|m\|_{M^p_{\alpha;\alpha}}\,,$$

where $\nu \leq (2\alpha+1)|1/p-1/2|$ and $1/p + 1/p' = 1$.

c) If m satisfies the condition

$$\|m\|_\infty + \sup_{N\in\mathbf{N}_0} \left(\sum_{k=N}^{2N} |\Delta_2 m_k| \right) \leq D^* < \infty\,,$$

then $m \in M^p_{\alpha;\alpha}$ if $1 \leq (4\alpha+4)/(2\alpha+3) < p < (4\alpha+4)/(2\alpha+1) \leq \infty$ and $\|m\|_{M^p_{\alpha;\alpha}} \leq C\,D^*$.

The constants C in the above statements are independent of the sequences m.

Remarks 1. The smoothness gap between the sufficient conditions in a) and the necessary ones in b) is essentially the gap which occurs in Sobolev embedding theorems (for the analogous result in the case of difference operators with increment 1 see [8, Theorem 5 b]).

2. We note that, in particular, Corollary 1.2 b) for half–integers $\alpha = (n-2)/2$ contains necessary conditions for zonal multipliers for spherical harmonic expansions (for the relevant notation and sufficient criteria see Strichartz [17]).

3. Using the method which leads to Theorem 1.1 and its Corollary one can easily deduce weighted analogs. In the case $p = 2$ and $c > 0$ there holds

$$\|m\|_{M^2_{\alpha;\alpha+c}} \approx \|m_e\|_{M^2_{(\alpha,-1/2);(\alpha+c,-1/2)}} + \|m_o\|_{M^2_{(\alpha,-1/2);(\alpha+c,-1/2)}}$$

provided the hypotheses of Muckenhoupt's transplantation theorem are satisfied, i.e., $(c+1-\alpha)/2$ is not a positive integer and the multipliers are defined on those subspaces of $L^p_{\alpha+c}$–functions for which $\hat{f}_{(\alpha,\alpha)}(k) = 0,\ 0 \le k \le \max\{0,\ [(c+1-\alpha)/2]\} - 1$. Now a characterization of multipliers for trigonometric series on weighted $L^2(-\pi,\pi)$–spaces, due to Muckenhoupt, Wheeden and Young [16, Theorems 10.1 and 10.2], (restricted to even functions) can be used to give

$$\|m\|_{M^2_{\alpha;\alpha+c}} \approx \|m\|_\infty + \sup_{N\in\mathbf{N}_0} \Big(\sum_{k=N}^{2N} |(k+1)^c \Delta_2^c m_k|^2 \frac{1}{k+1} \Big)^{1/2},$$

provided c satisfies the condition $l + 1/2 < c < l + 3/2,\ l \in \mathbf{N}_0$.

2 Proof of Theorem 1.1

The reduction of ultraspherical multipliers to Jacobi multipliers with $\beta = -1/2$ or $1/2$ is accomplished by the transformation formulas in [18, (4.1.5)],

$$R^{(\alpha,\alpha)}_{2k}(\cos\theta) = R^{(\alpha,-1/2)}_k(\cos 2\theta), \quad R^{(\alpha,\alpha)}_{2k+1}(\cos\theta) = \cos\theta\, R^{(\alpha,1/2)}_k(\cos 2\theta). \tag{7}$$

The relevant Fourier–Jacobi coefficients are connected in the following way:

$$2^{2\alpha+1}(f_e)\hat{}_{(\alpha,\alpha)}(2k) = [f_e(\theta/2)]\hat{}_{(\alpha,-1/2)}(k) =: A_k, \quad k \in \mathbf{N}_0, \tag{8}$$

$$2^{2\alpha+1}(f_o)\hat{}_{(\alpha,\alpha)}(2k+1) = [f_o(\theta/2)/\cos(\theta/2)]\hat{}_{(\alpha,1/2)}(k) =: B_k, \quad k \in \mathbf{N}_0. \tag{9}$$

Furthermore, elementary computations give

$$h^{(\alpha,\alpha)}_{2k} = 2^{2\alpha+1} h^{(\alpha,-1/2)}_k, \quad \|f_e\|_{L^p_\alpha} \approx \|f_e(\theta/2)\|_{L^p_{(\alpha,-1/2)}}, \tag{10}$$

$$h^{(\alpha,\alpha)}_{2k+1} = 2^{2\alpha+1} h^{(\alpha,1/2)}_k, \quad \|f_o\|_{L^p_\alpha} \approx \|f_o(\theta/2)/(\cos(\theta/2))^{2/p}\|_{L^p_{(\alpha,1/2)}}. \tag{11}$$

This inserted in (4) and (5) leads for $f = f_e + f_o$ a cosine polynomial (i.e., a polynomial in powers of $\cos\theta$) to

$$f_e(\theta) = \sum_{k=0}^\infty A_k h^{(\alpha,-1/2)}_k R^{(\alpha,-1/2)}_k(\cos 2\theta), \tag{12}$$

$$f_o(\theta) = \sum_{k=0}^\infty B_k h^{(\alpha,1/2)}_k \cos\theta\, R^{(\alpha,1/2)}_k(\cos 2\theta). \tag{13}$$

Thus it follows from (12) and (10) that

$$\begin{aligned}
\|T_m f_e\|_{L^p_\alpha} &\approx \|\sum_{k=0}^{\infty} m_{2k} A_k h_k^{(\alpha,-1/2)}\, R_k^{(\alpha,-1/2)}(\cos 2\theta)\|_{L^p_\alpha} \\
&\approx \|\sum_{k=0}^{\infty} m_{2k} A_k h_k^{(\alpha,-1/2)}\, R_k^{(\alpha,-1/2)}(\cos \theta)\|_{L^p_{(\alpha,-1/2)}} \\
&\leq C\|m_e\|_{M^p_{(\alpha,-1/2);(\alpha,-1/2)}} \|f_e(\theta/2)\|_{L^p_{(\alpha,-1/2)}} \\
&\approx \|m_e\|_{M^p_{(\alpha,-1/2);(\alpha,-1/2)}} \|f_e\|_{L^p_\alpha}
\end{aligned}$$

which implies

$$\|\{m_k\}\|_{M^p_{\alpha;\alpha}\big|_{\text{even}}} \leq C\|m_e\|_{M^p_{(\alpha,-1/2);(\alpha,-1/2)}}\,.$$

The converse is proved analogously by just starting with $\|\sum m_{2k}\dots\|_{L^p_{(\alpha,-1/2)}}$; thus the even case of part a) is established.
Concerning the odd case, (13) and (11) give analogously

$$\begin{aligned}
\|T_m f_o\|_{L^p_\alpha} &\approx \|\sum_{k=0}^{\infty} m_{2k+1} B_k h_k^{(\alpha,1/2)} \left(\cos\frac{\theta}{2}\right)^{1-2/p} R_k^{(\alpha,1/2)}(\cos\theta)\|_{L^p_{(\alpha,1/2)}} \\
&\leq C\|m_o\|_{M^p_{(\alpha,1/2);(\alpha,(p-1)/2)}} \|f_o(\theta/2)(\cos(\theta/2))^{-2/p}\|_{L^p_{(\alpha,1/2)}} \\
&\approx \|m_o\|_{M^p_{(\alpha,1/2);(\alpha,(p-1)/2)}} \|f_o\|_{L^p_\alpha},
\end{aligned}$$

thus

$$\|m\|_{M^p_{\alpha;\alpha}\big|_{\text{odd}}} \leq C\|m_o\|_{M^p_{(\alpha,1/2);(\alpha,(p-1)/2)}}\,.$$

The converse inequality is shown along the same lines, thus Theorem 1.1 a) is established.

Concerning part b) we apply Muckenhoupt's transplantation theorem [14, p. 4] twice to obtain for any sequence $\{c_k\}$ of compact support, $\alpha \geq -1/2$, and $1 < p < \infty$, that

$$\|\sum_{k=0}^{\infty} c_k h_k^{(\alpha,-1/2)} R_k^{(\alpha,-1/2)}(\cos\theta)\|_{L^p_{(\alpha,-1/2)}} \approx \|\sum_{k=0}^{\infty} c_k h_k^{(\alpha,1/2)} R_k^{(\alpha,1/2)}(\cos\theta)\|_{L^p_{(\alpha,(p-1)/2)}}$$

which in particular implies

$$M^p_{(\alpha,-1/2);(\alpha,-1/2)} = M^p_{(\alpha,1/2);(\alpha,(p-1)/2)}\,, \quad 1 < p < \infty.$$

A combination of (6) with part a) now gives part b).

3 Necessary conditions for Jacobi multipliers

Here we give a second proof of Corollary 1.2 b) which has the advantage that it also gives an extension to the general Jacobi case (α,β), $-1/2 < \beta \le \alpha$; we note that the case $\beta = -1/2$ has already been discussed in [11]. On account of the duality $M^p_{(\alpha,\beta);(\alpha,\beta)} = M^{p'}_{(\alpha,\beta);(\alpha,\beta)}$, $1 < p < \infty$, we can restrict ourselves to the case $1 < p < 2$ without loss of generality in the following (the case $p = 2$ is trivial).

Theorem 3.1 *Let* $-1/2 < \beta \le \alpha$, $1 < p < 2$, $\nu = (2\beta+1)(1/p-1/2)$, *and* $\mu + \nu = (2\alpha+1)(1/p-1/2)$.

a) If $f \in L^p_{(\alpha,\beta)}$ *is a cosine polynomial, then for some constant* C *independent of* f *there holds*

$$\Big(\sum_{k=0}^{\infty} |\Delta_2^{\nu}\Delta^{\mu}(s_k^{(\alpha,\beta)}\hat f_{(\alpha,\beta)}(k))|^{p'}\Big)^{1/p'} \le C\|\sum_{k=0}^{\infty} h_k^{(\alpha,\beta)}\hat f_{(\alpha,\beta)}(k)R_k^{(\alpha,\beta)}(\cos\theta)\|_{L^p_{(\alpha,\beta)}} ,$$

where $s_k^{(\alpha,\beta)} = (h_k^{(\alpha,\beta)})^{1/2}$.

b) If $m \in M^p_{(\alpha,\beta);(\alpha,\beta)}$, *then*

$$\|m\|_\infty + \sup_{N\in\mathbf{N}}\Big(\sum_{k=N}^{2N} |(k+1)^{\mu+\nu}\Delta_2^{\nu}\Delta^{\mu}m_k|^{p'}\frac{1}{k+1}\Big)^{1/p'} \le C\|m\|_{M^p_{(\alpha,\beta);(\alpha,\beta)}} .$$

Remark 4. Part b) is a nearly best possible necessary multiplier condition: one regains (up to the critical index) the right unboundedness domain for the Cesàro means (see also the following remark); but this example leaves open the possiblity to increase ν at the expense of the exponent p', which would have to be replaced by some $q < p'$ (Sobolev embedding). That this is not possible is shown by the further example $m = \{i^k(k+1)^{-\sigma}\}$. An application of part b) to m (with p' replaced by $q \le p'$) yields that m cannot generate a bounded operator on L^p_α if $p < (2\alpha+1)/(\sigma+\alpha+1/2)$, which coincides with a result of Askey and Wainger [2, Theorem 4, ii)]. There it is also proved that m generates a bounded operator when $(2\alpha+1)/(\sigma+\alpha+1/2) < p \le 2$.

5. We recall that the particular case $\alpha = a > -1/2$, $\beta = b = -1/2$, $p < (4\alpha+4)/(2\alpha+3)$ of the general Cohen type inequality for Jacobi multipliers due to Dreseler and Soardi [6] is an immediate consequence of formula (3.8) in [11]. So it is not surprising that Theorem 3.1 b) in the case of a finite sequence $\{m_k\}_{k=0}^N$ now implies the corresponding result for $-1/2 < \beta \le \alpha$. Obviously, the dyadic sum in part b) can be estimated from below by the single term $k = N$ and $\Delta_2^\nu\Delta^\mu m_N = m_N$. A computation of the occurring $(N+1)$ powers immediately leads to

$$(N+1)^{(2\alpha+2)(1/p-1/2)-1/2}|m_N| \le C\|m\|_{M^p_{(\alpha,\beta);(\alpha,\beta)}} , \quad 1 < p < (4\alpha+4)/(2\alpha+3). \tag{14}$$

We mention that, by a different method, Kalneĭ [13] has obtained a lower bound for finite sequences in the case $p = 1,\ \alpha > -1/2,\ \alpha \geq \beta > -1$, which even reflects logarithmic divergence and in particular implies the missing case $p = 1$ in (14). Kalneĭ's lower bound is of different type than the one given in Theorem 3.1 b).

Proof. First we note (cf. [11, p. 249]) that for $\mu \geq 0$ and $0 \leq \theta \leq \pi$

$$\Delta^\mu \cos k\theta = \frac{1}{2}\sum_{j=0}^{\infty} A_j^{-\mu-1}(e^{i(k+j)\theta} + e^{-i(k+j)\theta})$$

$$= \frac{1}{2}(e^{ik\theta}(1-e^{i\theta})^\mu + e^{-ik\theta}(1-e^{-i\theta})^\mu) = (2\sin\frac{\theta}{2})^\mu \cos\left((k+\mu/2)\theta - \mu\pi/2\right)$$

$$= (2\sin\frac{\theta}{2})^\mu\{\cos(k+\mu/2)\theta\cos\mu\pi/2 + \sin(k+\mu/2)\theta\sin\mu\pi/2\}.$$

Analogously it follows that for $\nu \geq 0$ there holds

$$\Delta_2^\nu \cos(k+\mu/2)\theta = (2\sin\theta)^\nu \cos\left((k+\mu/2+\nu)\theta - \nu\pi/2\right),$$

$$\Delta_2^\nu \sin(k+\mu/2)\theta = (2\sin\theta)^\nu \sin\left((k+\mu/2+\nu)\theta - \nu\pi/2\right).$$

Hence we obtain the following (L^1, l^∞)–estimate for a trigonometric polynomial f

$$|\Delta_2^\nu\Delta^\mu a_k| \leq C\int_0^\pi |f(\theta)|\Big(\sin\frac{\theta}{2}\Big)^{\mu+\nu}\Big(\cos\frac{\theta}{2}\Big)^\nu d\theta\,, \tag{15}$$

where $a_k = \int_0^\pi f(\theta)\cos k\theta\, d\theta$. For a corresponding (L^2, l^2)–estimate we observe that

$$\Delta_2^\nu\Delta^\mu a_k = \int_0^\pi f(\theta)\Big(\sin\frac{\theta}{2}\Big)^{\mu+\nu}\Big(\cos\frac{\theta}{2}\Big)^\nu (g_1(\theta)\cos k\theta + g_2(\theta)\sin k\theta)\, d\theta \ =: C_k + D_k,$$

where the continuous functions g_l are linear combinations of $\sin c\theta$ and $\cos c\theta$, c denoting different constants depending only upon μ and ν. If we now consider functions f with $f(\theta)\Big(\sin\frac{\theta}{2}\Big)^{\mu+\nu}\Big(\cos\frac{\theta}{2}\Big)^\nu \in L^2(0,\pi)$ and observe that the systems $\{\sin k\theta\}$ and $\{\cos k\theta\}$ are essentially orthonormal, it follows by the Parseval formula that

$$\sum_{k=0}^\infty |\Delta_2^\nu\Delta^\mu a_k|^2 \leq C\left(\sum_{k=0}^\infty |C_k|^2 + \sum_{k=0}^\infty |D_k|^2\right) \leq C\int_0^\pi |f(\theta)\Big(\sin\frac{\theta}{2}\Big)^{\mu+\nu}\Big(\cos\frac{\theta}{2}\Big)^\nu|^2 d\theta\,. \tag{16}$$

The Riesz–Thorin interpolation theorem applied to (15) and (16) gives for $1 \leq p \leq 2$ the Hausdorff–Young type inequality

$$\Big(\sum_{k=0}^\infty |\Delta_2^\nu\Delta^\mu a_k|^{p'}\Big)^{1/p'} \leq C\Big(\int_0^\pi |\sum_{k=0}^\infty a_k\cos k\theta\Big(\sin\frac{\theta}{2}\Big)^{\mu+\nu}\Big(\cos\frac{\theta}{2}\Big)^\nu|^p d\theta\Big)^{1/p}. \tag{17}$$

If one transplants this inequality for the cosine expansion (which corresponds in the Jacobi setting to the parameters $(-1/2,-1/2)$) to arbitrary Jacobi expansions with parameters $(\alpha,\beta), -1/2<\beta\le\alpha$, then in L^p–spaces with natural weights one has to check the hypotheses in Muckenhoupt's transplantation theorem [14, p. 4]. For $1<p\le 2$ this leads to the restrictions $\nu=(2\beta+1)(1/p-1/2)>0$ (hence $\beta=-1/2$ is not admitted) and $\mu+\nu>0,\ \mu=2(\alpha-\beta)(1/p-1/2)$. Now choose

$$a_k=\int_0^\pi f(\theta)\Big(\sin\frac{\theta}{2}\Big)^{\alpha+1/2}\Big(\cos\frac{\theta}{2}\Big)^{\beta+1/2}\phi_k^{(\alpha,\beta)}(\theta)\,d\theta=s_k^{(\alpha,\beta)}\hat f_{(\alpha,\beta)}(k)\,,$$

where we use the Muckenhoupt notation [14, (2.2)]

$$\phi_k^{(\alpha,\beta)}(\theta)=t_k^{(\alpha,\beta)}P_k^{(\alpha,\beta)}(\cos\theta)\Big(\sin\frac{\theta}{2}\Big)^{\alpha+1/2}\Big(\cos\frac{\theta}{2}\Big)^{\beta+1/2}$$

with $t_k^{(\alpha,\beta)}=s_k^{(\alpha,\beta)}/P_k^{(\alpha,\beta)}(1)$. Then Muckenhoupt's transplantation theorem gives

$$\Big(\int_0^\pi\Big|\sum_{k=0}^\infty a_k\cos k\theta\Big|^p\Big(\sin\frac{\theta}{2}\Big)^{p(\mu+\nu)}\Big(\cos\frac{\theta}{2}\Big)^{p\nu}d\theta\Big)^{1/p}$$

$$\le C\Big(\int_0^\pi\Big|\sum_{k=0}^\infty a_k\phi_k^{(\alpha,\beta)}(\theta)\Big|^p\Big(\sin\frac{\theta}{2}\Big)^{p(\mu+\nu)}\Big(\cos\frac{\theta}{2}\Big)^{p\nu}d\theta\Big)^{1/p}$$

$$=C\Big(\int_0^\pi\Big|\sum_{k=0}^\infty \hat f_{(\alpha,\beta)}(k)h_k^{(\alpha,\beta)}R_k^{(\alpha,\beta)}(\cos\theta)\Big|^p\Big(\sin\frac{\theta}{2}\Big)^{2\alpha+1}\Big(\cos\frac{\theta}{2}\Big)^{2\beta+1}d\theta\Big)^{1/p}.$$

A combination with (17) gives part a) of Theorem 3.1.

Concerning part b), consider a C^∞–function $\chi(x)$ with

$$\chi(x)=\begin{cases}0 & \text{if}\quad 0\le x\le 1/2\\ 1 & \text{if}\quad 1\le x\le 4\\ 0 & \text{if}\quad x\ge 8\end{cases}\quad,\quad \chi_i(x)=\chi(2^{-i}x)\,,$$

and an associated test sequence $\{(s_k^{(\alpha,\beta)})^{-1}\chi_i(k)\}$. Then, by [3, Theorem 2], it is not hard to see that

$$\|\sum_{k=0}^\infty(s_k^{(\alpha,\beta)})^{-1}\chi_i(k)h_k^{(\alpha,\beta)}R_k^{(\alpha,\beta)}(\cos\theta)\|_{L^p_{(\alpha,\beta)}}\le C(2^i)^{(2\alpha+2)/p'-\alpha-1/2}\,. \tag{18}$$

By part a) and the hypothesis $m\in M^p_{(\alpha,\beta);(\alpha,\beta)}$ we have that

$$\Big(\sum_{k=2^i}^{2^{i+1}}|\Delta_2^\nu\Delta^\mu(m_k\chi_i(k))|^{p'}\Big)^{1/p'}\le\Big(\sum_{k=0}^\infty|\Delta_2^\nu\Delta^\mu(m_k\chi_i(k))|^{p'}\Big)^{1/p'}$$

$$\leq C\| \sum_{k=0}^{\infty} m_k (s_k^{(\alpha,\beta)})^{-1} \chi_i(k) h_k^{(\alpha,\beta)} R_k^{(\alpha,\beta)}(\cos\theta)\|_{L^p_{(\alpha,\beta)}}$$

$$\leq C\|m\|_{M^p_{(\alpha,\beta);(\alpha,\beta)}} \| \sum_{k=0}^{\infty} (s_k^{(\alpha,\beta)})^{-1} \chi_i(k) h_k^{(\alpha,\beta)} R_k^{(\alpha,\beta)}(\cos\theta)\|_{L^p_{(\alpha,\beta)}}$$

whence by (18)

$$\Big(\sum_{k=2^i}^{2^{i+1}} |(k+1)^{\mu+\nu} \Delta_2^{\nu} \Delta^{\mu} (m_k \chi_i(k))|^{p'} \frac{1}{k+1} \Big)^{1/p'} \leq C\|m\|_{M^p_{(\alpha,\beta);(\alpha,\beta)}}$$

with the right side independent of i. The final statement b) now follows along the lines of the proof of [12, Lemma 2.3].

Remark 6. Of course one can state analogous to part b) the same necessary conditions for the above considered cosine expansions in weighted L^p–spaces by applying once more Muckenhoupt's transplantation theorem. Observe that then multipliers only make sense for those functions whose first N coefficients of the cosine expansion vanish; here $N = \max\{[(\alpha+1/2)(1/p-1/2)+1/2p+1/2], 0\} + \max\{[(\beta+1/2)(1/p-1/2)+1/2p+1/2], 0\}$.

4 Criteria for integrable functions

First we consider the problem: Given a sequence $\{f_k\}$, what are sufficient conditions satisfied by $\{f_k\}$ such that the f_k are Fourier–Jacobi coefficients of an L^1_α–function f? Via the transformation formulas in (7) we briefly give improvements of known criteria. We start with a Parseval relation.

Proposition 4.1 *Fix* $\alpha \geq -1/2$ *and let* $f(\theta) = \sum \hat{f}_{(\alpha,\alpha)}(k) h_k^{(\alpha,\alpha)} R_k^{(\alpha,\alpha)}(\cos\theta)$ *be a finite sum (i.e., a polynomial in* $\cos\theta$*).*

a) If $-1/2 < \mu < \alpha + 2$*, then*

$$\int_0^{\pi} |f(\theta)|^2 (\sin\theta)^{2(\alpha+\mu)+1} d\theta \leq C \sum_{k=0}^{\infty} |\Delta_2^{\mu} \hat{f}_{(\alpha,\alpha)}(k)|^2 h_k^{(\alpha,\alpha)} .$$

b) If $\mu > -1$*, then the converse holds, i.e.*

$$\sum_{k=0}^{\infty} |\Delta_2^{\mu} \hat{f}_{(\alpha,\alpha)}(k)|^2 h_k^{(\alpha,\alpha)} \leq C \int_0^{\pi} |f(\theta)|^2 (\sin\theta)^{2(\alpha+\mu)+1} d\theta .$$

For the proof we have only to observe that by (12) and (13)

$$\int_0^{\pi} |f(\theta)|^2 (\sin\theta)^{2(\alpha+\mu)+1} d\theta \approx \int_0^{\pi/2} \Big(|f_e(\theta)|^2 + |f_o(\theta)|^2\Big)(\sin\theta)^{2(\alpha+\mu)+1} d\theta$$

$$\approx \int_0^{\pi} \Big|\sum_{k=0}^{\infty} A_k h_k^{(\alpha,-1/2)} R_k^{(\alpha,-1/2)}(\cos\theta)\Big|^2 \Big(\sin\frac{\theta}{2}\Big)^{2(\alpha+\mu)+1} d\theta$$

$$+ \int_0^{\pi} \Big|\sum_{k=0}^{\infty} B_k h_k^{(\alpha,1/2)} R_k^{(\alpha,1/2)}(\cos\theta)\Big|^2 \Big(\sin\frac{\theta}{2}\Big)^{2(\alpha+\mu)+1} \Big(\cos\frac{\theta}{2}\Big)^2 d\theta\,.$$

Then [10, Theorem 1], whose proof extends to the case $\alpha, \beta \geq -1/2$, can be applied to the two terms of the right side, and the assertion follows after noting that

$$\sum_{k=0}^{\infty} |\Delta^{\mu} A_k|^2 h_k^{(\alpha,-1/2)} + \sum_{k=0}^{\infty} |\Delta^{\mu} B_k|^2 h_k^{(\alpha,1/2)} \approx \sum_{k=0}^{\infty} |\Delta_2^{\mu} \hat{f}_{(\alpha,\alpha)}(k)|^2 h_k^{(\alpha,\alpha)}\,.$$

Theorem 4.2 *Let $\alpha \geq -1/2$ and $\mu > \alpha + 1$. If $\{c_k\}$ is a bounded sequence with $\lim_{k\to\infty} c_k = 0$ and*

$$\sum_{j=1}^{\infty} \Big(\sum_{k=2^{j-1}}^{2^j-1} k^{-1}|c_k|^2\Big)^{1/2} + \sum_{j=1}^{\infty} \Big(\sum_{k=2^{j-1}}^{2^j-1} k^{-1}|k^{\mu}\Delta_2^{\mu} c_k|^2\Big)^{1/2} \leq K_{\{c_k\}}\,,$$

then there exists an $f \in L^1_\alpha$ with $\hat{f}_{(\alpha,\alpha)}(k) = c_k$ for all $k \in \mathbf{N}_0$ and $\|f\|_{L^1_\alpha} \leq C K_{\{c_k\}}$.

The proof follows from Proposition 4.1 a), analogous to that of [10, Theorem 2 a], or directly from [10, Theorem 2 a] by the same method used for Proposition 4.1.

Next we give another simple sufficient multiplier condition which is not comparable with Theorem 4.2 — see the discussion in [10].

Theorem 4.3 *Let $\alpha \geq -1/2$ and $\mu > \alpha + 1/2$. If $\{c_k\}$ is a bounded sequence with $\lim_{k\to\infty} c_k = 0$ and*

$$\sum_{k=0}^{\infty} (k+1)^{\mu} |\Delta_2^{\mu+1} c_k| \leq K_{\{c_k\}}\,,$$

then there exists an $f \in L^1_\alpha$ with $\hat{f}_{(\alpha,\alpha)}(k) = c_k$ for all $k \in \mathbf{N}_0$ and $\|f\|_{L^1_\alpha} \leq C K_{\{c_k\}}$.

Split the sequence $\{c_k\}$ into the two subsequences $\{a_k\}$ and $\{b_k\}$, where $a_k = c_{2k}$, $b_k = c_{2k+1}$, and observe that

$$\sum_{k=0}^{\infty} (k+1)^{\mu} |\Delta_2^{\mu+1} c_k| \approx \sum_{k=0}^{\infty} (k+1)^{\mu} |\Delta^{\mu+1} a_k| + \sum_{k=0}^{\infty} (k+1)^{\mu} |\Delta^{\mu+1} b_k|.$$

Now one can follow for each subsequence the proof of [11, Lemma 1]. The assumption there that the sequence has compact support is not used in [11, (3.5)]. First observe that

$$\int_0^\pi \Big|\sum_{j=0}^k (A^\mu_{k-j}/A^\mu_k) h_j^{(\alpha,-1/2)} R_j^{(\alpha,-1/2)}(\cos\theta)\Big| \Big(\sin\frac{\theta}{2}\Big)^{2\alpha+1} d\theta \le K, \quad \mu > \alpha+1/2,$$

which is proved in [18, (9.41.1)]. Thus the series

$$\sum_{k=0}^\infty A^\mu_k \Delta^{\mu+1} a_k \sum_{j=0}^k (A^\mu_{k-j}/A^\mu_k) h_j^{(\alpha,-1/2)} R_j^{(\alpha,-1/2)}(\cos\theta)$$

converges a.e. to a function $f_1 \in L^1_{(\alpha,-1/2)}$ with coefficients $(f_1)\hat{}_{(\alpha,-1/2)}(k) = a_k = c_{2k}$. Analogously one deals with the sequence $\{b_k\}$ to which one associates the function

$$f_2(\theta) = \sum_{k=0}^\infty A^\mu_k \Delta^{\mu+1} b_k \sum_{j=0}^k (A^\mu_{k-j}/A^\mu_k) h_j^{(\alpha,1/2)} R_j^{(\alpha,1/2)}(\cos\theta).$$

To deduce that $f_2 \in L^1_{(\alpha,0)}$ one needs the following boundedness result concerning the Cesàro kernel

$$\int_0^\pi \Big|\sum_{j=0}^k (A^\mu_{k-j}/A^\mu_k) h_j^{(\alpha,1/2)} R_j^{(\alpha,1/2)}(\cos\theta)\Big| \Big(\sin\frac{\theta}{2}\Big)^{2\alpha+1} \cos\frac{\theta}{2}\, d\theta \le K, \quad \mu > \alpha+1/2,$$

which follows by a slight modification of Szegö's proof — note that by the third case of [18, (7.34.1)], $\int_{\pi/2}^\pi |P_n^{(\alpha,1/2)}(\cos\theta)| \cos\theta/2\, d\theta = O(n^{-1/2})$, so that the right side estimate in [18, (9.41.2)]) remains valid, as does the rest of the proof in [18]. To complete the proof it suffices to set $f = f_e + f_o$ with $f_e(\theta) = f_1(2\theta)$ and $f_o(\theta) = \cos\theta\, f_2(2\theta)$, and to use (3) and (10) – (13).

Let us turn to the question of necessary conditions. As in Sec. 1 decompose a cosine polynomial $f \in L^1_\alpha$ into its even and odd parts with respect to the line $\theta = \pi/2$: $f = f_e + f_o$, and set $f_1(\theta) = f_e(\theta/2)$. Then, by (8), $A_k := (f_1)\hat{}_{(\alpha,-1/2)}(k) = 2^{2\alpha+1}(f_e)\hat{}_{(\alpha,\alpha)}(2k)$, and [11, (3.2)] gives for $\alpha \ge -1/2$, $\nu \ge 0$, that

$$\sup_k |(h_k^{(\alpha,-1/2)})^{1/2} \Delta^\nu A_k| \approx \sup_k |(h_{2k}^{(\alpha,\alpha)})^{1/2} \Delta_2^\nu (f_e)\hat{}_{(\alpha,\alpha)}(2k)|$$
$$\le C \int_0^\pi |(\sin\theta)^{\alpha+\nu+1/2} f_e(\theta)| d\theta.$$

Similarly, from the case $\alpha \ge -1/2, \beta = 1/2$ of [11, (3.2)] (it extends immediately to this case) applied to the function $f_2(\theta) = f_o(\theta/2)/\cos(\theta/2)$ and the fact that $B_k := (f_2)\hat{}_{(\alpha,1/2)}(k) = 2^{2\alpha+1}(f_o)\hat{}_{(\alpha,\alpha)}(2k+1)$ we get

$$\sup_k |(h_k^{(\alpha,1/2)})^{1/2} \Delta^\nu B_k| \approx \sup_k |(h_{2k+1}^{(\alpha,\alpha)})^{1/2} \Delta_2^\nu (f_o)\hat{}_{(\alpha,\alpha)}(2k+1)|$$

$$\leq C \int_0^\pi |(\sin\theta)^{\alpha+\nu+1/2} f_o(\theta)|\, d\theta.$$

Combining these two estimates gives the inequality

$$\sup_k |(h_k^{(\alpha,\alpha)})^{1/2} \Delta_2^\nu \hat{f}_{(\alpha,\alpha)}(k)| \leq C \int_0^\pi |(\sin\theta)^{\alpha+\nu+1/2} f(\theta)| d\theta. \tag{19}$$

Since $h_k^{(\alpha,\alpha)} \approx (k+1)^{2\alpha+1}$, application of the Riesz–Thorin theorem to (19) and Proposition 4.1 b) yields part a) of

Theorem 4.4 *a) Let $1 \leq p \leq 2, \alpha \geq -1/2$, and $\nu \geq 0$. If f is a cosine polynomial, then the Hausdorff–Young type inequality*

$$\Big(\sum_{k=0}^\infty |(k+1)^{\alpha+1/2} \Delta_2^\nu \hat{f}_{(\alpha,\alpha)}(k)|^{p'} \Big)^{1/p'} \leq C \Big(\int_0^\pi |(\sin\theta)^{\alpha+\nu+1/2} f(\theta)|^p \, d\theta \Big)^{1/p}$$

holds. Also, if $f \in L^1_\alpha$ then

$$\sup_{k\in\mathbf{N}_0} |(k+1)^{\alpha+1/2} \Delta_2^{\alpha+1/2} \hat{f}_{(\alpha,\alpha)}(k)| \leq C \|f\|_{L^1_\alpha},$$

which gives a necessary condition for a sequence to be the sequence of Fourier–Jacobi coefficients of an L^1_α–function.

b) If $0 < \nu < \alpha + 1/2$, then

$$\sum_{k=0}^\infty (k+1)^{\nu-1} |\Delta_2^\nu \hat{f}_{(\alpha,\alpha)}(k)| \leq C \|f\|_{L^1_\alpha}.$$

Remark 7. The case $p = 1$ of Part a) contains a Cohen type inequality for ultraspherical expansions. The assertion in part b) does not follow from part a): Observe that in general for $0 < \nu < \alpha + 1/2$ there only holds

$$\sup_{k\in\mathbf{N}_0} |(k+1)^\nu \Delta_2^\nu \hat{f}_{(\alpha,\alpha)}(k)| \leq C \sup_{k\in\mathbf{N}_0} |(k+1)^{\alpha+1/2} \Delta_2^{\alpha+1/2} \hat{f}_{(\alpha,\alpha)}(k)|$$

(consider e.g. the sequence $\{k^{i\gamma}\}, \gamma \in \mathbf{R}$, fixed); then it is clear that the estimate of part a) would lead to the diverging harmonic series $\sum 1/(k+1)$.

The proof of Theorem 4.4 b) is an immediate consequence of

Lemma 4.5 *Let $\alpha > -1/2$, $0 < \nu < \alpha + 1/2$, and $0 < \theta < \pi/2$. Then there holds*

a) $$|\Delta_2^\nu R_k^{(\alpha,\alpha)}(\cos\theta)| \leq C\, (\sin\theta)^\nu,$$

b) $$|\Delta_2^\nu R_k^{(\alpha,\alpha)}(\cos\theta)| \leq C\, (\sin\theta)^{\nu-\alpha-1/2} (k+1)^{-\alpha-1/2},$$

where in b) it is additionally assumed that $\nu \geq [\alpha+1/2]^*$ if $\alpha > 1/2$. Here we use the notation $[a]^*$, $a \in \mathbf{R}$, for the greatest integer smaller than a, $[a]^* < a$.

For suppose that Lemma 4.5 is true. Obviously,

$$\sum_{k=0}^{\infty}(k+1)^{\nu-1}|\Delta_2^{\nu}\hat{f}_{(\alpha,\alpha)}(k)| \le C\|f\|_{L^1_\alpha} \sup_{0\le\theta\le\pi}\sum_{k=0}^{\infty}(k+1)^{\nu-1}|\Delta_2^{\nu}R_k^{(\alpha,\alpha)}(\cos\theta)|\,.$$

On account of the symmetry of the ultraspherical polynomials (in each difference all the polynomials are even or all are odd with respect to the line $\theta = \pi/2$) we may take the supremum over $0 \le \theta \le \pi/2$. Now decompose the interval $[0,\pi/2]$ into intervals $I_j := [2^{-j-1}\pi, 2^{-j}\pi]\,,\ j \in \mathbf{N}$, and consider $\theta \in I_j$. If we set $N_j = [2^j/\pi]$, then $0 < C' \le N_j \sin\theta \le C < \infty$ uniformly in j. It then follows by the above two estimates that

$$\sup_{\theta\in I_j}\sum_{k=0}^{\infty}(k+1)^{\nu-1}|\Delta_2^{\nu}R_k^{(\alpha,\alpha)}(\cos\theta)|$$

$$\le C(\sin\theta)^{\nu}\sum_{k=0}^{N_j}(k+1)^{\nu-1} + C(\sin\theta)^{\nu-\alpha-1/2}\sum_{k=N_j}^{\infty}(k+1)^{\nu-\alpha-3/2} \le C.$$

The convergence of the last series is ensured by the hypothesis $\nu < \alpha + 1/2$; thus the assertion holds for $\alpha_0 := [\alpha+1/2]^* \le \nu < \alpha + 1/2$. The extension to all $\nu,\ 0 < \nu < \alpha+1/2$, is straightforward. It is clear that $|\hat{f}_{(\alpha,\alpha)}(k)| \le \|f\|_{L^1_\alpha}$ since $|R_k^{(\alpha,\alpha)}(\cos\theta)| \le 1$ for $\alpha > -1/2$. Thus proceeding as in [8, Lemma 1], in particular using the Andersen formula for bounded sequences $\{a_k\}$

$$\Delta^{\lambda+\kappa}a_k = \Delta^{\lambda}(\Delta^{\kappa}a_k), \quad \kappa \ge 0,\ \lambda > -1,\ \kappa+\lambda > 0,$$

one obtains for $\min\{0,\ \alpha_0 - 1\} < \nu < \alpha_0$

$$\sum_{k=0}^{\infty}(k+1)^{\nu-1}|\Delta^{\nu}\hat{f}_{(\alpha,\alpha)}(k)| \le C\sum_{k=0}^{\infty}A_k^{\nu-1}|\sum_{j=k}^{\infty}A_{j-k}^{\alpha_0-\nu-1}\Delta^{\alpha_0}\hat{f}_{(\alpha,\alpha)}(j)|$$

$$\le C\sum_{j=0}^{\infty}|\Delta^{\alpha_0}\hat{f}_{(\alpha,\alpha)}(j)|\sum_{k=0}^{j}A_k^{\nu-1}A_{j-k}^{\alpha_0-\nu-1} \le C\sum_{j=0}^{\infty}(k+1)^{\alpha_0-1}|\Delta^{\alpha_0}\hat{f}_{(\alpha,\alpha)}(j)|\,.$$

Iteration of this procedure gives the assertion.

Proof of Lemma 4.5. Mehler's integral [7, 10.9 (32)] and the formula for the fractional difference $\Delta_2^{\nu}\cos(k+\mu/2)\theta$ preceding (15) give

$$\Delta_2^{\nu}R_k^{(\alpha,\alpha)}(\cos\theta)$$

$$= C_\alpha(\sin\theta)^{-2\alpha}\int_0^{\theta}(\cos\phi-\cos\theta)^{\alpha-1/2}(2\sin\phi)^{\nu}\cos((k+\alpha+\nu+1/2)\phi-\nu\pi/2)\,d\phi\,.$$

Hence

$$|\Delta_2^\nu R_k^{(\alpha,\alpha)}(\cos\theta)| \le C(\sin\theta)^{-2\alpha}\int_0^\theta \Big(\sin\frac{\theta+\phi}{2}\,\sin\frac{\theta-\phi}{2}\Big)^{\alpha-1/2}(\sin\phi)^\nu d\phi$$

$$\le C(\sin\theta)^{-\alpha-1/2}\Big[\int_0^{\theta/2}+\int_{\theta/2}^\theta\Big]\Big(\sin\frac{\theta-\phi}{2}\Big)^{\alpha-1/2}(\sin\phi)^\nu d\phi$$

$$\le C(\sin\theta)^{-1}\int_0^{\theta/2}\phi^\nu d\phi + C(\sin\theta)^{\nu-\alpha-1/2}\int_{\theta/2}^\theta(\theta-\phi)^{\alpha-1/2}d\phi \le C(\sin\theta)^\nu$$

since $\alpha > -1/2$ and $0 < \theta < \pi/2$; thus part a) is established.

The case $\nu \in \mathbf{N}$ of part b) has already been shown by Kalneĭ [13, Lemma 3]. Since it is clear by part a) that b) holds for $0 < \theta \le c/(k+1)$ for fixed $c > 0$, without loss of generality we may assume $\pi/k < \theta \le \pi/2,\ k \ge 3$. Obviously,

$$\Delta_2^\nu R_k^{(\alpha,\alpha)}(\cos\theta) = C(\sin\theta)^{\nu-\alpha-1/2} I_{\alpha,\nu}(\theta;k),$$

where

$$I_{\alpha,\nu}(\theta;k) = \int_0^\theta \Big(\frac{\cos\phi-\cos\theta}{\sin\theta}\Big)^{\alpha-1/2}\Big(\frac{\sin\phi}{\sin\theta}\Big)^\nu \cos((k+\alpha+\nu+1/2)\phi-\nu\pi/2)\,d\phi\,.$$

Now write

$$\begin{aligned}&\cos((k+\alpha+\nu+1/2)\phi-\nu\pi/2)\\ &= \cos((\alpha+\nu+1/2)\phi-\nu\pi/2)\,\cos k\phi - \sin((\alpha+\nu+1/2)\phi-\nu\pi/2)\,\sin k\phi\,.\end{aligned}$$

The idea for obtaining the $(k+1)^{-\alpha-1/2}$ decrease is to interpret the preceding integral as cosine and sine coefficients of functions which satisfy appropriate L^1–Lipschitz conditions. Then formula (4.2) in [19, Chap. II] and iterated integrations by parts of sufficiently high order give the desired $(k+1)$–decrease.

Let us first look at the case $-1/2 < \alpha \le 1/2$ and set

$$G_{\alpha,\nu;\theta}(\phi) = \begin{cases} \Big(\frac{\cos\phi-\cos\theta}{\sin\theta}\Big)^{\alpha-1/2}\Big(\frac{\sin\phi}{\sin\theta}\Big)^\nu & \text{if} \quad 0\le\phi\le\theta\le\pi/2 \\ 0 & \text{if} \quad \theta<\phi<\pi\,. \end{cases}$$

Then

$$\begin{aligned} I_{\alpha,\nu}(\theta;k) &= \int_0^\pi G_{\alpha,\nu;\theta}(\phi)\cos((\alpha+\nu+1/2)\phi-\nu\pi/2)\cos k\phi\,d\phi \\ &\quad - \int_0^\pi G_{\alpha,\nu;\theta}(\phi)\sin((\alpha+\nu+1/2)\phi-\nu\pi/2)\sin k\phi\,d\phi\,. \end{aligned}$$

Since $\cos((\alpha+\nu+1/2)\phi-\nu\pi/2)$ and $\sin((\alpha+\nu+1/2)\phi-\nu\pi/2)$ are bounded C^∞–functions we can clearly neglect them when discussing the smoothness of $G_{\alpha,\nu;\theta}(\phi)$. Elementary, though tedious, computations show that

$$\int_0^\pi |G_{\alpha,\nu;\theta}(\phi+\delta)-G_{\alpha,\nu;\theta}(\phi)|\,d\phi \le C\delta^{\alpha+1/2}, \quad 0<\delta\le\theta, \quad \theta\ge\pi/k\,.$$

Now formula (4.2) in [19, Chap. II] (adapted for sine and cosine expansions) gives

$$I_{\alpha,\nu}(\theta;k)\le C(k+1)^{-\alpha-1/2}\,,$$

which yields assertion b) in the case $-1/2<\alpha\le 1/2$. If $1/2<\alpha\le 3/2$, an integration by parts leads to

$$I_{\alpha,\nu}(\theta;k) = -\frac{1}{k+\alpha+\nu+1/2}\int_0^\theta G'_{\alpha,\nu;\theta}(\phi)\sin((k+\alpha+\nu+1/2)\phi-\nu\pi/2)\,d\phi.$$

An examination of the derivative $G'_{\alpha,\nu;\theta}(\phi)$ shows that it has at least the same smoothness as $G_{\alpha-1,\nu;\theta}(\phi)$. Hence again $I_{\alpha,\nu}(\theta;k)\le C(k+1)^{-\alpha-1/2}$. An iteration of this procedure finally shows the assertion b) of Lemma 4.5 to be true for all $\alpha>-1/2$.

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