# On necessary multiplier conditions for Laguerre expansions

by

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*Dedicated to P.G. Rooney on the occasion of his 65th birthday.*

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Abstract

Necessary multiplier conditions for Laguerre expansions are derived and discussed within the framework of weighted Lebesgue spaces.

## 1 Introduction

The purpose of this paper is to enlighten the structure of multipliers for Laguerre expansions on $L^p$ spaces from the point of view of necessary conditions. From the theory of Hankel and Jacobi multipliers (see Gasper and Trebels [6], [7] ) it is known that necessary conditions may very well reflect the behavior of multipliers in so far as they are (up to a natural smoothness gap) comparable with sufficient conditions. Following Görlich and Markett [9] we consider the Lebesgue spaces

$$L^p_{w(\gamma)} = \{f :\| f \|_{L^p_{w(\gamma)}} = (\int_0^\infty |f(x)e^{-x/2}|^p x^\gamma \, dx)^{1/p} < \infty\} \, , \quad 1 \le p < \infty;$$

in particular, for $\gamma = \alpha p/2$, these are the $L^p_{u(\alpha)}$-spaces in [9]:

$$L^p_{u(\alpha)} = \{f :\| f \|_{L^p_{u(\alpha)}} = \left(\int_0^\infty |f(x)u(x,\alpha)|^p \, dx\right)^{1/p} < \infty\}, \; 1 \le p < \infty,$$

where $u(x,\alpha) = x^{\alpha/2}e^{-x/2}$. Let $L^\alpha_n(x),\ \alpha > -1,\ n \in \mathbf{N}_0$, be the classical Laguerre polynomials (see Szegö [19, p. 100]),

$$R^\alpha_n(x) = L^\alpha_n(x)/L^\alpha_n(0), \;\; L^\alpha_n(0) = A^\alpha_n = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$

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Define the Fourier Laguerre coefficients of a function $f \in L^p_{w(\gamma)}$ with respect to the orthogonal system $\{R^\alpha_n\}$ by

$$\hat{f}_\alpha(n) = \int_0^\infty f(x) R^\alpha_n(x) x^\alpha e^{-x}\, dx,$$

if the integrals exist. Then the formal Laguerre expansion of $f$ is given by

$$f(x) \sim (\Gamma(\alpha+1))^{-1} \sum_{k=0}^\infty \hat{f}_\alpha(k) L^\alpha_k(x).$$

A sequence $\{m_k\}$ is called a multiplier on $L^p_{w(\gamma)}$, notation $\{m_k\} \in M^p_{w(\gamma)}$, if

$$\| \sum_{k=0}^\infty m_k \hat{f}_\alpha(k) L^\alpha_k \|_{L^p_{w(\gamma)}} \le C \| f \|_{L^p_{w(\gamma)}}$$

for all polynomials $f$; the smallest constant $C$ for which this holds is called the multiplier norm $\| \{m_k\} \|_{M^p_{w(\gamma)}}$. Generic positive constants that are independent of the functions (and sequences) will be denoted by $C$. In the case of Laguerre multipliers on $L^p_{w(\alpha)}$ there seems to occur a surprising phenomenon: whereas for $4/3 \le p < 2$ the necessary conditions quite well reflect the boundedness behavior of the well understood example of the Cesàro means, there is a broadening (towards $p = 1$) growth/smoothness gap between our (at $p = 1$ best possible) necessary conditions and the Cesàro multiplier; it seems that the space $L^{4/3}_{w(\alpha)}$ plays a crucial role for the theory of Fourier Laguerre multipliers. The boundedness of the Cesàro means of the Laguerre expansion of $f$

$$(C,\delta)^\alpha_n(f,x) = (A^\delta_n)^{-1} \sum_{k=0}^n A^\delta_{n-k} \hat{f}_\alpha(k) L^\alpha_k(x)$$

is discussed in a number of papers by Askey and Wainger [2], Muckenhoupt [16], Poiani [17], Markett [12], and Görlich and Markett [9] ; e.g. there holds for $\alpha \ge 0$ and $\delta \ge 0$

$$\| (C,\delta)^\alpha_n f \|_{L^p_{w(\gamma)}} \le C \| f \|_{L^p_{w(\gamma)}}, \quad \delta > \begin{cases} 2|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2} & \text{if } \gamma = \alpha p/2 \\ (2\alpha+2)|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2} & \text{if } \gamma = \alpha \end{cases} \tag{1}$$

uniformly in $n$; by interpolation one easily gets results also for other $\gamma$-values. By Trebels [20, p.21] this implies in particular that any sequence $\{m_k\}$, converging to zero and being sufficiently smooth, is a multiplier on $L^p_{w(\gamma)}$, more precisely,

$$\| \sum_{k=0}^\infty m_k \hat{f}_\alpha(k) L^\alpha_k \|_{L^p_{w(\gamma)}} \le C \sum_{k=0}^\infty A^\delta_k |\Delta^{\delta+1} m_k| \, \| f \|_{L^p_{w(\gamma)}} \tag{2}$$

for all polynomials $f$ when $\delta$ and $\gamma$ satisfy the conditions in (1). Here the fractional difference of order $\delta$ is defined by

$$\Delta^\delta m_k = \sum_{j=0}^{\infty} A_j^{-\delta-1} m_{k+j}$$

whenever the sum converges. Within the setting of the $L^p_{w(\alpha)}$-spaces our main result reads

**Theorem 1.1** *If $f \in L^p_{w(\alpha)}$, $1 \leq p \leq 2$, and $\alpha > -1$, then*

$$\sup_k |(k+1)^{\lambda+(\alpha+1)/q} \Delta^\lambda \hat{f}_\alpha(k)| \leq C \parallel f \parallel_{L^p_{w(\alpha)}},$$

*provided*

*a)* $0 < \lambda \leq (2\alpha + \frac{4}{3}) \left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{3}$ *if* $1 \leq p < \frac{4}{3}$,

*b)* $0 < \lambda < (2\alpha + 2) \left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2}$ *if* $\frac{4}{3} \leq p < 2$.

This theorem and an extension of it are proved in Section 2. The proof relies heavily on the particularly simple formula for fractional differences of the $R_n^\alpha$ polynomials

$$\Delta^\lambda R_k^\alpha(x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\lambda+1)} x^\lambda R_k^{\alpha+\lambda}(x), \quad x > 0,\ \lambda > -(\alpha+1/2)/2, \tag{3}$$

which is just formula 6.15(4) in [4] when setting $c = \alpha+1, c' = \alpha+\lambda+1$ and observing that $R_k^\alpha(x) = {}_1F_1(-k; \alpha+1; x) = \Phi(-k, \alpha+1; x)$.

**Corollary 1.1** *Let $1 \leq p \leq 2$, $\lambda > 0$, and $\{m_k\}$ be a Fourier Laguerre multiplier sequence on $L^p_{w(\alpha)}$. Then*

$$\sup_k |(k+1)^\lambda \Delta^\lambda m_k| \leq C \parallel \{m_k\} \parallel_{M^p_{w(\alpha)}},$$

*provided $\lambda$ satisfies the conditions in Theorem 1.1.*

**Remarks.** 1) This result is best possible for $p = 1$ and $\alpha \geq 0$ in the sense that there is a uniformly bounded multiplier family which satisfies the above necessary condition only for $\lambda \leq \alpha + 1/3$. For consider the multiplier sequence $\{m_k(t)\}, m_k(t) = e^{-t/2} R_k^\alpha(t)$, which is uniformly bounded in $t > 0$ (see [9]). By (3) it follows that

$$|k^\lambda \Delta^\lambda m_k(t)| = C|k^\lambda t^\lambda e^{-t/2} R_k^{\alpha+\lambda}(t)| \approx |k^{-\alpha} t^\lambda e^{-t/2} L_k^{\alpha+\lambda}(t)|.$$

The sup-norm over $t$ of the last expression behaves like $k^{\lambda-\alpha-1/3}$ by the fourth case of Markett's Lemma 1 in [14], hence it diverges when $\lambda > \alpha + 1/3$.

2) Corollary 1.1 gives unboundedness of the Cesàro means in the $p$ interval $1 \leq p < 4/3$ only for $\delta < (2\alpha + 4/3)(1/p - 1/2) - 1/3$, whereas the correct critical index $\delta_c$ at which still divergence happens is $\delta_c = (2\alpha + 2)(1/p - 1/2) - 1/2$ (see [9]), i.e., there is a considerable gap between the real range of unboundedness and the one given by Corollary 1.1 in the case $1 \leq p < 4/3$ for the Cesàro test multiplier. This is in contrast to the Jacobi and Hankel multiplier case (see [6], [7]) where, except for the endpoint, the correct range for the unboundedness of the Cesàro means is given by the general necessary conditions. We note that for $4/3 \leq p \leq 2$ Corollary 1.1 gives divergence for $\delta < \delta_c$ with the right divergence order.

3) In summability theory for numerical series the following result is well known (see [22, p. 105]): The factor sequence $\{m_k\}$ maps each $C_\delta$ summable series $\sum u_k$ into a $C_\delta$ summable series $\sum m_k\, u_k$ if and only if the sequence is bounded and

$$\sum_{k=0}^{\infty} A_k^\delta |\Delta^{\delta+1} m_k| < \infty.$$

If one wants to discuss this problem in a Banach space setting (see [20]) one may decompose the Banach space $X$ when assuming the existence of a sequence of projections $\{P_k\}_{k\in\mathbf{N}_0} \subset [X]$, where $[X]$ is the space of all bounded linear operators from $X$ to $X$, with the following properties:

i) the projections are mutually orthogonal: $P_k P_j = \delta_{j,k} P_k$,

ii) they are total: $P_k f = 0$ for all $k$ implies $f = 0$,

iii) the linear span of the ranges $P_k(X)$ is dense in $X$.

iv) the Cesàro means

$$(C,\delta)_n f = (A_n^\delta)^{-1} \sum_{k=0}^{n} A_{n-k}^\delta P_k f$$

are uniformly bounded for some $\delta \geq 0$ :

$$\| (C,\delta)_n f \| \leq C \| f \| \quad \forall f \in X. \tag{4}$$

If we introduce multipliers analogous to the above Laguerre case, then an analog to the sufficient direction holds for such Cesàro bounded expansions (see [20, p. 21]). But one cannot expect that the converse is also true since concrete orthogonal expansions in general satisfy additional properties, e.g. they are $(C,\delta)$ bounded for all $\delta$ greater than a critical index but not for the critical index itself. Nevertheless, motivated by the case of Jacobi expansions (or Hankel transforms) one may look at the following problem in the above Banach space setting:

Suppose that

a) (4) holds for all $\delta > \delta_c > 0$,

b) for some $f \in X$ one has

$$\limsup_{n\to\infty} \| (C,\delta_c)_n f \| = \infty.$$

Is it true that the multiplier norm of a sequence $\{m_k\}$ can, up to a constant, be estimated from below by $\sup_k |k^\lambda \Delta^\lambda m_k|$ for all $\lambda,\ 0 < \lambda < \delta_c$?

Corollary 1.1 answers this question with no: the $(C, \alpha+1/2)^\alpha_n$ means of the above Laguerre expansion are not uniformly bounded (see [9]) so that according to (1) the critical index in $L^1_{w(\alpha)}$ is $\alpha+1/2$, whereas only $\lambda \le \alpha+1/3$ is admitted by the example in Remark 1.

4) According to a written communication of C. Markett there exists, apart from the obvious sufficient condition of type (2), the following unpublished result due to V. Dietrich, E. Görlich, G. Hinsen, and C. Markett

$$\| \sum_{k=0}^\infty m_k \hat{f}_\alpha(k) L_k^\alpha \|_{L^p_{w(\alpha)}} \le C \left\{ \sup_k |m_k| + \sup_n \left( \sum_{k=n}^{2n} |A_k^\gamma \Delta^\gamma m_k|^2 \frac{1}{k} \right)^{1/2} \right\} \| f \|_{L^p_{w(\alpha)}}$$

provided $1 < p < \infty$ and $\gamma \ge \alpha + 1 \ge 1$. This condition is comparable with the necessary one in Corollary 1.1 (see [5]); in particular, their combination gives

**Corollary 1.2** *If the sequence* $\{m_k\} \in M^p_{w(\alpha)}$ *for all* $\alpha \ge 0$ *and some fixed* $p \ne 2$, *then* $\{m_k\} \in M^p_{w(\alpha)}$ *for all* $\alpha \ge 0$ *and for all* $p$, $1 < p < \infty$.

For the proof observe that by duality one can assume without loss of generality that $1 < p < 2$. For fixed $p < 2$ and fixed $\alpha' \ge 0$ the necessary condition guarantees for the multiplier sequence in question a $\lambda$ smoothness of order greater than $\alpha' + 1$ since by hypothesis $\alpha$ may be chosen sufficiently large, and so application of the sufficient condition with respect to the parameter $\alpha'$ gives the assertion.

Better sufficient conditions would allow better transplantation theorems for multipliers with respect to Laguerre expansions of different parameters.

5) Corollary 1.1 may be extended by considering multipliers acting on $L^p_{w(\alpha)}$ into $L^r_{w(\alpha)}$, $p \le r$, i.e., more precisely, we say $m \in M^{p,r}_{w(\alpha)}$ if

$$\| \sum_{k=0}^\infty m_k \hat{f}_\alpha(k) L_k^\alpha \|_{L^r_{w(\alpha)}} \le C \| f \|_{L^p_{w(\alpha)}}$$

for all polynomials $f$, and define $\| m \|_{M^{p,r}_{w(\alpha)}}$ to be the smallest constant $C$ for which the preceding inequality holds.

**Corollary 1.3** *Let* $1 \le p \le r \le 2,\ \lambda > 0,$ *and* $\{m_k\} \in M^{p,r}_{w(\alpha)}$. *Then*

$$\sup_k |(k+1)^{\lambda+\sigma}\Delta^\lambda m_k| \le C \parallel \{m_k\} \parallel_{M^{p,r}_{w(\alpha)}},$$

*where* $1/r = 1/p - \sigma/(\alpha+1)$ *and*

*a)* $0 < \lambda \le (2\alpha + \frac{4}{3})\left(\frac{1}{r} - \frac{1}{2}\right) - \frac{1}{3}$ *if* $1 \le r < \frac{4}{3}$,

*b)* $0 < \lambda < (2\alpha + 2)\left(\frac{1}{r} - \frac{1}{2}\right) - \frac{1}{2}$ *if* $\frac{4}{3} \le r < 2$.

Corollary 1.3 nicely indicates how fractional integration (with multiplier sequence $\{(k+1)^{-\sigma}\}$) should work.

Theorem 1.1, Corollary 1.1, and Corollary 1.3 are proved in Section 2 along with some extensions. In Section 3 expansions with respect to the orthonormalized Laguerre functions

$$\mathcal{L}^\alpha_n(x) = (n!/\Gamma(n+\alpha+1))^{1/2} x^{\alpha/2} e^{-x/2} L^\alpha_n(x)$$

will be considered. We define modified Fourier Laguerre coefficients

$$\hat{f}_n = \int_0^\infty f(x)\mathcal{L}^\alpha_n(x) x^{\alpha/2} e^{-x/2}\, dx$$

(whenever the integrals exist, e.g., when $f \in L^p_{u(\alpha)},\ 1 \le p < \infty,\ \alpha \ge 0$) and have the expansion

$$f(x)x^{\alpha/2}e^{-x/2} \sim \sum_{n=0}^\infty \hat{f}_n \mathcal{L}^\alpha_n(x).$$

Since $\hat{f}_n = (A^\alpha_n/\Gamma(\alpha+1))^{1/2}\hat{f}_\alpha(n)$ we may state the standard Parseval formula in the following form

$$\frac{1}{\Gamma(\alpha+1)}\sum_{k=0}^\infty A^\alpha_k|\hat{f}_\alpha(k)|^2 = \sum_{k=0}^\infty |\hat{f}_k|^2 = \int_0^\infty |f(x)u(x,\alpha)|^2\, dx = \int_0^\infty |f(x)e^{-x/2}|^2 x^\alpha\, dx \tag{5}$$

whenever $f \in L^2_{u(\alpha)} = L^2_{w(\alpha)}$. In Section 3 it is shown that even though the associated multiplier spaces $M^p_{u(\alpha)}$ and $\mathcal{M}^p_{u(\alpha)}$ for expansions of functions in $L^p_{u(\alpha)}$ with respect to $L^\alpha_n$ and, respectively, to $\mathcal{L}^\alpha_n$ coincide, there is an interesting different $L^1_{u(\alpha)}$ behavior of the Cesàro kernel $\chi^{\alpha,\delta}_n(x)$ in (7) and the modified Cesàro kernel $k^{\alpha,\delta}_n(x)$ defined in Section 3.

## 2 Proofs and extensions

Theorem 1.1 is an immediate consequence of the $\gamma = \alpha$ case of the following basic

**Lemma 2.1** *Let* $\alpha > -1$, $f \in L^p_{w(\gamma)}$, $1 \le p \le 2$, *and* $\gamma > (\alpha+1)p/2 - 2/3$. *Then*

*a)*

$$\sup_k |(k+1)^{\gamma/p+1/(3p)} \Delta^{(2\gamma+4/3)/p-(\alpha+1)} \hat{f}_\alpha(k)| \le C \parallel f \parallel_{L^p_{w(\gamma)}}, \quad 1 \le p < 4/3,$$

*provided* $\gamma \ge -1/3$ *if* $p = 1$ *or* $\gamma > -1/3$ *if* $1 < p < 4/3$.

*b)*

$$\sup_k |(k+1)^{\alpha+\lambda+1-(\gamma+1)/p} \Delta^\lambda \hat{f}_\alpha(k)| \le C \parallel f \parallel_{L^p_{w(\gamma)}}, \quad 1 \le p \le 2,$$

*provided*

$$0 < \lambda < \begin{cases} (2\gamma+4/3)/p - (\alpha+1) & \text{if } 1 \le p < 4/3 \\ (2\gamma+2)/p - (\alpha+3/2) & \text{if } 4/3 \le p \le 2. \end{cases}$$

**Proof.** By the definition of the Fourier Laguerre coefficients, formula (3), and Hölder's inequality it follows that $(1/p + 1/q = 1)$

$$\begin{aligned} \Delta^\lambda \hat{f}_\alpha(k) &= \int_0^\infty f(t) \Delta^\lambda R_k^\alpha(t) e^{-t} t^\alpha \, dt \\ &= C \int_0^\infty f(t) R_k^{\alpha+\lambda}(t) e^{-t} t^{\alpha+\lambda} \, dt \ \left(= C \hat{f}_{\alpha+\lambda}(k)\right) \\ &\le C \parallel f \parallel_{L^p_{w(\gamma)}} \begin{cases} \left(\int_0^\infty |R_k^{\alpha+\lambda}(t) e^{-t/2} t^{\alpha+\lambda-\gamma}|^q t^\gamma \, dt\right)^{1/q} & \text{if } p > 1 \\ \sup_{t>0} |R_k^{\alpha+\lambda}(t) e^{-t/2} t^{\alpha+\lambda-\gamma}| & \text{if } p = 1 \ . \end{cases} \end{aligned}$$

The observation that $R_k^{\alpha+\lambda} = L_k^{\alpha+\lambda}/A_k^{\alpha+\lambda}$ and a direct application of Lemma 1 in [14] now give for $\lambda \ge 0$

$$\begin{aligned} &|\Delta^\lambda \hat{f}_\alpha(k)| \le C \parallel f \parallel_{L^p_{w(\gamma)}} \times \qquad (6) \\ &\times \begin{cases} k^{-\gamma/p-1/(3p)} & \text{if } \lambda \ge (2\gamma+4/3)/p - (\alpha+1),\ 1 \le p < 4/3 \\ k^{-(\alpha+\lambda+1)+(\gamma+1)/p} & \text{if } \lambda < (2\gamma+4/3)/p - (\alpha+1),\ 1 \le p < 4/3 \\ k^{1/2-(\gamma+1)/p} & \text{if } \lambda > (2\gamma+2)/p - (\alpha+3/2),\ 4/3 < p \le \infty \\ k^{1/2-(\gamma+1)/p} (\log k)^{1-1/p} & \text{if } \lambda = (2\gamma+2)/p - (\alpha+3/2),\ 4/3 \le p \le \infty \\ k^{-(\alpha+\lambda+1)+(\gamma+1)/p} & \text{if } \lambda < (2\gamma+2)/p - (\alpha+3/2),\ 4/3 \le p \le \infty, \end{cases} \end{aligned}$$

where, as usual, $k$ and $\log k$ on the right hand side are replaced by positive constants when $k = 0$ or $1$. The assertion of the Lemma is now evident.

In order to deduce necessary multiplier conditions from Lemma 2.1 we need on the one hand boundedness of the multipliers involved and on the other control over suitable test functions; the latter will be guaranteed by

**Lemma 2.2** *Let* $1 \leq p \leq 2, \gamma > -1$, *and let* $N$ *be a fixed integer greater than* $2(\gamma+1)/p - (\alpha+3/2)$. *If* $\{g_k\} \in l^\infty$ *has compact support and* $g(x) = \sum_{k=0}^\infty g_k L_k^\alpha(x)$, *then*

$$\| g \|_{L^p_{w(\gamma)}} \leq C \sum_{k=0}^\infty (k+1)^{N+(\alpha+1)-(\gamma+1)/p} |\Delta^{N+1} g_k|.$$

**Proof.** Start with the Cesàro kernel given by

$$\chi_n^{\alpha,\delta}(x) = (A_n^\delta \Gamma(\alpha+1))^{-1} \sum_{k=0}^n A_{n-k}^\delta L_k^\alpha(x) = (A_n^\delta \Gamma(\alpha+1))^{-1} L_n^{\alpha+\delta+1}(x). \tag{7}$$

Then $g$ may be represented as

$$g(x) = \Gamma(\alpha+1) \sum_{k=0}^\infty A_k^N (\Delta^{N+1} g_k) \chi_k^{\alpha,N}(x).$$

Since the third case of Lemma 1 in [14] gives

$$\| \chi_k^{\alpha,\delta} \|_{L^p_{w(\gamma)}} \leq C\,(k+1)^{(\alpha+1)-(\gamma+1)/p}, \quad \delta > 2(\gamma+1)/p - (\alpha+3/2), \tag{8}$$

when $1 \leq p \leq 2, \alpha+\delta > -2$, and $\gamma > -1$, Lemma 2.2 follows after taking the $L^p_{w(\gamma)}$-norm of $g(x)$.

Consider a monotone decreasing $C^\infty$-function $\phi(x)$ with

$$\phi(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 2 \\ 0 & \text{if } x \geq 4 \end{cases}, \quad \phi_i(x) = \phi(x/2^i).$$

Then

$$\sum_{j=0}^\infty (j+1)^{N+(\alpha+1)-(\gamma+1)/p} |\Delta^{N+1} \phi_i(j)| \leq C\,(2^i)^{(\alpha+1)-(\gamma+1)/p},$$

which can be easily verified by using a slight modification of Lemma 3.6 in [20], and it follows by applying Lemma 2.2 to the function

$$\Phi^{(i)}(x) = \sum_{j=0}^\infty \phi_i(j) L_j^\alpha(x)$$

that

$$\| \Phi^{(i)} \|_{L^p_{w(\gamma)}} \leq C\,(2^i)^{(\alpha+1)-(\gamma+1)/p} \tag{9}$$

when $1 \leq p \leq 2, \alpha > -1$ and $\gamma > -1$.

Let us turn to the problem of dominating the $l^\infty$-norm of the multiplier sequence in question by its multiplier norm. First observe that by the second and fifth case of formula (6) there holds

$$|\hat{f}_\alpha(k)| \leq C(k+1)^{(\gamma+1)/p-(\alpha+1)} \| f \|_{L^p_{w(\gamma)}}$$

if $\gamma > (\alpha+1)p/2 - 2/3$ when $1 \leq p < 4/3$, and if $\gamma > (\alpha+3/2)p/2 - 1$ when $4/3 \leq p \leq 2$. So if one considers $m = \{m_k\} \in M^p_{w(\gamma)}$ and replaces $\hat{f}_\alpha(k)$ by $m_k\phi_i(k)$ one obtains by the previous estimate and (9) that

$$\sup_{k \leq 2^{i+1}} |m_k| \leq \sup_k |m_k \phi_i(k)| \leq C \parallel m \parallel_{M^p_{w(\gamma)}}$$

with constant independent of $i$, thus

$$M^p_{w(\gamma)} \subset l^\infty \quad \text{when} \quad \gamma > \begin{cases} (\alpha+1)p/2 - 2/3 & \text{if } 1 \leq p < 4/3 \\ (\alpha+3/2)p/2 - 1 & \text{if } 4/3 \leq p \leq 2 \end{cases} \tag{10}$$

in the sense of continuous embedding.

**Lemma 2.3** *If $\alpha > -1$ and $m \in M^p_{w(\gamma)}$, $1 \leq p \leq 2$, then*

*a)* $\sup_k |(k+1)^{(2\gamma+4/3)/p-(\alpha+1)} \Delta^{(2\gamma+4/3)/p-(\alpha+1)} m_k| \leq C \parallel m \parallel_{M^p_{w(\gamma)}}$

*when $1 \leq p < 4/3$ and $\gamma > (\alpha+1)p/2 - 2/3$, and*

*b)* $\sup_k |(k+1)^\lambda \Delta^\lambda m_k| \leq C \parallel m \parallel_{M^p_{w(\gamma)}}$

*when $\lambda$ satisfies the conditions of Lemma 2.1 and $\gamma$ those of (10) .*

**Proof.** Set $\lambda = (2\gamma+4/3)/p - (\alpha+1)$. From Part a) of Lemma 2.1, it follows that

$$\begin{aligned} C(2^i)^{(\alpha+1)-(\gamma+1)/p} \parallel m \parallel_{M^p_{w(\gamma)}} &\geq C \parallel \sum m_k \phi_i(k) L^\alpha_k \parallel_{L^p_{w(\gamma)}} \\ \geq \sup_{2^{i-1} \leq k \leq 2^i} |k^{\gamma/p+1/3p} \Delta^\lambda (m_k \phi_i(k))| &\geq \sup_{2^{i-1} \leq k \leq 2^i} |k^{\gamma/p+1/3p} \Delta^\lambda m_k| \\ &- \sup_{2^{i-1} \leq k \leq 2^i} |k^{\gamma/p+1/3p} \sum_{j=2^i}^\infty A_j^{-\lambda-1} \{\phi_i(k+j) - 1\} m_{k+j}|. \end{aligned}$$

Hence

$$\parallel m \parallel_{M^p_{w(\gamma)}} \geq C 2^{i\lambda} \sup_{2^{i-1} \leq k \leq 2^i} |\Delta^\lambda m_k| - C \sup_k |m_k|$$

and therefore, by (10) ,

$$\sup_{2^{i-1} \leq k \leq 2^i} |k^\lambda \Delta^\lambda m_k| \leq C \parallel m \parallel_{M^p_{w(\gamma)}}$$

uniformly in $i$, whence Part a); Part b) follows analogously.

**Remarks.** 1) Corollary 1.1 is the $\gamma = \alpha$ case of Lemma 2.3. Corollary 1.3 can be derived analogously from Lemma 2.1 (with $\gamma = \alpha$) and (9) when observing that

$$\sup_k |(k+1)^{\lambda+\alpha+1-(\alpha+1)/r}\Delta^\lambda(m_k\phi_i(k))|$$
$$\leq C \parallel m \parallel_{M^{p,r}_{w(\alpha)}} \parallel \Phi^{(i)} \parallel_{L^p_{w(\alpha)}} \leq C(2^i)^{\alpha+1-(\alpha+1)/p} \parallel m \parallel_{M^{p,r}_{w(\alpha)}} .$$

For historical reasons (e.g., see the convolution structure in [8]) and for later use we state a special case ($\gamma = \alpha p/2$) of Lemmas 2.1 and 2.3 (using the notation $M^p_{u(\alpha)} := M^p_{w(\alpha p/2)}$ and $1/p + 1/q = 1$)

**Corollary 2.4** *If $\alpha > -1$ and $1 \leq p < 4/3$, then*

*a)* $\sup_k |(k+1)^{\lambda+\alpha/2+1/q}\Delta^\lambda \hat{f}_\alpha(k)| \leq C \parallel f \parallel_{L^p_{u(\alpha)}}, \quad 0 < \lambda \leq \frac{1}{3} - \frac{4}{3q}$,

*b)* $\sup_k |(k+1)^\lambda \Delta^\lambda m_k| \leq C \parallel m \parallel_{M^p_{u(\alpha)}}, \quad 0 < \lambda \leq \frac{1}{3} - \frac{4}{3q}$.

For sufficient multiplier conditions on $L^p_{u(\alpha)}$ comparable with (of the same type as) the necessary ones, see the Corollary for $n = 1$ in Długosz [3]. Using the transplantation result of Kanjin [10] one can improve Długosz's result to

$$\parallel \sum_{k=0}^\infty m_k \hat{f}(k) L_k^\alpha \parallel_{L^p_{u(\alpha)}} \leq C \left\{ \sup_k |m_k| + \sup_n \left( \sum_{k=n}^{2n} |(k+1)\Delta^1 m_k|^2 \frac{1}{k} \right)^{1/2} \right\} \parallel f \parallel_{L^p_{u(\alpha)}} .$$

for all $\alpha \geq 0$ and $1 < p < \infty$; namely, Kanjin's result implies

$$M^p_{u(\alpha)} = M^p_{u(0)} = M^p_{w(0)}, \quad \alpha \geq 0, \ 1 < p < \infty,$$

and the assertion follows by the above mentioned result of Dietrich, Görlich, Hinsen, and Markett.

2) There arises the question whether Lemma 2.3 can be improved by interpolation. Observe that from Lemma 2.1 with $p = 1$, $\gamma = (\alpha+\lambda)/2 - 1/6$ and $\lambda > 0$, we have

$$\sup_k |\sqrt{A_k^{\alpha+\lambda}}(k+1)^{1/6}\Delta^\lambda \hat{f}_\alpha(k)| \leq C \int_0^\infty |f(x)e^{-x/2}x^{(\alpha+\lambda)/2-1/6}|\, dx,$$

and from (5) with $\alpha$ replaced by $\alpha + \lambda$ and the formula in the first lines of the proof to Lemma 2.1 we have

$$\left( \sum_{k=0}^\infty |\sqrt{A_k^{\alpha+\lambda}}\Delta^\lambda \hat{f}_\alpha(k)|^2 \right)^{1/2} \leq C \left( \int_0^\infty |f(x)e^{-x/2}x^{(\alpha+\lambda)/2}|^2\, dx \right)^{1/2}.$$

Then application of the Stein and Weiss interpolation theorem (see [18]), where we set $Tf = \{Tf(k)\}$, $Tf(k) = \sqrt{A_k^{\alpha+\lambda}}\Delta^\lambda \hat{f}_\alpha(k)$, gives

$$\left( \sum_{k=0}^\infty |(k+1)^{(2/p-1)/6} Tf(k)|^q \right)^{1/q} \leq C \left( \int_0^\infty |f(x)e^{-x/2}x^{(\alpha+\lambda)/2-(2/p-1)/6}|^p\, dx \right)^{1/p}.$$

In particular this implies Part a) of the following

**Corollary 2.5** *Let $\alpha > 0$ , $1 \le p \le 2$, and $(\alpha+1)(1/p-1/2) > 1/4$. Then, with $\lambda = (2\alpha + 2/3)(1/p - 1/2)$,*

*a)* $$\left(\sum_{k=0}^{\infty} |(k+1)^{\lambda+\alpha/q}\Delta^{\lambda}\hat{f}_{\alpha}(k)|^q\right)^{1/q} \le C \parallel f \parallel_{L^p_{w(\alpha)}},$$

*b)* $$\sup_n \left(\sum_{k=n}^{2n} |(k+1)^{\lambda}\Delta^{\lambda} m_k|^q \frac{1}{k}\right)^{1/q} \le C \parallel m \parallel_{M^p_{w(\alpha)}} .$$

Part b) follows along the lines of the proof of Lemma 2.3; observe that the restriction on $p$ comes from (10). Part b) does not contain Corollary 1.1 and vice versa, which may be seen by the examples of the Cesàro multiplier family $\{A^{\delta}_{n-k}/A^{\delta}_n\}$ (where, e.g., at $p = 4/3$ Corollary 1.1 gives a greater $\delta$-domain where divergence happens) and sequences of type $\{e^{ik}/(k+1)^{\nu}\}$ (where, e.g., at $p = 4/3$ Corollary 2.5 leads to a greater $\nu$-domain in which this sequence cannot be a multiplier). The embedding results in [5] lead to the conjecture that $\lambda = (2\alpha + 1/3)(1/p - 1/2) + 1/6$ for $1 < p < 4/3$ and $\lambda = (2\alpha + 1)(1/p - 1/2)$ for $4/3 \le p < 2$ should be the best possible $\lambda$-parameters. One possibility to get these is to try to improve the inequality in Corollary 2.5 a) at the point $p = 4/3$.

3) Formula (3) is equivalent to the Laguerre expansion (9) in Askey [1] after the latter is corrected by replacing the ratio $\Gamma(n-k+\gamma-\alpha+1)/\Gamma(\gamma-\alpha+1)$ in it by $\Gamma(n-k+\gamma-\alpha)/\Gamma(\gamma-\alpha)$. By arguing as on pages 251 – 252 of Tricomi [21] it can be shown that the fractional difference formula in (3) also holds for $x > 0$ when $\lambda > -1 - \min(\alpha, \alpha/2 - 1/4)$. When the more restrictive condition $\lambda > -(\alpha+1/2)/2$ is satisfied, the infinite series for the function $\Delta^{\lambda} R^{\alpha}_k(x)$ on the left side of (3) converges absolutely for $x > 0$ .

# 3 Expansions with respect to the orthonormalized Laguerre functions

The orthonormalized Laguerre functions were introduced at the end of the Introduction. A multiplier sequence in this new setting, notation $\{m_k\} \in \mathcal{M}^p_{u(\alpha)}$, satisfies

$$\left(\int_0^{\infty} |\sum_{k=0}^{\infty} m_k \hat{f}_k \mathcal{L}^{\alpha}_k(x)|^p \, dx\right)^{1/p} \le C \parallel f \parallel_{L^p_{u(\alpha)}}$$

for all polynomials $f$. Since $\Gamma(\alpha+1)\hat{f}_k\mathcal{L}^{\alpha}_k(x) = \hat{f}_{\alpha}(k)L^{\alpha}_k(x)u(x,\alpha)$ it is clear that $\mathcal{M}^p_{u(\alpha)} = M^p_{u(\alpha)}$ and thus, that Corollary 2.4 b) holds. But it is not obvious that an analogue of Corollary 2.4 a) holds with $\hat{f}_\alpha(k)$ replaced by $\hat{f}_k$ . For consider the modified Cesàro kernel

$$k_n^{\alpha,\delta}(x) = (A_n^\delta)^{-1} \sum_{j=0}^n A_{n-j}^\delta \mathcal{L}_j^\alpha(x)$$

which differs from (7), apart from the weight $u(x,\alpha)$, by the additional factor $(\Gamma(\alpha+1)A_j^\alpha)^{-1/2}$ inside the sum. Since (8) implies

$$\sup_n (A_n^\alpha)^{-1/2} \parallel \chi_n^{\alpha,\delta} \parallel_{L^1_{u(\alpha)}} \leq C, \quad \delta > 1/2, \tag{11}$$

the following lemma comes as a surprise.

**Lemma 3.1** *For $\alpha > 0$ and $\delta \geq 0$ there holds*

$$\sup_n \int_0^\infty |k_n^{\alpha,\delta}(x)|\, dx = \infty.$$

Observe however that on account of (1) there still holds $\{A_{n-j}^\delta / A_n^\delta\} \in \mathcal{M}^1_{u(\alpha)} = M^1_{u(\alpha)}$ with its multiplier norm uniformly bounded in $n$. Let us first give an upper bound for $\int_0^\infty |k_n^{\alpha,\delta}(x)|\, dx$ in the case $\delta \geq 1$. Since

$$\begin{aligned} k_n^{\alpha,\delta}(x) &= \sum_{k=0}^n (A_{n-k}^\delta / A_n^\delta) \mathcal{L}_k^\alpha(x) \\ &= \sum_{k=0}^n m_k (A_{n-k}^\delta / A_n^\delta) L_k^\alpha(x) u(x,\alpha), \end{aligned}$$

with $m_k = (\Gamma(\alpha+1)A_k^\alpha)^{-1/2}$, we have that

$$k_n^{\alpha,\delta}(x) = \Gamma(\alpha+1)(A_n^\delta)^{-1} \sum_{k=0}^n A_k^1 \Delta^2(m_k A_{n-k}^\delta) \chi_k^{\alpha,1}(x) u(x,\alpha),$$

where $\chi_n^{\alpha,\delta}$ is defined by (7). Hence taking the $L^1(0,\infty)$-norm and observing (11) leads to

$$\int_0^\infty |k_n^{\alpha,\delta}(x)|\, dx \leq C(A_n^\delta)^{-1} \sum_{k=0}^n A_k^1 (k+1)^{\alpha/2} |\Delta^2(m_k A_{n-k}^\delta)|.$$

Leibniz' formula for differences gives

$$|\Delta^2(m_k A_{n-k}^\delta)| \approx (k+1)^{-\alpha/2-2} A_{n-k}^\delta + (k+1)^{-\alpha/2-1} A_{n-k}^{\delta-1} + (k+1)^{-\alpha/2} A_{n-k}^{\delta-2}$$

and the hypothesis $\delta \geq 1$ guarantees that we have only positive terms. Split up the resulting three sums into $0 \leq k \leq n/2$ and $n/2 \leq k \leq n$ summations. Then the first term with summation over $0 \le k \le n/2$ gives a $\log(n+1)$ contribution, and all other terms only give (uniformly in $n$) bounded contributions. Hence, for $\delta \ge 1$,

$$\int_0^\infty |k_n^{\alpha,\delta}(x)|\, dx \le C \log(n+1).$$

Of course, this is no proof of Lemma 3.1; but by a similar argument its proof can be reduced to the problem of showing that when $\alpha > 0$ the modified Poisson kernel

$$p_r^\alpha(x) = \sum_{j=0}^\infty r^j \mathcal{L}_j^\alpha(x) \tag{12}$$

has no uniformly in $r,\ 0 < r < 1$, bounded $L^1(0,\infty)$-norm, i.e.

$$\sup_{0<r<1} \int_0^\infty |p_r^\alpha(x)|\, dx = \infty, \quad \alpha > 0. \tag{13}$$

Take (13) for the moment for granted and assume that Lemma 3.1 is not true for some $\alpha$ and $\delta$. Since

$$p_r^\alpha(x) = \sum_{j=0}^\infty A_j^\delta (\Delta^{\delta+1} r^j) k_j^{\alpha,\delta}(x)$$

and $\sum A_j^\delta |\Delta^{\delta+1} r^j| \le C$ for all $r,\ 0 < r < 1$ (see Chapter 3 in [20]), we immediately get a contradiction, for if we take $L^1(0,\infty)$-norms on both sides of the last equation, the right hand side is uniformly bounded by assumption, whereas the left hand side is not bounded.

In order to prove (13), we first observe that from the generating function [4, 10.12(17)]

$$\sum_{n=0}^\infty r^n L_n^\alpha(x) = (1-r)^{-\alpha-1} e^{-xr/(1-r)}, \quad |r| < 1,$$

and the special case of the beta integral [4, 1.5(1)]

$$\int_0^1 r^n (1-r)^{a-1} dr = \frac{\Gamma(n+1)\Gamma(a)}{\Gamma(n+a+1)}, \quad a > 0, n \ge 0,$$

it follows, formally, by termwise integration that

$$\sum_{n=0}^\infty (A_n^a)^{-1} L_n^\alpha(x) = a \int_0^1 (1-r)^{a-\alpha-2} e^{-xr/(1-r)} dr = g^{a,\alpha}(x)$$

with

$$g^{a,\alpha}(x) = a e^x \int_1^\infty t^{\alpha-a} e^{-xt} dt = a e^x x^{a-\alpha-1} \int_0^\infty t^{\alpha-a} e^{-t} dt,$$

where $x > 0$ and $0 < a < \alpha + 1$. Notice that since, by use of the Laplace transform [4, 10.12(32)],

$$\int_0^\infty g^{a,\alpha}(x) L_n^\alpha(x) e^{-x} x^\alpha dx = a \int_1^\infty t^{\alpha - a} (\int_0^\infty L_n^\alpha(x) e^{-xt} x^\alpha dx) dt$$

$$= \frac{a\Gamma(n+\alpha+1)}{\Gamma(n+1)} \int_1^\infty (t-1)^n t^{-a-n-1} dt = \Gamma(\alpha+1) A_n^\alpha / A_n^a$$

for $x > 0$ and $0 < a < \alpha + 1$, we have that

$$g^{a,\alpha}(x) \sim \sum_{n=0}^\infty (A_n^a)^{-1} L_n^\alpha(x).$$

Also notice that, from the above integral representations for $g^{a,\alpha}$,

$$g^{a,\alpha}(x) = O(x^{a-\alpha-1}) \quad as\ x \to 0^+,$$

and, by [4, 6.9(21)] and [11, 4.7(2)],

$$g^{a,\alpha}(x) = a x^{a-\alpha-1} \psi(a-\alpha, a-\alpha; x) = O(1/x) \quad as\ x \to \infty.$$

From these estimates it follows that we have

**Lemma 3.2** *Let* $0 < a < \alpha + 1$, $c > 0$, *and* $0 < p < \infty$. *Then*

$$\int_0^\infty |g^{a,\alpha}(x)|^p e^{-cx} x^\gamma dx < \infty \quad if\ and\ only\ if \quad \gamma > (1 + \alpha - a)p - 1.$$

*In particular,* $g^{a,\alpha} \in L^1_{u(\alpha)}$, *i.e.*

$$\int_0^\infty |g^{a,\alpha}(x)| e^{-x/2} x^{\alpha/2}\, dx < \infty,$$

*if and only if* $a > \alpha/2$.

Let $\alpha > 0$. We will now use Lemma 3.2 to show that

$$\sum_{n=0}^\infty (\Gamma(\alpha+1) A_n^\alpha)^{-1/2} L_n^\alpha(x)$$

is the Laguerre series of a function $g^\alpha$ that is not in $L^1_{u(\alpha)}$. Observe that, by [4, 1.18(4)], $\Gamma(\alpha+1) A_n^\alpha \simeq n^\alpha$ and

$$(\Gamma(\alpha+1) A_n^\alpha)^{-1/2} = (\Gamma(\alpha/2+1) A_n^{\alpha/2})^{-1} + c_\alpha (A_n^{\alpha/2+1})^{-1} + E_n^\alpha$$

with $E_n^\alpha = O((n+1)^{-\alpha/2-2})$. From the above lemma, the function $g_1^\alpha = (\Gamma(\alpha/2+1))^{-1} g^{\alpha/2,\alpha}$ is in $L^1_{w(\alpha)}$ , but it is not in $L^1_{u(\alpha)}$. The function $g_2^\alpha = c_\alpha g^{\alpha/2+1,\alpha}$ is in both $L^1_{w(\alpha)}$ and $L^1_{u(\alpha)}$, and a termwise use of [14, Lemma 1] shows that the function

$$g_3^\alpha = \sum_{n=0}^{\infty} E_n^\alpha L_n^\alpha(x)$$

is also in both $L^1_{w(\alpha)}$ and $L^1_{u(\alpha)}$. Hence, the function $g^\alpha = g_1^\alpha + g_2^\alpha + g_3^\alpha$ is in $L^1_{w(\alpha)}$, it has the Laguerre expansion

$$\sum_{n=0}^{\infty} (\Gamma(\alpha+1)A_n^\alpha)^{-1/2} L_n^\alpha(x),$$

but it is not in $L^1_{u(\alpha)}$. By Lemma 4 and Theorem 3 in Muckenhoupt [15], the Poisson integral $g^\alpha(r,x)$ of $g^\alpha(x)$ has the Laguerre expansion

$$\sum_{n=0}^{\infty} r^n (\Gamma(\alpha+1)A_n^\alpha)^{-1/2} L_n^\alpha(x)$$

and tends to $g^\alpha(x)$ almost everywhere as $r \to 1^-$. In view of Parseval's formula (5),

$$\sum_{n=0}^{\infty} r^n (\Gamma(\alpha+1)A_n^\alpha)^{-1/2} L_n^\alpha(x)$$

is the Laguerre series of an $L^2_{u(\alpha)}$ function when $0 \le r < 1$. Application of the asymptotic formula [19, 8.22.1)] shows that the above series converges for $x > 0$ when $0 \le r < 1$. Since, by $L^2$ theory, it converges to $g^\alpha(r,x)$ in the $L^2_{u(\alpha)}$-norm, it must also converge to $g^\alpha(r,x)$ almost everywhere when $0 \le r < 1$. Then, using Fatou's Lemma,

$$\infty = \int_0^\infty |g^\alpha(x)| e^{-x/2} x^{\alpha/2}\, dx \le \liminf_{r\to 1^-} \int_0^\infty |p_r^\alpha(x)|\, dx \le \sup_{0<r<1} \int_0^\infty |p_r^\alpha(x)|\, dx$$

when $\alpha > 0$, which proves (13) and hence completes the proof of Lemma 3.1.

So it is not obvious that the following analogue of Corollary 2.4 a) holds.

**Theorem 3.1** *If $f \in L^p_{u(\alpha)}, 1 \le p < 4/3, \alpha \ge 0$ when $p = 1$, and $\alpha > 2/p - 2$ when $p > 1$, then $(1/p + 1/q = 1)$*

$$\sup_n |(n+1)^{\lambda+1/q} \Delta^\lambda \hat{f}_n| \le C \parallel f \parallel_{L^p_{u(\alpha)}}, \quad \lambda = \frac{1}{3} - \frac{4}{3q}.$$

**Proof.** Use (3) to write

$$\begin{aligned}\Delta^\lambda \hat{f}_n &= \int_0^\infty f(x)\Delta^\lambda \mathcal{L}_n^\alpha(x)u(x,\alpha)\,dx\\ &= (\Gamma(\alpha+1))^{-1/2}\int_0^\infty f(x)\Delta^\lambda\left(\sqrt{A_n^\alpha}R_n^\alpha(x)\right)u^2(x,\alpha)\,dx\\ &= C\int_0^\infty f(x)u(x,\alpha)\sum_{j=0}^\infty A_j^{-\lambda-1}R_{n+j}^\alpha(x)\sqrt{A_n^\alpha}\,u(x,\alpha)\,dx\\ &+ C\int_0^\infty f(x)u(x,\alpha)\sum_{j=0}^\infty A_j^{-\lambda-1}R_{n+j}^\alpha(x)\left(\sqrt{A_{n+j}^\alpha}-\sqrt{A_n^\alpha}\right)u(x,\alpha)\,dx\\ &= I+II.\end{aligned}$$

From (3) and Hölder's inequality it follows that if $1 \le p < 4/3$ and $\lambda \ge 1/3 - 4/3q$ then

$$\begin{aligned}|I| &\le C \parallel f \parallel_{L^p_{u(\alpha)}} \sqrt{A_n^\alpha} \parallel x^\lambda R_n^{\alpha+\lambda}(x) \parallel_{L^q_{u(\alpha)}}\\ &\le C(n+1)^{-\alpha/2-\lambda} \parallel f \parallel_{L^p_{u(\alpha)}} \parallel L_n^{\alpha+2\lambda-\lambda} \parallel_{L^q_{u(\alpha+2\lambda)}}\\ &\le C(n+1)^{-\alpha/2-\lambda} \parallel f \parallel_{L^p_{u(\alpha)}} (n+1)^{(\alpha+2\lambda)/2-1/3+1/3q}\end{aligned}$$

where the latter inequality follows from the fourth case of Lemma 1 in Markett [14]. Hence, if $f \in L^p_{u(\alpha)}$, $1 \le p < 4/3$ and $\lambda \ge 1/3 - 4/3q$, then

$$|I| \le C(n+1)^{-1/3+1/3q} \parallel f \parallel_{L^p_{u(\alpha)}}.$$

In order to estimate $II$ first note that $|\sqrt{A_{n+j}^\alpha} - \sqrt{A_n^\alpha}| \approx j(n+j+1)^{\alpha/2-1}$. Then, by Hölder's inequality and the fifth case of Lemma 1 in [14],

$$\begin{aligned}|II| &\le C\int_0^\infty |f(x)|u(x,\alpha)\sum_{j=1}^\infty j^{-1-\lambda}|L_{n+j}^\alpha(x)|(n+j)^{-\alpha}|\sqrt{A_{n+j}^\alpha}-\sqrt{A_n^\alpha}|\,u(x,\alpha)\,dx\\ &\le C \parallel f \parallel_{L^p_{u(\alpha)}} \sum_{j=1}^\infty j^{-\lambda}(n+j)^{-\alpha/2-1} \parallel L_{n+j}^\alpha \parallel_{L^q_{u(\alpha)}}\\ &\le C(n+1)^{-\lambda-1/q} \parallel f \parallel_{L^p_{u(\alpha)}}.\end{aligned}$$

If we now set $\lambda = 1/3 - 4/3q$, then the combination of the above inequalities completes the proof of Theorem 3.1.

There is the question in how far supplementary necessary conditions exist which reflect a behavior as shown by the modified Cesàro kernel; this is closely connected with the problem to gain control over additional test multipliers as one has, e.g., in the case of radial Fourier multipliers.

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