Problem 19 in the SIAM Activity Group on Orthogonal Polynomials and Special Functions Newsletter 8 (3) (1998), pp. 11–12

19. **Uniform bounds for shifted Jacobi multiplier sequences.** For Fourier series the following is immediate: Suppose the real or complex sequence $\{m_k\}$ generates a bounded operator on $L^p(\mathbf{T})$, $1 \le p \le \infty$, i.e., for polynomial f

$$\| \sum m_k \hat{f}_k e^{ik\varphi} \|_{L^p(\mathbf{T})} \le \|m\|_{M^p(\mathbf{T})} \| \sum \hat{f}_k e^{ik\varphi} \|_{L^p(\mathbf{T})} \,,$$

then one has for the shifted sequence $\{m_{k+j}\}_{k\in\mathbf{Z}}$ that

$$\sup_{j\in\mathbf{N}_0} \|\{m_{k+j}\}\|_{M^p(\mathbf{T})} \le C\,\|m\|_{M^p(\mathbf{T})}\,,\ 1 \le p \le \infty\,. \quad (1)$$

Looking at cosine expansions on $L^p(0,\pi)$ one easily derives the analog of (1) via the addition formula

$$\cos(k \pm j)\theta = \cos k\theta \cos j\theta \mp \sin k\theta \sin j\theta$$

provided the periodic Hilbert transform is bounded, i.e., for $1 < p < \infty$. More generally, by Muckenhoupt's transplantation theorem [2, Theorem 1.6],

$$\Big(\int_0^\pi \Big| \sum m_{k+j}\, a_k\, P_k^{(\alpha,\beta)}(\cos\theta) \Big|^p \sin^{2\alpha+1}\frac{\theta}{2} \cos^{2\beta+1}\frac{\theta}{2}\, d\theta \Big)^{1/p}$$

$$\equiv \Big(\int_0^\pi \Big| \sum m_{k+j}\, b_k\, \phi_k^{(\alpha,\beta)}(\cos\theta) \Big|^p w_{\alpha,\beta,p}(\theta)\, d\theta \Big)^{1/p}$$

$$\approx \Big(\int_0^\pi \Big| \sum m_{k+j}\, b_k\, \cos k\theta \Big|^p w_{\alpha,\beta,p}(\theta)\, d\theta \Big)^{1/p},$$

where $P_k^{(\alpha,\beta)}$ are the Jacobi polynomials, $\phi_k^{(\alpha,\beta)}(\cos\theta)$ are the orthonormalized Jacobi functions with respect to $d\theta$, and

$$w_{\alpha,\beta,p}(\theta) = \sin^{(2-p)(\alpha+1/2)}\frac{\theta}{2} \cos^{(2-p)(\beta+1/2)}\frac{\theta}{2}\,.$$

Therefore, the above argument for cosine expansions also applies to Jacobi expansions provided the periodic Hilbert transform is bounded with respect to the weight function $w_{\alpha,\beta,p}$; hence, the analog of (1) holds for Jacobi expansions when

$$\frac{2\alpha+2}{\alpha+3/2} < p < \frac{2\alpha+2}{\alpha+1/2}\,, \qquad \alpha \ge \beta \ge -\frac{1}{2}\,.$$

(i) Can the above p-range be extended? By Muckenhoupt [2, (1.3)], a fixed shift is bounded for all p, $1 < p < \infty$.

(ii) Consider the corresponding problem for Laguerre expansions (for the appropriate setting see [1]); a fixed shift is easily seen to be bounded for all $p \ge 1$.

Both questions are of course trivial for $p = 2$ since $\ell^\infty = M^2$ by Parseval's formula.

References

[1] Gasper, G. and W. Trebels: On necessary multiplier conditions for Laguerre expansions, Canad. J. Math. 43 (1991), 1228 – 1242.

[2] Muckenhoupt, B.: Transplantation Theorems and Multiplier Theorems for Jacobi Series, Memoirs Amer. Math. Soc., Vol. 64, No. 356, Providence, R.I., 1986.

(Submitted on May 19, 1998)

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