On necessary multiplier conditions for Laguerre expansions II

GEORGE GASPER[1] AND WALTER TREBELS[2]

Dedicated to Dick Askey and Frank Olver

(Published in SIAM J. Math. Anal. 25(1994), 384–391)

Abstract. The necessary multiplier conditions for Laguerre expansions derived in Gasper and Trebels [3] are supplemented and modified. This allows us to place Markett's Cohen type inequality [6] (up to the log–case) in the general framework of necessary conditions.

Key words. Laguerre polynomials, necessary multiplier conditions, Cohen type inequalities, fractional differences, weighted Lebesgue spaces

AMS(MOS) subject classifications. 33C65, 42A45, 42C10

1. Introduction. The purpose of this sequel to [3] is to obtain a better insight into the structure of Laguerre multipliers on L^p spaces from the point of view of necessary conditions. We recall that in [3] there occurs the annoying phenomenon that, e.g., the optimal necessary conditions in the case $p = 1$ do not give the "right" unboundedness behavior of the Cesàro means. By slightly modifying these conditions we can not only remedy this defect but can also derive Markett's Cohen type inequality [6] (up to the log–case) as an immediate consequence.
For the convenience of the reader we briefly repeat the notation; we consider the Lebesgue spaces

$$L^p_{w(\gamma)} = \{f : \| f \|_{L^p_{w(\gamma)}} = (\int_0^\infty |f(x)e^{-x/2}|^p x^\gamma \, dx)^{1/p} < \infty\}, \quad 1 \le p < \infty,$$

denote the classical Laguerre polynomials by $L_n^\alpha(x)$, $\alpha > -1$, $n \in \mathbf{N}_0$ (see Szegö [8, p. 100]), and set

$$R_n^\alpha(x) = L_n^\alpha(x)/L_n^\alpha(0), \quad L_n^\alpha(0) = A_n^\alpha = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$

Associate to f its formal Laguerre series

$$f(x) \sim (\Gamma(\alpha+1))^{-1} \sum_{k=0}^\infty \hat{f}_\alpha(k) L_k^\alpha(x),$$

where the Fourier Laguerre coefficients of f are defined by

$$\hat{f}_\alpha(n) = \int_0^\infty f(x) R_n^\alpha(x) x^\alpha e^{-x} \, dx \tag{1}$$

[1] Department of Mathematics, Northwestern University, Evanston, IL 60208, USA. The work of this author was supported in part by the National Science Foundation under grant DMS-9103177.
[2] Fachbereich Mathematik, TH Darmstadt, D–6100 Darmstadt, Germany.

(if the integrals exist). A sequence $m = \{m_k\}$ is called a (bounded) multiplier on $L^p_{w(\gamma)}$, notation $m \in M^p_{w(\gamma)}$, if

$$\| \sum_{k=0}^{\infty} m_k \hat{f}_\alpha(k) L_k^\alpha \|_{L^p_{w(\gamma)}} \leq C \| f \|_{L^p_{w(\gamma)}}$$

for all polynomials f; the smallest constant C for which this holds is called the multiplier norm $\| m \|_{M^p_{\alpha,\gamma}}$. The necessary conditions will be given in certain "smoothness" properties of the multiplier sequence in question. To this end we introduce a fractional difference operator of order δ by

$$\Delta^\delta m_k = \sum_{j=0}^{\infty} A_j^{-\delta-1} m_{k+j}$$

(whenever the sum converges), the first order difference operator Δ_2 with increment 2 by

$$\Delta_2 m_k = m_k - m_{k+2},$$

and the notation

$$\Delta_2 \Delta^\delta m_k = \Delta^{\delta+1} m_k + \Delta^{\delta+1} m_{k+1}.$$

Generic positive constants that are independent of the functions (and sequences) will be denoted by C. Within the setting of the $L^p_{w(\gamma)}$-spaces our main results now read (with $1/p + 1/q = 1$):

Theorem 1.1. *Let α, $a > -1$ and $\alpha + a > -1$. If $f \in L^p_{w(\gamma)}$, $1 \leq p < 2$, $\gamma > -1$, then*

$$\left(\sum_{k=0}^{\infty} |(k+1)^{(\gamma+1)/p-1/2} \Delta_2 \Delta^a \hat{f}_\alpha(k)|^q \right)^{1/q} \leq C \| f \|_{L^p_{w(\gamma)}}, \tag{2}$$

provided

$$\frac{\gamma+1}{p} \leq \frac{\alpha+a}{p} + 1 \qquad \text{if } \alpha + a \leq 1/2,$$

$$\frac{\gamma+1}{p} \leq \frac{\alpha+a}{2} + 1 + \frac{1}{2}\left(\frac{1}{p} - \frac{1}{2}\right) \quad \text{if } \alpha + a > 1/2.$$

If we note that

$$|m_k| \| L_k^\alpha \|_{L^p_{w(\gamma)}} = \| m_k L_k^\alpha \|_{L^p_{w(\gamma)}} \leq \| m \|_{M^p_{\alpha,\gamma}} \| L_k^\alpha \|_{L^p_{w(\gamma)}}, \quad \gamma > -1,$$

implies $M^p_{w(\gamma)} \subset l^\infty$, we immediately obtain as in [3] (see there the proof of Lemma 2.3)

Theorem 1.2. *Let $m = \{m_k\} \in M^p_{w(\gamma)}$, $1 \leq p < 2$, and let α, γ, and a be as in Theorem 1.1. Then*

$$\sup_n \left(\sum_{k=n}^{2n} |(k+1)^{(2\gamma+1)/p-(2\alpha+1)/2} \Delta_2 \Delta^a m_k|^q \frac{1}{k+1} \right)^{1/q} \leq C \| m \|_{M^p_{\alpha,\gamma}}. \tag{3}$$

An extension of Theorem 1.2 to $2 < p < \infty$ easily follows by duality

$$M^p_{w(\gamma)} = M^q_{w(\alpha q - \gamma q/p)}\,, \quad -1 < \gamma < p(\alpha+1) - 1, \quad 1 < p < \infty.$$

In view of the results in [6], [3] and for an easy comparison we want to emphasize the cases $\gamma = \alpha$ and $\gamma = \alpha p/2$. Therefore, we state

COROLLARY 1.3.

a) Let $m \in M^p_{w(\alpha)}$, $1 \le p < 2$, $\alpha > -1$, and let $\lambda := (2\alpha+1)(1/p - 1/2)$. Then, with $\lambda > 0$,

$$\sup_n \left(\sum_{k=n}^{2n} |(k+1)^\lambda \Delta_2 \Delta^{\lambda-1} m_k|^q \frac{1}{k+1} \right)^{1/q} \le C \parallel m \parallel_{M^p_{w(\alpha)}}$$

if $\alpha + \lambda \ge 3/2$; if $\alpha + \lambda < 3/2$ it has additionally to be assumed that $\lambda \ge 2 - p$. In the case $\alpha + \lambda < 3/2$ and $\lambda < 2 - p$ an analogous result holds when the difference operator $\Delta_2 \Delta^{\lambda-1}$ is replaced by Δ_2.

b) Let $m \in M^p_{w(\alpha p/2)}$, $1 \le p < 4/3$, and $(\alpha - 1)(1/p - 1/2) \ge -1/2$. Then

$$\sup_n \left(\sum_{k=n}^{2n} |(k+1)^{1/p-1/2} \Delta_2 m_k|^q \frac{1}{k+1} \right)^{1/q} \le C \parallel m \parallel_{M^p_{w(\alpha p/2)}}.$$

Remarks. 1) For polynomial $f(x) = \sum_{k=0}^n c_k L_k^\alpha(x)$ Theorem 1.1 yields, by taking only the $(k = n)$–term on the left hand side of (2),

$$|c_n|(n+1)^{(\gamma+1)/p-1/2} \le C \parallel f \parallel_{L^p_{w(\gamma)}}, \quad 1 \le p < 2$$

(under the restrictions on γ of Theorem 1.1). In particular, if we choose $\gamma = \alpha$, this comprises formula (1.13) in Markett [6] for his basic case $\beta = \alpha$. For $\gamma = \alpha p/2$, it even extends formula (1.14) in [6] to negative α's as described in Corollary 1.3, b). The case $2 < p < \infty$ can be done by an application of a Nikolskii inequality, see [6].

2) Analogously, Cohen type inequalities follow from Theorem 1.2; in particular, Corollary 1.3 yields

COROLLARY 1.4. *Let $m = \{m_k\}_{k=0}^n$ be a finite sequence, $1 \le p < 2$, and $\alpha > -1$.*

a) If $m \in M^p_{w(\alpha)}$ then

$$(n+1)^{(2\alpha+2)(1/p-1/2)-1/2}|m_n| \le C \parallel m \parallel_{M^p_{w(\alpha)}}, \quad 1 \le p < \frac{4\alpha+4}{2\alpha+3}.$$

b) If $m \in M^p_{w(\alpha p/2)}$ and $(\alpha - 1)(1/p - 1/2) \ge -1/2$, then

$$(n+1)^{2/p-3/2}|m_n| \le C \parallel m \parallel_{M^p_{w(\alpha p/2)}}, \quad 1 \le p < 4/3.$$

With the exception of the crucial log–case, i.e. $p_0 = (4\alpha+4)/(2\alpha+3)$ or $p_0 = 4/3$, resp., Corollary 1.4 contains Markett's Theorem 1 in [6] and extends it to negative

α's. In particular we obtain for the Cesàro means of order $\delta \geq 0$, represented by its multiplier sequence $m^\delta_{k,n} = A^\delta_{n-k}/A^\delta_n$, the "right" unboundedness behavior (see [4])

$$\| \{m^\delta_{k,n}\} \|_{M^p_{w(\alpha)}} \geq C(n+1)^{(2\alpha+2)(1/p-1/2)-1/2-\delta}, \quad 1 \leq p < \frac{4\alpha+4}{2\alpha+3+2\delta}.$$

3) There arises the question, in how far the type of necessary conditions in [3] are comparable with the present ones. Let $\lambda > 1$. Since $\Delta_2 m_k = \Delta m_k + \Delta m_{k+1}$ we obviously have

$$\sup_n \left(\sum_{k=n}^{2n} |(k+1)^{(2\gamma+1)/p-(2\alpha+1)/2} \Delta_2 \Delta^{\lambda-1} m_k|^q \frac{1}{k+1} \right)^{1/q} \tag{4}$$

$$\leq C \sup_n \left(\sum_{k=n}^{2n} |(k+1)^{(2\gamma+1)/p-(2\alpha+1)/2} \Delta^{\lambda} m_k|^q \frac{1}{k+1} \right)^{1/q}.$$

In general, a converse cannot hold as can be seen by the following example: choose $\gamma = \alpha$, $\lambda = (2\alpha+1)(1/p-1/2)$ and $m_k = (-1)^k(k+1)^{-\varepsilon}$, $0 < \varepsilon < 1$. Then

$$\sup_n \left(\sum_{k=n}^{2n} |(k+1)\Delta m_k|^q \frac{1}{k+1} \right)^{1/q} = \infty$$

and hence by the embedding properties of the wbv–spaces, see [2], the right hand side of (4) cannot be finite for all $\lambda > 1$. But since $\Delta_2 \Delta^{\lambda-1} m_k = \Delta^{\lambda-1}\Delta_2 m_k \sim (k+1)^{-\varepsilon-\lambda}$, the left hand side of (4) is finite for all $\lambda > 1$.

Theorem 1.1 will be proved in Section 2 by interpolating between (L^1, l^∞)– and (L^2, l^2)–estimates. The $a \neq 0$ case is an easy consequence of the case $a = 0$ when one uses the basic formula (see formula (3) in [3] and Remark 3 preceding Section 3 there)

$$\Delta^a R^\alpha_k(x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+a+1)} x^a R^{\alpha+a}_k(x), \quad x > 0,\ a > -1 - \min\{\alpha, \alpha/2 - 1/4\}, \tag{5}$$

where in the case $a > -(2\alpha+1)/4$ the series for the fractional difference converges absolutely. In Section 3, a necessary (L^1, l^1)–estimate is derived and it is compared with a corresponding sufficient (l^1, L^1)–estimate.

2. Proof of Theorem 1.1. Let us first handle the (L^2, l^2)–estimate. Since

$$\Delta_2 \Delta^a \hat{f}_\alpha(k) = \Delta^{1+a} \hat{f}_\alpha(k) + \Delta^{1+a} \hat{f}_\alpha(k+1)$$

it follows from the Parseval formula preceding Corollary 2.5 in [3] that

$$\left(\sum_{k=0}^{\infty} |\sqrt{A^{\alpha+1+a}_k} \Delta_2 \Delta^a \hat{f}_\alpha(k)|^2 \right)^{1/2} \leq C \left(\int_0^\infty |f(t) e^{-t/2} t^{(\alpha+1+a)/2}|^2 \, dt \right)^{1/2}. \tag{6}$$

Concerning the (L^1, l^∞)–estimate we first restrict ourselves to the case $a = 0$. Define $\mu \in \mathbf{R}$ by

$$2\left(\frac{1}{p} - \frac{1}{2}\right)\mu = \frac{\gamma}{p} - \frac{\alpha+1}{2};$$

with the notation $\mathcal{L}_k^\alpha(t) = (A_k^\alpha/\Gamma(\alpha+1))^{1/2} R_k^\alpha(t) e^{-t/2} t^{\alpha/2}$ it follows that

$$|\Delta_2 \hat{f}_\alpha(k)| = C|\int_0^\infty f(t)\{\mathcal{L}_k^\alpha(t)/\sqrt{A_k^\alpha} - \mathcal{L}_{k+2}^\alpha(t)/\sqrt{A_{k+2}^\alpha}\}e^{-t/2}t^{\alpha/2}\,dt|$$

$$\leq C(k+1)^{-1-\alpha/2}\int_0^\infty |f(t)||t^{-\mu-1/2}\mathcal{L}_k^\alpha(t)|e^{-t/2}t^{(\alpha+1)/2+\mu}\,dt$$

$$+C(k+1)^{-\alpha/2}\int_0^\infty |f(t)||t^{-\mu-1/2}\{\mathcal{L}_k^\alpha(t) - \mathcal{L}_{k+2}^\alpha(t)\}|e^{-t/2}t^{(\alpha+1)/2+\mu}\,dt = I + II.$$

We distinguish the two cases $\alpha \leq 1/2$ and $\alpha > 1/2$:

First consider the case $\alpha \leq 1/2$. By the asymptotic estimates for $\mathcal{L}_k^\alpha(t) - \mathcal{L}_{k+2}^\alpha(t)$ in Askey and Wainger [1, p. 699], see formula (2.12) in [6], it follows for $\gamma \leq \alpha + p - 1$ that

$$\| t^{-\mu-1/2}\{\mathcal{L}_k^\alpha(t) - \mathcal{L}_{k+2}^\alpha(t)\} \|_\infty \leq C(k+1)^{-1-\mu}$$

so that

$$II \leq C(k+1)^{-1-\mu-\alpha/2}\int_0^\infty |f(t)|e^{-t/2}t^{(\alpha+1)/2+\mu}\,dt, \quad \gamma \leq \alpha + p - 1. \tag{7}$$

By Lemma 1, 4th case, in [5]

$$\| t^{-\mu-1/2}\mathcal{L}_k^\alpha(t) \|_\infty \leq C(k+1)^{-\mu-5/6}$$

so that trivially

$$I \leq C(k+1)^{-1-\mu-\alpha/2}\int_0^\infty |f(t)|e^{-t/2}t^{(\alpha+1)/2+\mu}\,dt, \quad \frac{\gamma+1}{p} \leq \frac{\alpha+1}{2} - \frac{1}{3p} + \frac{2}{3}.$$

By Lemma 1, 5th case, in [5]

$$\| t^{-\mu-1/2}\mathcal{L}_k^\alpha(t) \|_\infty \leq C(k+1)^{\mu+1/2}$$

so that

$$I \leq C(k+1)^{\mu-1/2-\alpha/2}\int_0^\infty |f(t)|e^{-t/2}t^{(\alpha+1)/2+\mu}\,dt,$$

$$\leq C(k+1)^{-1-\mu-\alpha/2}\int_0^\infty |f(t)|e^{-t/2}t^{(\alpha+1)/2+\mu}\,dt, \quad \frac{\gamma+1}{p} > \frac{\alpha+1}{2} - \frac{1}{3p} + \frac{2}{3},$$

provided that $\mu - (\alpha+1)/2 \leq -1 - \mu - \alpha/2$ which is equivalent to $\mu \leq -1/4$ or $\gamma \leq 3p/4 - 1/2 + \alpha p/2$. But this is no further restriction since for $\alpha \leq 1/2$ there holds

$\alpha + p - 1 \le 3p/4 - 1/2 + \alpha p/2$. Summarizing, for $-1 < \alpha \le 1/2$, $\gamma \le \alpha + p - 1$ and $\mu = (\gamma/p - (\alpha+1)/2)/2(1/p - 1/2)$ we have that

$$\sup_k |(k+1)^{1+\mu+\alpha/2} \Delta_2 \hat{f}_\alpha(k)| \le C \int_0^\infty |f(t)| e^{-t/2} t^{(\alpha+1)/2+\mu}\, dt. \tag{8}$$

Now consider the case $\alpha > 1/2$. Then, by formula (2.12) in [6], (7) is obviously true when $(\gamma+1)/p \le \alpha/2 + 1 + (1/p - 1/2)/2$. Again, the application of Lemma 1 in [5] requires $\gamma \le \alpha + p - 1$, which for $\alpha > 1/2$ is less restrictive than $(\gamma+1)/p \le \alpha/2 + 1 + (1/p - 1/2)/2$. Its 4th case now leads to

$$I \le C(k+1)^{-11/6-\mu-\alpha/2} \int_0^\infty |f(t)| e^{-t/2} t^{(\alpha+1)/2+\mu}\, dt, \quad \frac{\gamma+1}{p} \le \frac{\alpha+1}{2} - \frac{1}{3p} + \frac{2}{3},$$

and its 5th case to

$$I \le C(k+1)^{\mu-1/2-\alpha/2} \int_0^\infty |f(t)| e^{-t/2} t^{(\alpha+1)/2+\mu}\, dt, \quad (\gamma+1)/p > \frac{\alpha+1}{2} - \frac{1}{3p} + \frac{2}{3}.$$

But $\mu - 1/2 - \alpha/2 \le -\mu - 1 - \alpha/2$ if $(\gamma+1)/p \le \alpha/2 + 1 + (1/p - 1/2)/2$; so that, summarizing, (8) also holds under this restriction for $\alpha > 1/2$.

Now an application of the Stein and Weiss interpolation theorem (see [7]) with $Tf = \{Tf(k)\}$and $Tf(k) = \sqrt{A_k^{\alpha+1}}\Delta_2 \hat{f}_\alpha(k)$ gives the assertion of Theorem 1.1 in the case $a = 0$.

If $a \ne 0$ then by (1), the definition of $\Delta_2 \Delta^a$, and by (5)

$$\Delta_2 \Delta^a \hat{f}_\alpha(k) = C\{\Delta \hat{f}_{\alpha+a}(k) + \Delta \hat{f}_{\alpha+a}(k+1)\} = C\Delta_2 \hat{f}_{\alpha+a}(k),$$

since already the condition $\gamma < \alpha + a + 1$ (which implies no new restriction) gives absolute convergence of the infinite sum and integral involved (see the formula following (9) in [3]) and Fubini's Theorem can be applied. Hence all the previous estimates remain valid when α is replaced by $\alpha + a$.

3. A variant for integrable functions. Theorem 1.1 gives a necessary condition for a sequence $\{f_k\}$ to generate with respect to L_k^α an $L^1_{w(\gamma)}$–function. But this condition is hardly comparable with the following sufficient one which is a slight modification of Lemma 2.2 in [3].

THEOREM 3.1. *Let $\alpha > -1$ and $\delta > 2\gamma - \alpha + 1/2 \ge 0$. If $\{f_k\}$ is a bounded sequence with $\lim_{k\to\infty} f_k = 0$ and*

$$\sum_{k=0}^\infty (k+1)^{\delta+\alpha-\gamma} |\Delta^{\delta+1} f_k| \le K_{\{f_k\}},$$

then there exists a function $f \in L^1_{w(\gamma)}$ with $\hat{f}_\alpha(k) = f_k$ for all $k \in \mathbf{N}_0$ and

$$\| f \|_{L^1_{w(\gamma)}} \le C\, K_{\{f_k\}}$$

for some constant C independent of the sequence $\{f_k\}$.

The proof follows along the lines of Lemma 2.2 in [3] since the norm of the Cesàro kernel

$$\chi_n^{\alpha,\delta}(x) = (A_n^\delta \Gamma(\alpha+1))^{-1} \sum_{k=0}^{n} A_{n-k}^\delta L_k^\alpha(x) = (A_n^\delta \Gamma(\alpha+1))^{-1} L_n^{\alpha+\delta+1}(x)$$

can be estimated with the aid of Lemma 1 in [5] by

$$\| \chi_k^{\alpha,\delta} \|_{L^1_{w(\gamma)}} \le C(k+1)^{\alpha-\gamma}, \quad \delta > 2\gamma - \alpha + 1/2.$$

The variant of Theorem 1.1 in the case $p = 1$ is

THEOREM 3.2. *If* $\alpha > -1$ *and* $\gamma > \max\{-1/3, \alpha/2 - 1/6\}$, *then*

$$\sum_{k=0}^{\infty} (k+1)^{\gamma-2/3} |\Delta^{2\gamma-\alpha+1/3} \hat{f}_\alpha(k)| \le C \| f \|_{L^1_{w(\gamma)}} .$$

A comparison of the sufficient condition and the necessary one nicely shows where the $L^1_{w(\gamma)}$–functions live; in particular we see that the "smoothness" gap (the difference of the orders of the difference operators) is just greater than $7/6$. It is clear that Theorem 3.2 can be modified by using the Δ_2–operator. Theorem 3.2 does not follow from the $p = 1$ case of Lemma 2.1 in [3] since that estimate would lead to the divergent sum $\sum_{k=0}^{\infty} (k+1)^{-1} \| f \|_{L^1_{w(\gamma)}}$.

Proof

By formula (5) we have

$$\Delta^{2\gamma-\alpha+1/3} \hat{f}_\alpha(k) = C \int_0^\infty f(t) R_k^{2\gamma+1/3}(t) t^{2\gamma+1/3} e^{-t} \, dt$$

$$= C(k+1)^{-\gamma-1/6} \int_0^\infty f(t) \mathcal{L}_k^{2\gamma+1/3}(t) t^{\gamma+1/6} e^{-t/2} \, dt$$

and hence

$$\sum_{k=0}^{\infty} (k+1)^{\gamma-2/3} |\Delta^{2\gamma-\alpha+1/3} \hat{f}_\alpha(k)|$$

$$\le C \int_0^\infty |f(t)| \sum_{k=0}^{\infty} (k+1)^{-5/6} |t^{1/6} \mathcal{L}_k^{2\gamma+1/3}(t)| t^\gamma e^{-t/2} \, dt$$

if the right hand side converges. To show this we discuss for $j \in \mathbf{Z}$

$$\sup_{2^j \le t \le 2^{j+1}} \sum_{k=0}^{\infty} (k+1)^{-5/6} |t^{1/6} \mathcal{L}_k^{2\gamma+1/3}(t)|$$

and prove that this quantity is uniformly bounded in j, whence the assertion.

First consider those $j \ge 0$ for which there exists a nonnegative integer n such that $0 \le k \le 2^n$ implies $3\nu/2 := 3(2k + 2\gamma + 4/3) \le 2^j$ but such that this inequality fails to hold for $k \geq 2^{n+1}$; the latter assumption in particular implies that essentially $\nu/2 \geq 2^{j+1}$ for $k \geq 2^{n+4}$. Since $\| t^{1/6}\mathcal{L}_k^{2\gamma+1/3}(t) \|_\infty \leq C(k+1)^{-1/6}$ by Lemma 1 in [5], we obviously have

$$(9) \qquad \sum_{k=0}^{\infty}(k+1)^{-5/6}|t^{1/6}\mathcal{L}_k^{2\gamma+1/3}(t)| \leq \left(\sum_{k=0}^{2^n} + \sum_{k=2^{n+4}}^{\infty}\right)\ldots + O(1).$$

For $k = 0, \ldots, 2^n$ we can now apply the fourth case of formula (2.5) in [5] to obtain $|t^{1/6}\mathcal{L}_k^{2\gamma+1/3}(t)| \leq Ce^{-\mu 2^j}$ for some positive constant μ and the first sum on the right hand side of (9) is bounded uniformly in j. In consequence of the choice of n the second case of formula (2.5) in [5] can be used for $k \geq 2^{n+4}$, giving

$$\sum_{k=2^{n+4}}^{\infty} (k+1)^{-5/6}|t^{1/6}\mathcal{L}_k^{2\gamma+1/3}(t)| \leq Ct^{-1/12} \sum_{k=2^{n+4}}^{\infty} (k+1)^{-13/12} = O(1)$$

since $2^j \leq t \leq 2^{j+1}$ and j and n are comparable.
Now consider the remaining j's: We have to split up the sum $\sum_{k=0}^{\infty}\ldots$ into two parts, one where k is such that $2^j\nu \geq 1$ (this contribution has just been seen to be uniformly bounded in j), the other where k is such that $2^j\nu \leq 1$. To deal with the last case choose again n to be the greatest integer such that $2^{n+2} + 4\gamma + 8/3 \leq 2^{-j}$; this time, n and $-j$ are comparable and we obtain by the first case of (2.5) in [5]

$$\sum_{k=0}^{2^n}(k+1)^{-5/6}|t^{1/6}\mathcal{L}_k^{2\gamma+1/3}(t)| \leq Ct^{\gamma+1/3}\sum_{k=0}^{2^n}(k+1)^{\gamma-2/3} = O(1)$$

if $2^j \leq t \leq 2^{j+1}$, $\gamma > -1/3$, which completes the proof.

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