q-EXTENSIONS OF CLAUSEN'S FORMULA AND OF THE INEQUALITIES USED BY DE BRANGES IN HIS PROOF OF THE BIEBERBACH, ROBERTSON, AND MILIN CONJECTURES*

George Gasper†

(Published in SIAM. J. Math. Anal. 20 (1989), 1019–1034)

Abstract. A q-extension of the terminating form of Clausen's ${}_3F_2$ series representation for the square of a ${}_2F_1(a, b; a + b + 1/2; z)$ series is derived. It is used to prove the nonnegativity of certain basic hypergeometric series and to derive q-extensions of the inequalities and differential equations de Branges used in his proof of the Bieberbach, Robertson, and Milin conjectures.

Key words. Clausen's formula, basic hypergeometric series, nonnegative polynomials, inequalities, q-difference equations, Bieberbach, Robertson, and Milin conjectures

AMS(MOS) subject classifications. Primary 33A99; secondary 30C50

1. Introduction. In 1828 Clausen [15] used second- and third-order differential equations to prove the formula

$$(1.1)\qquad \left\{ {}_2F_1\left[\begin{matrix} a, b \\ a+b+1/2 \end{matrix}; z\right]\right\}^2 = {}_3F_2\left[\begin{matrix} 2a, 2b, a+b \\ 2a+2b, a+b+1/2 \end{matrix}; z\right], \qquad |z| < 1.$$

The above ${}_2F_1$ and ${}_3F_2$ are special cases of ${}_rF_s$ hypergeometric series defined by

$${}_rF_s\left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{n!(b_1)_n \cdots (b_s)_n} z^n,$$

where $(a)_n$ is the shifted factorial defined by

$$(a)_n = \prod_{k=0}^{n-1} (a+k).$$

*Received by the editors May 2, 1988; accepted for publication June 21, 1988.

†Department of Mathematics, Northwestern University, Evanston, Illinois 60208. This work was supported in part by the National Science Foundation under grant DMS-8601901.

Almost 150 years later, Clausen's formula was used in Askey and Gasper [4] to prove that

$$(1.2) \qquad {}_3F_2\left[\begin{matrix}-n, n+\alpha+2, (\alpha+1)/2\\ \alpha+1, (\alpha+3)/2\end{matrix}; \frac{1-x}{2}\right] \ge 0, \qquad -1 \le x \le 1,$$

when $\alpha > -2$ and $n = 0, 1, 2, \ldots$, and then this inequality was used to prove the positivity of certain important kernels involving sums of Jacobi polynomials (also see Askey [3, Lecture 8] and the extensions in Gasper [18], [19]). In 1984 the special cases $\alpha = 2, 4, 6, \ldots$ of (1.2) were used by de Branges [11], [12] to complete the last part of his proof of the Milin [30, p. 55] conjecture that if f is in the class S of functions

$$f(z) = z + c_2 z^2 + c_3 z^3 + \cdots$$

that are analytic and univalent in the unit disk $|z| < 1$ and if

$$\log \frac{f(z)}{z} = 2 \sum_{k=1}^{\infty} \gamma_k z^k,$$

then

$$(1.3) \qquad \sum_{k=1}^{n} k(n+1-k)|\gamma_k|^2 \le \sum_{k=1}^{n} \frac{n+1-k}{k}, \qquad n = 1, 2, \ldots.$$

It was already known that Milin's conjecture implied Robertson's [33] conjecture that if f is an odd function in S, then

$$(1.4) \qquad \sum_{k=1}^{n} |c_{2k-1}|^2 \le n, \qquad n = 2, 3, \ldots,$$

and that Robertson's conjecture implied Bieberbach's [9] conjecture that if f is in S, then

$$(1.5) \qquad |c_n| \le n, \qquad n = 2, 3, \ldots.$$

Since ${}_rF_s$ series are limit cases of ${}_r\phi_s$ basic hypergeometric series [25], [38]

$$(1.6) \qquad {}_r\phi_s\left[\begin{matrix}a_1, \ldots, a_r\\ b_1, \ldots, b_s\end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n,$$

where $\binom{n}{2} = n(n-1)/2$,

$$(a_1, a_2, \ldots, a_r; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n$$

and $(a; q)_n$ is the q-shifted factorial defined by

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k),$$

it is natural to search for q-extensions (also called q-analogues and, in the terminology of [10], quantum generalizations) of Clausen's formula (1.1), the inequalities (1.2), and of the other parts of de Branges' proof of the Milin conjecture.

In this paper we will derive a q-extension of Clausen's formula (1.1) for terminating series and various q-extensions of the inequalities (1.2) and of some other inequalities. In addition, since the existence of decreasing solutions of de Branges' differential equations

$$\sigma_n(t) + \frac{t}{n}\,\sigma_n'(t) = \sigma_{n+1}(t) - \frac{t}{n+1}\,\sigma_{n+1}'(t), \qquad 1 \le t < \infty, \tag{1.7}$$

played a crucial role in his proof of the Milin conjecture, we derive q-extensions of (1.7) and show that they have solutions which have negative first q-derivatives. Some prospects for further research are pointed out.

2. q-Extensions of Clausen's formula (1.1).

In 1940 Jackson [27] derived a general theorem about solutions of q^θ equations, where q^θ is the operator $\exp\big((\log q)x\frac{d}{dx}\big)$, which gives [28, p. 171] the product formula

$$\begin{aligned} &{}_2\phi_1\left[\begin{matrix} q^{2a}, q^{2b} \\ q^{2a+2b+1} \end{matrix}; q^2, z\right] {}_2\phi_1\left[\begin{matrix} q^{2a}, q^{2b} \\ q^{2a+2b+1} \end{matrix}; q^2, qz\right] \\ &\quad = {}_4\phi_3\left[\begin{matrix} q^{2a}, q^{2b}, q^{a+b}, -q^{a+b} \\ q^{2a+2b}, q^{a+b+1/2}, -q^{a+b+1/2} \end{matrix}; q, z\right], \qquad |z|<1, |q|<1. \end{aligned} \tag{2.1}$$

Since

$$\lim_{q\uparrow 1} \frac{(q^a;q)_n}{(1-q)^n} = (a)_n, \qquad \lim_{q\uparrow 1}(-q^a;q)_n = 2^n, \tag{2.2}$$

Clausen's formula (1.1) is a limit case of Jackson's product formula (2.1). However, unlike in (1.1), the left side of (2.1) is not a square and so (2.1) cannot be used to write sums of basic hypergeometric series as sums of squares of basic hypergeometric series as was done in [4], [18] for hypergeometric series to prove the nonnegativity of certain sums of hypergeometric series. Also, by considering negative integer values of a, we find that the series on the right side of (2.1) can assume negative values. It is natural to consider replacing the left side of (1.1) by

$$\left\{ {}_2\phi_1\left[\begin{matrix} q^a, q^b \\ q^{a+b+1/2} \end{matrix}; q, z\right] \right\}^2 \tag{2.3}$$

but, unfortunately, this square of a ${}_2\phi_1$ series does not equal a basic hypergeometric series of the type in (1.6) as can be easily seen by computing the coefficient of z^2 in its power series expansion. Thus, in order to find a basic hypergeometric series which is the square of a basic hypergeometric series we are forced to look for another q-extension of (1.1).

One way to proceed is to recall that in [4] Clausen's formula was used to write (1.2) as a sum of squares of ultraspherical polynomials

$$C_n^\lambda(x) = \frac{(2\lambda)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n, n+2\lambda\\ \lambda+1/2\end{matrix}; \frac{1-x}{2}\right] \tag{2.4}$$

$$= \frac{(\lambda)_n}{n!} e^{in\theta}\,{}_2F_1\left[\begin{matrix}-n, \lambda\\ 1-n-\lambda\end{matrix}; e^{-2i\theta}\right], \qquad x = \cos\theta,$$

and to recall that in his work [34]–[36] during the 1890's on the now famous Rogers–Ramanujan identities, Rogers [36] considered the q-extension

$$C_n(x;\beta \mid q) = \frac{(\beta;q)_n}{(q;q)_n}\, e^{in\theta}\,{}_2\phi_1\left[\begin{matrix}q^{-n}, \beta\\ q^{1-n}\beta^{-1}\end{matrix}; q, q\beta^{-1}e^{-2i\theta}\right], \quad x = \cos\theta, \tag{2.5}$$

of (2.4). Askey and Ismail [6] showed that these polynomials were orthogonal on $(-1,1)$ with respect to an absolutely continuous weight function and called them the continuous q-ultraspherical polynomials to distinguish them from the (discrete) q-ultraspherical polynomials

$$C_n^\lambda(x;q) = \frac{(q^{2\lambda};q)_n}{(q;q)_n}\,{}_2\phi_1\left[\begin{matrix}q^{-n}, q^{n+2\lambda}\\ q^{\lambda+1/2}\end{matrix}; q, qx\right] \tag{2.6}$$

which are orthogonal [2, (3.8)] with respect to a discrete measure with point masses at $x = q^k$, $k = 0, 1, 2, \ldots$. They also showed that

$$C_n^\lambda(x) = \lim_{q\uparrow 1} C_n(x; q^\lambda \mid q) \tag{2.7}$$

and

$$C_n(\cos\theta;\beta \mid q) = \frac{(\beta^2;q)_n}{\beta^{n/2}(q;q)_n}\,{}_4\phi_3\left[\begin{matrix}q^{-n}, \beta^2 q^n, \beta^{1/2}e^{i\theta}, \beta^{1/2}e^{-i\theta}\\ \beta q^{1/2}, -\beta q^{1/2}, -\beta\end{matrix}; q, q\right]. \tag{2.8}$$

In 1895 Rogers [36, p. 29] employed an induction argument to prove the linearization formula

$$C_m(x;\beta \mid q)C_n(x;\beta \mid q) = \sum_{k=0}^{\min(m,n)} \frac{(q;q)_{m+n-2k}(\beta;q)_{m-k}(\beta;q)_{n-k}}{(q;q)_k(q;q)_{m-k}(q;q)_{n-k}} \cdot \frac{(\beta;q)_k(\beta^2;q)_{m+n-k}(1-\beta q^{m+n-2k})}{(\beta q;q)_{m+n-k}(\beta^2;q)_{m+n-2k}(1-\beta)}\, C_{m+n-2k}(x;\beta \mid q). \tag{2.9}$$

A simple computational proof of (2.9) was given by the author in [20]. For additional proofs, see Bressoud [14] and Rahman [31]. Note that if we use (2.5) or (2.8) on the right side of (2.9), we get a double sum that does not reduce to a single sum when $n = m$, even after changing the order of summation and trying to apply known summation formulas. This is also true when (2.8) is replaced by the formulas obtained by applying Sears' transformation formula [37, (8.3)]

$$(2.10)\qquad {}_4\phi_3\left[\begin{matrix} a, b, c, q^{-n} \\ d, e, f \end{matrix}; q, q\right] = \frac{(de/ab, df/ab; q)_n}{(e, f; q)_n} \left(\frac{ab}{d}\right)^n {}_4\phi_3\left[\begin{matrix} d/a, d/b, c, q^{-n} \\ d, de/ab, df/ab \end{matrix}; q, q\right]$$

where def $= abcq^{1-n}$ and $n = 0, 1, 2, \ldots$, to the ${}_4\phi_3$ series in (2.8).

The right side of (2.8) suggests that we should still look at expansions involving $e^{i\theta}$ and $e^{-i\theta}$ among the parameters. Since the polynomials on the right side of (2.9) are of even degree in x when $n = m$, we could try to use the expansion [7, p. 41]

$$(2.11)\qquad C_{2n}(\cos\theta; \beta \mid q) = \frac{(\beta^2; q)_{2n}(-q, -q^{1/2}; q)_n}{(q; q)_{2n}(-\beta, -\beta q^{1/2}; q)_n} q^{-n/2} \cdot {}_4\phi_3\left[\begin{matrix} q^{-n}, \beta q^n, q^{1/2}e^{2i\theta}, q^{1/2}e^{-2i\theta} \\ \beta q^{1/2}, -q^{1/2}, -q \end{matrix}; q, q\right],$$

in which the ${}_4\phi_3$ series terminates after $n+1$ terms, even though $C_{2n}(x; \beta|q)$ is a polynomial of degree $2n$ in x. But, by using (2.11) in the right side of (2.9) and changing the order of summation we get a sum of terminating very well poised ${}_8\phi_7$ series that are not balanced and hence not summable by Jackson's formula [38, (3.3.1.1)].

However, if we apply (2.10) to (2.11) to get

$$(2.12)\qquad C_{2n}(\cos\theta; \beta \mid q) = \frac{(\beta^2; q)_{2n}}{\beta^n (q; q)_{2n}} {}_4\phi_3\left[\begin{matrix} q^{-n}, \beta q^n, \beta e^{2i\theta}, \beta e^{-2i\theta} \\ \beta q^{1/2}, -\beta q^{1/2}, -\beta \end{matrix}; q, q\right]$$

and use (2.12) in the right side of (2.9) we obtain

$$\begin{aligned}(2.13)\quad &\{C_n(\cos\theta; \beta \mid q)\}^2 \\ &= \sum_{k=0}^{n} \frac{(\beta, \beta; q)_{n-k}(\beta; q)_k(\beta^2; q)_{2n-k}(1-\beta q^{2n-2k})}{(q, q; q)_{n-k}(q; q)_k(\beta q; q)_{2n-k}(1-\beta)} \\ &\quad \cdot \beta^{k-n} \sum_{j=0}^{n-k} \frac{(q^{k-n}, \beta q^{n-k}, \beta e^{2i\theta}, \beta e^{-2i\theta}; q)_j}{(q, \beta q^{1/2}, -\beta q^{1/2}, -\beta; q)_j} q^j \\ &= \frac{(\beta^2; q)_{2n}(\beta, \beta; q)_n}{\beta^n (\beta; q)_{2n}(q, q; q)_n} \sum_{j=0}^{n} \frac{(q^{-n}, \beta q^n, \beta e^{2i\theta}, \beta e^{-2i\theta}; q)_j}{(q, \beta q^{1/2}, -\beta q^{1/2}, -\beta; q)_j} q^j \\ &\quad \cdot {}_6\phi_5\left[\begin{matrix} \beta^{-1}q^{-2n}, \beta^{-1/2}q^{1-n}, -\beta^{-1/2}q^{1-n}, \beta, q^{j-n}, q^{-n} \\ \beta^{-1/2}q^{-n}, -\beta^{-1/2}q^{-n}, \beta^{-2}q^{1-2n}, \beta^{-1}q^{1-n-j}, \beta^{-1}q^{1-n} \end{matrix}; q, \beta^{-2}q^{1-j}\right]\end{aligned}$$

in which, fortunately, the ${}_6\phi_5$ series is summable by the summation formula [38, (3.3.1.4)]

$$(2.14)\qquad {}_6\phi_5\left[\begin{matrix} a, qa^{1/2}, -qa^{1/2}, b, c, q^{-n} \\ a^{1/2}, -a^{1/2}, aq/b, aq/c, aq^{n+1} \end{matrix}; q, \frac{aq^{n+1}}{bc}\right] = \frac{(aq, aq/bc; q)_n}{(aq/b, aq/c; q)_n}, \qquad n = 0, 1, 2, \ldots,$$

where four misprints have been corrected. Using (2.14) to sum the ${}_6\phi_5$ series in (2.13) gives

$$\{C_n(\cos\theta;\beta \mid q)\}^2 = \frac{(\beta^2,\beta^2;q)_n}{(q,q;q)_n}\,\beta^{-n}\,{}_5\phi_4\left[\begin{matrix} q^{-n},\beta^2q^n,\beta,\beta e^{2i\theta},\beta e^{-2i\theta} \\ \beta^2,\beta q^{1/2},-\beta q^{1/2},-\beta \end{matrix};q,q\right] \tag{2.15}$$

and hence, by (2.8), we have the following q-extension of the terminating case of Clausen's formula (1.1)

(2.16)

$$\left\{{}_4\phi_3\left[\begin{matrix} q^{-n},\beta^2q^n,\beta^{1/2}e^{i\theta},\beta^{1/2}e^{-i\theta} \\ \beta q^{1/2},-\beta q^{1/2},-\beta \end{matrix};q,q\right]\right\}^2 = {}_5\phi_4\left[\begin{matrix} q^{-n},\beta^2q^n,\beta,\beta e^{2i\theta},\beta e^{-2i\theta} \\ \beta^2,\beta q^{1/2},-\beta q^{1/2},-\beta \end{matrix};q,q\right],$$

where $n = 0,1,2,\ldots$. This formula was derived independently by Mizan Rahman. He and the author independently observed that it can be derived by using (2.8) inside the Rahman and Verma [32, (1.20)] integral representation for the product of two continuous q-ultraspherical polynomials and then integrating termwise to get (2.15) and hence (2.16).

By setting $a = \beta^{1/2}e^{i\theta}$, $b = \beta^{1/2}e^{-i\theta}$ and $z = \beta q^n$, formula (2.16) can be written in the form

$$\left\{{}_4\phi_3\left[\begin{matrix} a,b,abz,ab/z \\ abq^{1/2},-abq^{1/2},-ab \end{matrix};q,q\right]\right\}^2 = {}_5\phi_4\left[\begin{matrix} a^2,b^2,ab,abz,ab/z \\ a^2b^2,abq^{1/2},-abq^{1/2},-ab \end{matrix};q,q\right], \tag{2.17}$$

which holds when the series on both sides terminate. For, by (2.16), (2.17) holds when abz or ab/z is a negative integer power of q and, if a or b is a negative integer power of q, then both sides of (2.17) are rational functions of z which are equal for $z = abq^n$, $n = 0,1,2,\ldots$, and hence must be equal for all (complex) values of z. Notice that by replacing a, b, z in (2.17) by q^a, q^b, $e^{i\theta}$ with $x = \cos\theta$ and letting $q \uparrow 1$, we get Clausen's formula (1.1) with $z = (1-x)/2$ for the terminating case when a or b is a negative integer.

To see that (2.17) does not hold in the nonterminating case, it suffices to observe that if, e.g., (2.17) held for $b = 0$, then it would follow from the q-binomial theorem [38, (3.2.2.11)] that $((aq;q)_\infty)^2 = (a^2q;q)_\infty(q;q)_\infty$, which is clearly false for, e.g., $a = q^{-1/2}$. A nonterminating q-extension of (1.1) containing the square of an ${}_8\phi_7$ series and the sum of two ${}_5\phi_4$ series will be given in [26].

Note that, in addition to the formulas that follow when (2.10) is applied to the ${}_4\phi_3$ in (2.17), we can apply the quadratic transformation formula [7, (3.2)]

$${}_4\phi_3\left[\begin{matrix} a^2,b^2q,c,d \\ abq,-abq,-cd \end{matrix};q,q\right] = {}_4\phi_3\left[\begin{matrix} a^2,b^2q,c^2,d^2 \\ a^2b^2q^2,-cd,-cdq \end{matrix};q^2,q^2\right], \tag{2.18}$$

which holds when both series terminate, to obtain that (2.17) is equivalent to the formula

$$\left\{{}_4\phi_3\left[\begin{matrix} a^2,b^2,abz,ab/z \\ a^2b^2q,-ab,-abq \end{matrix};q^2,q^2\right]\right\}^2 = {}_5\phi_4\left[\begin{matrix} a^2,b^2,ab,abz,ab/z \\ a^2b^2,abq^{1/2},-abq^{1/2},-ab \end{matrix};q,q\right] \tag{2.19}$$

when both series terminate. Also, if we replace a, b, z, q in (2.17) by their squares and apply (2.18), we obtain that

$$\left\{{}_4\phi_3\left[\begin{matrix} a^2,b^2,abz,ab/z \\ abq^{1/2},-abq^{1/2},-a^2b^2 \end{matrix};q,q\right]\right\}^2 = {}_5\phi_4\left[\begin{matrix} a^4,b^4,a^2b^2,a^2b^2z^2,a^2b^2/z^2 \\ a^4b^4,a^2b^2q,-a^2b^2q,-a^2b^2 \end{matrix};q^2,q^2\right] \tag{2.20}$$

when both series terminate.

Another proof of (2.17) can be given by observing that from the product formulas (2.8) or (2.10) in Gasper and Rahman [24] it follows that if a, b, abz, or ab/z is a negative integer power of q, then

(2.21)

$$\left\{ {}_4\phi_3 \left[\begin{matrix} abz, ab/z, a, b \\ -ab, abq^{1/2}, -abq^{1/2} \end{matrix} ; q, q \right] \right\}^2$$

$$= \sum_{r\geq 0} \sum_{s \geq 0} \frac{(abz, ab/z; q)_{r+s} (a, b, -a, -b; q)_r}{(-ab, -ab; q)_{r+s} (q, -q, abq^{1/2}, -abq^{1/2}; q)_r}$$

$$\cdot \frac{(a, b, -a, -b; q)_s}{(q, -1, abq^{1/2}, -abq^{1/2}; q)_s} \frac{1+q^{r-s}}{1+q^s} q^{r+s}$$

$$= \sum_{k \geq 0} \frac{(abz, ab/z, a, -a, b, -b; q)_k}{(q, -1, abq^{1/2}, -abq^{1/2}, -ab, -ab; q)_k} q^k$$

$$\cdot \sum_{s=0}^{k} \frac{(q^{-k}, -q^{-k}, a, -a, b, -b, a^{-1}b^{-1}q^{1/2-k}, -a^{-1}b^{-1}q^{1/2-k}; q)_s}{(q, -q, a^{-1}q^{1-k}, -a^{-1}q^{1-k}, b^{-1}q^{1-k}, -b^{-1}q^{1-k}, abq^{1/2}, -abq^{1/2}; q)_s} \frac{1+q^{2s-k}}{1+q^{-k}} q^{2s}$$

$$= \sum_{k \geq 0} \frac{(abz, ab/z, a, b, -a, -b; q)_k}{(q, -1, abq^{1/2}, -abq^{1/2}, -ab, -ab; q)_k} q^k$$

$$\cdot {}_5\phi_4 \left[\begin{matrix} q^{-2k}, -q^{2-k}, a^2, b^2, a^{-2}b^{-2}q^{1-2k} \\ -q^{-k}, a^{-2}q^{2-2k}, b^{-2}q^{2-2k}, a^2b^2q \end{matrix} ; q^2, q^2 \right]$$

$$= {}_5\phi_4 \left[\begin{matrix} a^2, b^2, ab, abz, ab/z \\ a^2b^2, abq^{1/2}, abq^{1/2}, -ab \end{matrix} ; q, q \right],$$

since it can be shown that

(2.22)

$${}_5\phi_4 \left[q^{-2k}, -q^{2-k}, a^2, b^2, a^{-2}b^{-2}q^{1-2k} - q^{-k}, a^{-2}q^{2-2k}, b^{-2}q^{2-2k}, a^2b^2q \,; q^2, q^2 \right]$$
$$= \frac{(-1, a^2, b^2, ab, -ab; q)_k}{(a^2b^2, a, -a, b, -b; q)_k}, \qquad k = 0, 1, 2, \ldots,$$

by using the case $d = eq^{1/2} = (aq)^{1/2}$ of Jackson's summation formula [38, (3.3.1.1)]. A slightly more direct proof of (2.17) can be given by starting with the expansion [24, (2.2)] used to derive [24, (2.8)].

3. q-Extensions of (1.2).

At the last step in his proof of the Milin conjecture, de Branges [12] used the fact that for any positive integer r the functions

$$\sigma_n(t) = \frac{n\Gamma(n+r+2)}{\Gamma(2n+2)\Gamma(r+1-n)} \int_t^\infty {}_3F_2 \left[\begin{matrix} n-r, n+r+2, n+1/2 \\ 2n+1, n+3/2 \end{matrix} ; s^{-1} \right] s^{-n-1} ds \tag{3.1}$$

when $n = 1, \ldots, r$ and $\sigma_n(t) = 0$ when $n > r$, satisfy the differential equations (1.7) and

$$\sigma_n'(t)) = -\frac{n\Gamma(n+r+2)t^{-n-1}}{\Gamma(2n+2)\Gamma(r+1-n))} {}_3F_2\left[\begin{matrix} n-r, n+r+2, n+1/2 \\ 2n+1, n+3/2 \end{matrix}; t^{-1}\right] \leq 0 \tag{3.2}$$

for $t \geq 1$ when $n = 1, \ldots, r$. In [13], to estimate the coefficients of powers of unbounded Riemann mapping functions, he used the fact that the more general functions

(3.3)

$$\sigma_n(t) = \frac{\Gamma(n+1)\Gamma(n+r+2\nu+2\lambda+1)4^{-n}t^{2\nu}}{\Gamma(n+\nu+1)\Gamma(n+2\nu)\Gamma(n+\nu+\lambda+1)\Gamma(r+1-n)} \cdot \int_t^\infty {}_3F_2\left[\begin{matrix} n-r, n+r+2\nu+2\lambda+1, n+\nu+1/2 \\ 2n+2\nu+1, n+\nu+\lambda+1 \end{matrix}; s^{-1}\right] s^{-n-2\nu-1} ds$$

when $n = 1, \ldots, r$ and $\sigma_n(t) = 0$ when $n > r$, satisfy the differential equations

$$\frac{n}{n+2\nu}\sigma_n(t) + \frac{t\sigma_n'(t)}{n+2\nu} = \frac{n+2\nu+1}{n+1}\sigma_{n+1}(t) - \frac{t\sigma_{n+1}'(t)}{n+1} \tag{3.4}$$

and

(3.5)

$$\frac{d}{dt}[t^{-2\nu}\sigma_n(t)] = -\frac{\Gamma(n+1)\Gamma(n+r+2\nu+2\lambda+1)4^{-n}}{\Gamma(n+\nu+1)\Gamma(n+2\nu)\Gamma(n+\nu+\lambda+1)\Gamma(r+1-n)} \cdot {}_3F_2\left[\begin{matrix} n-r, n+r+2\nu+2\lambda+1, n+\nu+1/2 \\ 2n+2\nu+1, n+\nu+\lambda+1 \end{matrix}; t^{-1}\right] \leq 0$$

for $t \geq 1$ when $\nu > -1/2$, $\lambda \geq 0$, and $n = 1, \ldots, r$.

Since (3.5) reduces to (3.2) when $\nu = \lambda + 1/2 = 0$, in addition to deriving q-extensions of (1.2) we will derive q-extensions of the inequalities

$${}_3F_2\left[\begin{matrix} -n, n+a, b \\ 2b, (a+1)/2 \end{matrix}; \frac{1-x}{2}\right] \geq 0, \qquad -1 \leq x \leq 1, \tag{3.6}$$

where $a \geq 2b > -1$ and $n = 0, 1, \ldots$, which imply the inequalities in (3.5) and reduce to (1.2) when $a = \alpha + 2$ and $b = (\alpha+1)/2$.

Let $0 < q < 1$, $n = 0, 1, 2, \ldots$, and let a, b, α, β, γ, δ, θ be real parameters. Then

$$\lim_{q\to 1} {}_5\phi_4\left[\begin{matrix} q^{-n}, q^{n+a}, q^b, q^\alpha e^{i\theta}, q^\beta e^{-i\theta} \\ q^{2b}, q^{(a+1)/2}, -q^\gamma, -q^\delta \end{matrix}; q, q\right] = {}_3F_2\left[\begin{matrix} -n, n+a, b \\ 2b, (a+1)/2 \end{matrix}; \frac{1-x}{2}\right], \quad x = \cos\theta, \tag{3.7}$$

and so, in order to derive a q-extension of (3.6), it suffices to find values of α, β, γ, δ for which the ${}_6\phi_5$ series in (3.7) are nonnegative when $a \geq 2b > -1$.

Observe that from (2.16)

$$(3.8)\quad {}_5\phi_4\left[\begin{matrix} q^{-n}, q^{n+2b}, q^b, q^be^{i\theta}, q^be^{-i\theta} \\ q^{2b}, q^{b+1/2}, -q^{b+1/2}, -q^b \end{matrix}; q, q\right] = \left\{ {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{n+2b}, q^{b/2}e^{i\theta/2}, q^{b/2}e^{-i\theta/2} \\ q^{b+1/2}, -q^{b+1/2}, -q^b \end{matrix}; q, q\right]\right\}^2 \geq 0,$$

which shows that the ${}_5\phi_4$ series in (3.7) are nonnegative when $a = 2b$, $\alpha = \beta = \delta = b$, and $\gamma = b + 1/2$. In view of the "sums of squares" method [18, §8], we will consider sums of the nonnegative ${}_5\phi_4$ series in (3.8).

The author showed in [18, §8] that besides the sum of squares of ultraspherical polynomials used in [4, (1.16)] to prove (1.2) we could also use the sum of squares in [18, (8.17)] and observed that these two expansions are special cases of the expansion [18, (8.18)]

$$(3.9)\quad {}_3F_2\left[\begin{matrix} -n, n+\alpha+2, (\alpha+1)/2 \\ \alpha+1, (\alpha+3)/3 \end{matrix}; (1-x^2)(1-y^2)\right] = \sum_{j=0}^{n} \frac{n!(n+\alpha+2)_j\left(\frac{\alpha+2}{2}\right)_j}{j!(n-j)!\left(\frac{\alpha+3}{2}\right)_j(j+\alpha+1)_j}(1-y^2)^j \cdot\left\{\frac{j!(n-j)!}{(\alpha+1)_j(2j+\alpha+2)_{n-j}}C_j^{(\alpha+1)/2}(x)C_{n-j}^{j+(\alpha+2)/2}(y)\right\}^2,$$

which can be derived from the Fields and Wimp [17] expansion formula

$$(3.10)\quad {}_{r+t}F_{s+u}\left[\begin{matrix} a_R, c_T \\ b_S, d_U \end{matrix}; xw\right] = \sum_{j=0}^{\infty} \frac{(a_R)_j(\alpha)_j(\beta)_j}{(b_S)_j(\gamma+j)_j}\frac{(-x)^j}{j!} \cdot {}_{r+2}F_{s+1}\left[\begin{matrix} j+\alpha, j+\beta, j+a_R \\ 1+2j+\gamma, j+b_S \end{matrix}; x\right] {}_{t+2}F_{u+2}\left[\begin{matrix} -j, j+\gamma, c_T \\ \alpha, \beta, d_U \end{matrix}; w\right],$$

where we used the contracted notation of representing $a_1, a_2, \ldots, a_r$ by a_R, $(a_1)_j(a_2)_j\cdots(a_r)_j$ by $(a_R)_j$, and $j+a_1, j+a_2, \ldots, j+a_r$ by $j+a_R$. Recently the author derived a bibasic extension [22, (4.5)] of (3.10) which contained Verma's [40] q-analogues and gave the general expansion [22, (4.7)]

(3.11)

$$
\begin{aligned}
&{}_{r+t}\phi_{s+u}\left[\begin{matrix} a_R, c_T \\ b_S, d_U \end{matrix}; q, xw\right] \\
&\qquad = \sum_{j=0}^{\infty} \frac{(c_T, e_K, \sigma, \gamma q^{j+1}/\sigma; q)_j}{(q, d_U, f_M, \gamma q^j; q)_j} \left(\frac{x}{\sigma}\right)^j [(-1)^j q^{\binom{j}{2}}]^{u+m-t-k} \\
&\qquad \cdot {}_{t+k+4}\phi_{u+m+3}\left[\begin{matrix} \gamma q^{2j}/\sigma, q^{j+1}\sqrt{\gamma/\sigma}, -q^{j+1}\sqrt{\gamma/\sigma}, \sigma^{-1}, c_T q^j, e_K q^j \\ q^j\sqrt{\gamma/\sigma}, -q^j\sqrt{\gamma/\sigma}, \gamma q^{2j+1}, d_U q^j, f_M q^j \end{matrix}; q, xq^{j(u+m-t-k)}\right] \\
&\qquad \cdot {}_{r+m+2}\phi_{s+k+2}\left[\begin{matrix} q^{-j}, \gamma q^j, a_R, f_M \\ \gamma q^{j+1}/\sigma, q^{1-j}/\sigma, b_S, e_K \end{matrix}; q, wq\right],
\end{aligned}
$$

where we used a contracted notation analogous to that used in (3.10). Formulas (3.10) and (3.11) hold when the series terminate and when the parameters and variables are such that the series converge absolutely.

In this section we will use the following $\sigma \to \infty$ limit case of the $m = 2$, $f_1 = f_2 = 0$ case of (3.11)

$$
\begin{aligned}
(3.12)\quad &{}_{r+t}\phi_{s+u}\left[\begin{matrix} a_R, c_T \\ b_S, d_U \end{matrix}; q, xw\right] \\
&\qquad = \sum_{j=0}^{\infty} \frac{(c_T, e_K; q)_j}{(q, d_U, \gamma q^j; q)_j} x^j [(-1)^j q^{\binom{j}{2}}]^{u+3-t-k} \\
&\qquad \cdot {}_{t+k}\phi_{u+1}\left[\begin{matrix} c_T q^j, e_K q^j \\ \gamma q^{2j+1}, d_U q^j \end{matrix}; q, xq^{j(u+3-t-k)}\right] \\
&\qquad \cdot {}_{r+2}\phi_{s+k}\left[\begin{matrix} q^{-j}, \gamma q^j, a_R \\ b_S, e_K \end{matrix}; q, wq\right].
\end{aligned}
$$

which is equivalent to [39, (3.1)]. Set

$$
\begin{gathered}
\gamma = q^{2b}, a_1 = q^b, a_2 = q^b e^{i\theta}, a_3 = q^b e^{-i\theta}, b_1 = q^{2b}, b_2 = -q^b, \\
c_1 = q^{-n}, c_2 = q^{n+a}, d_1 = q^{(a+1)/2} = -d_2, e_1 = q^{b+1/2} = -e_2, x = q, w = 1,
\end{gathered}
$$

in the $r=3$, $s=t=u=k=2$ case of (3.12) to obtain

$$(3.13)\quad {}_5\phi_4\left[\begin{matrix} q^{-n}, q^{n+a}, q^b, q^be^{i\theta}, q^be^{-i\theta} \\ q^{2b}, q^{(a+1)/2}, -q^{(a+1)/2}, -q^b \end{matrix}; q, q\right]$$
$$= \sum_{j=0}^{n} \frac{(q^{-n}, q^{n+a}, q^{b+1/2}, -q^{b+1/2}; q)_j}{(q, q^{(a+1)/2}, -q^{(a+1)/2}, q^{j+2b}; q)_j} (-1)^j q^{j+\binom{j}{2}}$$
$$\cdot\, {}_4\phi_3\left[\begin{matrix} q^{j-n}, q^{j+n+a}, q^{j+b+1/2}, -q^{j+b+1/2} \\ q^{2j+2b+1}, q^{j+(a+1)/2}, -q^{j+(a+1)/2} \end{matrix}; q, q\right]$$
$$\cdot\, {}_5\phi_4\left[\begin{matrix} q^{-j}, q^{j+2b}, q^b, q^be^{i\theta}, q^be^{-i\theta} \\ q^{2b}, q^{b+1/2}, -q^{b+1/2}, -q^b \end{matrix}; q, q\right].$$

By Andrews' [1, Thm. 1] q-analogue of Watson's ${}_3F_2$ summation formula

$$(3.14)\quad {}_4\phi_3\left[\begin{matrix} a, b, c^{1/2}, -c^{1/2} \\ c, (abq)^{1/2}, -(abq)^{1/2} \end{matrix}; q, q\right] = \frac{a^{n/2}(aq, bq, cq/a, cq/b; q^2)_\infty}{(q, abq, cq, cq/ab; q^2)_\infty},$$

where $b = q^{-n}$ and n is a nonnegative integer, the ${}_4\phi_3$ series in (3.13) equals zero when $n-j$ is odd and equals

$$\frac{(q, q^{a-2b}; q^2)_k}{(q^{2n-4k+a+1}, q^{2n-4k+2b+2}; q^2)_k} q^{2k(n-2k+b+1/2)}$$

when $n-j=2k$ and $k=0,1,\ldots$. Hence, from (3.13) and (3.8),

$$(3.15)\quad {}_5\phi_4\left[\begin{matrix} q^{-n}, q^{n+a}, q^b, q^be^{i\theta}, q^be^{-i\theta} \\ q^{2b}, q^{(a+1)/2}, -q^{(a+1)/2}, -q^b \end{matrix}; q, q\right]$$
$$= \sum_{k=0}^{[n/2]} \frac{(-1)^n(q^{-n}, q^{n+a}, q^{b+1/2}, -q^{b+1/2}; q)_{n-2k}}{(q, q^{(a+1)/2}, -q^{(a+1)/2}, q^{n-2k+2b}; q)_{n-2k}}$$
$$\cdot \frac{(q, q^{a-2b}; q^2)_k}{(q^{2n-4k+a+1}, q^{2n-4k+2b+2}; q^2)_k} q^{2k(n-2k+b+1/2)+(n-2k)(n-2k+1)/2}$$
$$\cdot \left\{ {}_4\phi_3\left[\begin{matrix} q^{2k-n}, q^{n-2k+2b}, q^{b/2}e^{i\theta/2}, q^{b/2}e^{-i\theta/2} \\ q^{b+1/2}, -q^{b+1/2}, -q^b \end{matrix}; q, q\right]\right\}^2.$$

Since $(-1)^n(q^{-n}; q)_{n-2k} \geq 0$, it is clear from (3.15) that

$$(3.16)\quad {}_5\phi_4\left[\begin{matrix} q^{-n}, q^{n+a}, q^b, q^be^{i\theta}, q^be^{-i\theta} \\ q^{2b}, q^{(a+1)/2}, -q^{(a+1)/2}, -q^b \end{matrix}; q, q\right] \geq 0$$

when $a \geq 2b > -1$ and $0 < q < 1$, which gives a q-extension of (3.6) and hence of the inequalities (1.2) used by de Branges in his proof of the Bieberbach, Robertson, and Milin conjectures.

Another q-extension of (3.6) can be derived by observing that from (3.12) we have

$$
\begin{aligned}
(3.17)\quad {}_6\phi_5 &\left[\begin{matrix} q^{-n}, q^{n+a}, q^b, -q^b, q^{a/2}e^{i\theta}, q^{1/2\,a}e^{-i\theta} \\ q^{2b}, q^{(a+1)/2}, -q^{(a+1)/2}, -q^{a/2}, -q^{a/2} \end{matrix}; q, q\right] \\
&= \sum_{j=0}^{n} \frac{(q^{-n}, q^{n+a}, q^{a/2}, q^{a/2}e^{i\theta}, q^{a/2}e^{-i\theta}; q)_j}{(q, q^{(a+1)/2}, -q^{(a+1)/2}, -q^{a/2}, q^{j+a-1}; q)_j} (-1)^j q^{j+\binom{j}{2}} \\
&\quad \cdot {}_5\phi_4 \left[\begin{matrix} q^{j-n}, q^{n+j+a}, q^{j+a/2}, q^{j+a/2}e^{i\theta}, q^{j+a/2}e^{-i\theta} \\ q^{2j+a}, q^{j+(a+1)/2}, -q^{j+(a+1)/2}, -q^{j+a/2} \end{matrix}; q, q\right] \\
&\quad \cdot {}_4\phi_3 \left[\begin{matrix} q^{-j}, q^{j+a-1}, q^b, -q^b \\ q^{2b}, q^{a/2}, -q^{a/2} \end{matrix}; q, q\right] \\
&= \sum_{k=0}^{[n/2]} \frac{(q^{-n}, q^{n+a}, q^{a/2}, q^{a/2}e^{i\theta}, q^{a/2}e^{-i\theta}; q)_{2k}}{(q, q^{(a+1)/2}, -q^{(a+1)/2}, -q^{\frac{1}{2}a}, q^{2k+a-1}; q)_{2k}} \\
&\quad \cdot \frac{(q, q^{a-2b}; q^2)_k}{(q^a, q^{2b+1}; q^2)_k} q^{2k^2+k+2kb} \\
&\quad \cdot \left\{ {}_4\phi_3 \left[\begin{matrix} q^{2k-n}, q^{n+2k+a}, q^{k+a/4}e^{i\theta/2}, q^{k+a/4}e^{-i\theta/2} \\ q^{2k+(a+1)/2}, -q^{2k+(a+1)/2}, -q^{2k+a/2} \end{matrix}; q, q\right] \right\}^2
\end{aligned}
$$

by (3.14) and (3.8). Hence,

$$
(3.18)\qquad {}_6\phi_5 \left[\begin{matrix} q^{-n}, q^{n+a}, q^b, -q^b, q^{a/2}e^{i\theta}, q^{a/2}e^{-i\theta} \\ q^{2b}, q^{(a+1)/2}, -q^{(a+1)/2}, -q^{a/2}, -q^{a/2} \end{matrix}; q, q\right] \geq 0
$$

when $a \geq 2b > -1$ and $0 < q < 1$, which is a q-extension of (3.6) that is different from (3.16). The expansions (8.12) and (8.17) in [18] are special cases of the $q \uparrow 1$ limit cases of (3.15) and (3.17), respectively, when (2.15) and [18, (8.10)] are used.

A q-extension of (3.9) can be derived by using (3.12) and (2.15) to obtain the expansion

(3.19)

$$
{}_7\phi_6\left[\begin{matrix} q^{-n}, q^{n+\alpha+2}, q^{(\alpha+1)/2}, q^{(\alpha+1)/2}e^{2i\theta}, q^{(\alpha+1)/2}e^{-2i\theta}, q^{(\alpha+2)/2}e^{2i\tau}, q^{(\alpha+2)/2}e^{-2i\tau} \\ q^{\alpha+1}, q^{(\alpha+3)/2}, -q^{(\alpha+3)/2}, -q^{(\alpha+2)/2}, -q^{(\alpha+2)/2}, -q^{(\alpha+1)/2} \end{matrix}; q, q\right]
$$

$$
= \sum_{j=0}^{n} \frac{(q^{-n}, q^{n+\alpha+2}, q^{(\alpha+2)/2}, q^{(\alpha+2)/2}e^{2i\tau}, q^{(\alpha+2)/2}e^{-2i\tau}; q)_j}{(q, q^{(\alpha+3)/2}, -q^{(\alpha+3)/2}, -q^{(\alpha+2)/2}, q^{j+\alpha+1}; q)_j} (-1)^j q^{j+\binom{j}{2}}
$$

$$
\cdot \left\{ \frac{(q;q)_j (q;q)_{n-j}}{(q^{\alpha+1};q)_j (q^{2j+\alpha+2};q)_{n-j}} q^{1/2\,(j+\alpha+3/2)} \right.
$$

$$
\left. \cdot\, C_j(\cos\theta; q^{(\alpha+1)/2} \mid q) C_{n-j}(\cos\tau; q^{j+(\alpha+2)/2} \mid q) \right\}^2,
$$

which is clearly nonnegative for real θ and τ when $\alpha > -2$. The case $\alpha = -2$ can be handled as a limit case of $(q^{\alpha+2};q)_n$ times the ${}_7\phi_6$ series in (3.19); see [4, p. 720] for the hypergeometric case.

Additional nonnegative sums and, in particular, the nonnegativity of q-extensions of the sums of Jacobi polynomials in [18, (8.19), (8.20), (8.22)] will be considered in [23].

4. q-Extensions of de Branges' differential equations. From (3.3) and the identity $(a)_n = \Gamma(n+a)/\Gamma(a)$ it follows that

(4.1)

$$
\sigma_n(t) = c\frac{n!(2\nu+2\lambda+1)_{n+r}4^{-n}t^{-n}}{(r-n)!(\nu+1)_n(2\nu+1)_n(\nu+\lambda+1)_n}
$$

$$
\cdot\, {}_4F_3\left[\begin{matrix} n-r, n+r+2\nu+2\lambda+1, n+\nu+1/2, n+2\nu \\ 2n+2\nu+1, n+\nu+\lambda+1, n+2\nu+1 \end{matrix}; t^{-1}\right]
$$

for $n = 1, \ldots, r$ with $c = \Gamma(2\nu+2\lambda+1)/[\Gamma(\nu+1)\Gamma(2\nu+1)\Gamma(\nu+\lambda+1)]$. In view of the limit (3.7), we set $t = 2/(1-x)$ and consider the functions

$$
\tau_n(x) = c^{-1}\sigma_n\left(\frac{2}{1-x}\right). \tag{4.2}
$$

Then de Branges' differential equation (3.4) is equivalent to

$$
\frac{n}{n+2\nu}\,\tau_n(x) + \frac{1-x}{n+2\nu}\,\tau_n'(x) = \frac{n+2\nu+1}{n+1}\,\tau_{n+1}(x) - \frac{1-x}{n+1}\,\tau_{n+1}'(x) \tag{4.3}
$$

which can be rewritten in the form

$$
\frac{d}{dx}\left[(1-x)^{-n}\tau_n(x)\right] = -\frac{n+2\nu}{n+1}\,(1-x)^{-2n-2\nu-1}\frac{d}{dx}\left[(1-x)^{n+2\nu+1}\tau_{n+1}(x)\right]. \tag{4.4}
$$

Let $0 < q < 1$, $x = \cos\theta$ and let a, α, λ, ν, θ be real numbers. A q-extension of $(1-x)^\alpha$ can be obtained by extending the definition of the q-shifted factorial to

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \tag{4.5}$$

and observing that, by the q-binomial theorem [38, (3.2.2.11)],

$$\lim_{q\to 1} 2^{-\alpha}(q^a e^{i\theta}, q^a e^{-i\theta}; q)_\alpha = (1-x)^\alpha. \tag{4.6}$$

Hence, if we define

$$\begin{aligned} u_n(x) = A_{n,r}&(q^{\nu+1}e^{i\theta}, q^{\nu+1}e^{-i\theta}; q)_n \\ &\cdot {}_6\phi_5\left[\begin{matrix} q^{n-r}, q^{n+r+2\nu+2\lambda+1}, q^{n+\nu+1/2}, q^{n+2\nu}, q^{n+\nu+1}e^{i\theta}, q^{n+\nu+1}e^{-i\theta} \\ q^{2n+2\nu+1}, q^{n+\nu+\lambda+1}, q^{n+2\nu+1}, -q^{n+\nu+\lambda+1}, -q^{n+\nu+1/2} \end{matrix}; q, q\right] \end{aligned} \tag{4.7}$$

with

$$A_{n,r} = \frac{(q;q)_n (q^{2\nu+2\lambda+1};q)_{n+r} 4^{-2n}}{(q;q)_{r-n}(q^{\nu+1}, q^{2\nu+1}, q^{\nu+\lambda+1};q)_n}, \qquad n = 1, \ldots, r, \tag{4.8}$$

and $A_{n,r} = 0$ when $n > r$, then

$$\lim_{q\to 1} u_n(x) = \tau_n(x). \tag{4.9}$$

To obtain a q-extension of differentiation that plays the same role for $(ae^{i\theta}, ae^{-i\theta}; q)_n$ as d/dx does for x^n, Askey and Wilson [7, p. 33] defined the operators δ_q and D_q by

$$\delta_q f(e^{i\theta}) = f(q^{1/2}e^{i\theta}) - f(q^{-1/2}e^{i\theta}), \tag{4.10}$$

$$D_q h(x) = \frac{\delta_q h(x)}{\delta_q x}, \tag{4.11}$$

where $x = (e^{i\theta} + e^{-i\theta})/2 = \cos\theta$, and observed that

$$\delta_q(ae^{i\theta}, ae^{-i\theta}; q)_n = aq^{-1/2}(1-q^n)(e^{i\theta} - e^{-i\theta})(aq^{1/2}e^{i\theta}, aq^{1/2}e^{-i\theta}; q)_{n-1} \tag{4.12}$$

and

$$\frac{\delta_q \prod_{k=0}^{n-1}(1 - 2axq^k + a^2q^{2k})}{\delta_q x} = \frac{-2a(1-q^n)}{1-q}\prod_{k=0}^{n-2}(1 - 2axq^{k+1/2} + a^2q^{2k+1}). \tag{4.13}$$

They noted that when $q \to 1$ formula (4.13) becomes

$$\frac{d}{dx}(1 - 2ax + a^2)^n = -2an(1 - 2ax + a^2)^{n-1} \tag{4.14}$$

and more generally,

$$\lim_{q\to 1} D_q h(x) = \lim_{q\to 1} \frac{\delta_q h(x)}{\delta_q x} = \frac{df(x)}{dx}. \tag{4.15}$$

To derive a q-extension of (4.4) and the inequalities (3.5), first observe that (4.12) extends to

$$\delta_q(ae^{i\theta}, ae^{-i\theta}; q)_\alpha = aq^{-1/2}(1-q^\alpha)(e^{i\theta}-e^{-i\theta})(aq^{1/2}e^{i\theta}, aq^{1/2}e^{-i\theta}; q)_{\alpha-1}, \tag{4.16}$$

which gives

$$D_q(ae^{i\theta}, ae^{-i\theta}; q)_\alpha = \frac{-2a(1-q^\alpha)}{1-q}\,(aq^{1/2}e^{i\theta}, aq^{1/2}e^{-i\theta}; q)_{\alpha-1}. \tag{4.17}$$

Hence, corresponding to the inequality (3.5), we have that

$$\begin{aligned}
&D_q[(q^{1-\nu}e^{i\theta}, q^{1-\nu}e^{-i\theta}; q)_{2\nu} u_n(x)] \\
&\quad = A_{n,r} \sum_{k=0}^{r-n} \frac{(q^{n-r}, q^{n+r+2\nu+2\lambda+1}, q^{n+\nu+1/2}, q^{n+2\nu}; q)_k q^k}{(q, q^{2n+2\nu+1}, q^{n+\nu+\lambda+1}, q^{n+2\nu+1}, -q^{n+\nu+\lambda+1}, -q^{n+\nu+1/2}; q)_k} \\
&\qquad \cdot D_q[(q^{1-\nu}e^{i\theta}, q^{1-\nu}e^{-i\theta}; q)_{n+k+2\nu}] \\
&\quad = -\frac{2(1-q^{n+2\nu})}{1-q}\, q^{1-\nu} A_{n,r} (q^{\frac{3}{2}-\nu}e^{i\theta}, q^{\frac{3}{2}-\nu}e^{-i\theta}; q)_{n+2\nu-1} \\
&\qquad \cdot {}_5\phi_4 \left[\begin{matrix} q^{n-r}, q^{n+r+2\nu+2\lambda+1}, q^{n+\nu+1/2}, q^{n+\nu+1/2}e^{i\theta}, q^{n+\nu+1/2}e^{-i\theta} \\ q^{2n+2\nu+1}, q^{n+\nu+\lambda+1}, -q^{n+\nu+\lambda+1}, -q^{n+\nu+1/2} \end{matrix} ; q, q \right] \le 0
\end{aligned} \tag{4.18}$$

by (3.16), when $\nu > -1/2$, $\lambda \ge 0$ and $n = 1, \ldots, r$. This explains why we chose the ${}_6\phi_5$ in (4.7).

To derive a q-extension of (4.4) notice that, corresponding to the left side of (4.4), we have

(4.19)

$$\begin{aligned}
&D_q[(q^{n+\nu+1}e^{i\theta}, q^{n+\nu+1}e^{-i\theta}; q)_{-n} u_n(x)] = A_{n,r} \\
&\quad \cdot D_q\left({}_6\phi_5 \left[\begin{matrix} q^{n-r}, q^{n+r+2\nu+2\lambda+1}, q^{n+\nu+1/2}, q^{n+2\nu}, q^{n+\nu+1}e^{i\theta}, q^{n+\nu+1}e^{-i\theta} \\ q^{2n+2\nu+1}, q^{n+\nu+\lambda+1}, q^{n+2\nu+1}, -q^{n+\nu+\lambda+1}, -q^{n+\nu+1/2} \end{matrix} ; q, q \right] \right) \\
&= \frac{(1-q^{n-r})(1-q^{n+r+2\nu+2\lambda+1})(1-q^{n+\nu+1/2})(1-q^{n+2\nu})(-2)q^{n+\nu+2}}{(1-q)(1-q^{2n+2\nu+1})(1-q^{n+\nu+\lambda+1})(1-q^{n+2\nu+1})(1+q^{n+\nu+\lambda+1})(1+q^{n+\nu+1/2})} A_{n,r} \\
&\quad \cdot {}_6\phi_5 \left[\begin{matrix} q^{n+1-r}, q^{n+r+2\nu+2\lambda+2}, q^{n+\nu+3/2}, q^{n+2\nu+1}, q^{n+\nu+3/2}e^{i\theta}, q^{n+\nu+3/2}e^{-i\theta} \\ q^{2n+2\nu+2}, q^{n+\nu+\lambda+2}, q^{n+2\nu+2}, -q^{n+\nu+\lambda+2}, -q^{n+\nu+3/2} \end{matrix} ; q, q \right].
\end{aligned}$$

Similarly,

(4.20)

$$
\begin{aligned}
&(q^{n+\nu+3/2}e^{i\theta}, q^{n+\nu+3/2}e^{-i\theta}; q)_{-2n-2\nu-1} D_q[(q^{-n-\nu}e^{i\theta}, q^{-n-\nu}e^{-i\theta}; q)_{n+2\nu+1} u_{n+1}(x)] \\
&\qquad = -\frac{2(1-q^{2n+2\nu+2})}{1-q}\, q^{-n-\nu} A_{n+1,r} \\
&\qquad\quad \cdot {}_6\phi_5 \left[\begin{matrix} q^{n+1-r}, q^{n+r+2\nu+2\lambda+2}, q^{n+\nu+3/2}, q^{n+2\nu+1}, q^{n+\nu+3/2}e^{i\theta}, q^{n+\nu+3/2}e^{-i\theta} \\ q^{2n+2\nu+2}, q^{n+\nu+\lambda+2}, q^{n+2\nu+2}, -q^{n+\nu+\lambda+2}, -q^{n+\nu+3/2} \end{matrix} ; q, q \right],
\end{aligned}
$$

which, combined with (4.19), gives the following q-extension of (4.4)

$$
\begin{aligned}
(4.21)\quad & D_q[(q^{n+\nu+1}e^{i\theta}, q^{n+\nu+1}e^{-i\theta}; q)_{-n} u_n(x)] \\
&\qquad = -\frac{1-q^{n+2\nu}}{1-q^{n+1}}\, B_{n,r} (q^{n+\nu+3/2}e^{i\theta}, q^{n+\nu+3/2}e^{-i\theta}; q)_{-2n-2\nu-1} \\
&\qquad\quad \cdot D_q[(q^{-n-\nu}e^{i\theta}, q^{-n-\nu}e^{-i\theta}; q)_{n+2\nu+1} u_{n+1}(x)]
\end{aligned}
$$

with

$$
B_{n,r} = \frac{16 q^{3n-r+2\nu+2}}{(1+q^{n+\nu+\lambda+1})(1+q^{n+\nu+1})(1+q^{n+\nu+1/2})^2}. \tag{4.22}
$$

Clearly, $B_{n,r} \to 1$ and (4.21) tends to (4.4) as $q \to 1$; but, unlike (4.4), the difference equation (4.21) depends on r. However, if we consider following positive multiple of u_n

$$
U_n(x) = \frac{4^{2n} q^{3n^2/2+n(2\nu+1/2-r)}}{(-q^{\nu+\lambda+1}, -q^{\nu+1}, -q^{\nu+1/2}, -q^{\nu+1/2}; q)_n}\, u_n(x), \tag{4.23}
$$

we find that it satisfies the difference equation

$$
\begin{aligned}
(4.24)\quad & D_q[(q^{n+\nu+1}e^{i\theta}, q^{n+\nu+1}e^{-i\theta}; q)_{-n} U_n(x)] \\
&\qquad = -\frac{1-q^{n+2\nu}}{1-q^{n+1}} (q^{n+\nu+3/2}e^{i\theta}, q^{n+\nu+3/2}e^{-i\theta}; q)_{-2n-2\nu-1} \\
&\qquad\quad \cdot D_q[(q^{-n-\nu}e^{i\theta}, q^{-n-\nu}e^{-i\theta}; q)_{n+2\nu+1} U_{n+1}(x)],
\end{aligned}
$$

which is independent of r.

Similarly, setting

(4.25)

$$
\begin{aligned}
&V_n(x) = C_{n,r} (q^{\nu+\lambda+1}e^{i\theta}, q^{\nu+\lambda+1}e^{-i\theta}; q)_n \\
&\quad \cdot {}_7\phi_6 \left[\begin{matrix} q^{n-r}, q^{n+r+2\nu+2\lambda+1}, q^{n+\nu+1/2}, -q^{n+\nu+1/2}, q^{n+2\nu}, q^{n+\nu+\lambda+1}e^{i\theta}, q^{n+\nu+\lambda+1}e^{-i\theta} \\ q^{2n+2\nu+1}, q^{n+\nu+\lambda+1}, -q^{n+\nu+\lambda+1}, q^{n+2\nu+1}, -q^{n+\nu+\lambda+1/2}, -q^{n+\nu+\lambda+1/2} \end{matrix} ; q, q \right]
\end{aligned}
$$

with

$$(4.26)\quad C_{n,r}=\frac{(q;q)_n(q^{2\nu+2\lambda+1};q)_{n+r}q^{3n^2/2+n(2\nu+1/2-r)}}{(q;q)_{r-n}(q^{\nu+1},q^{2\nu+1},q^{\nu+\lambda+1},-q^{\nu+1},-q^{\nu+\lambda+1},-q^{\nu+\lambda+1/2},-q^{\nu+\lambda+1/2};q)_n}$$

when $n=1,\dots,r$ and $C_{n,r}=0$ when $n>r$, we obtain that $V_n(x)\to\tau_n(x)$ as $q\to 1$, $V_n(x)$ satisfies the following q-extension of (4.4)

$$(4.27)\quad \begin{aligned} &D_q[(q^{n+\nu+\lambda+1}e^{i\theta},q^{n+\nu+\lambda+1}e^{-i\theta};q)_{-n}V_n(x)]\\ &\quad=-\frac{1-q^{n+2\nu}}{1-q^{n+1}}\,(q^{n+\nu+\lambda+3/2}e^{i\theta},q^{n+\nu+\lambda+3/2}e^{-i\theta};q)_{-2n-2\nu-1}\\ &\quad\cdot D_q[(q^{-n-\nu+\lambda}e^{i\theta},q^{-n-\nu+\lambda}e^{-i\theta};q)_{n+2\nu+1}V_{n+1}(x)]\end{aligned}$$

and, by (3.18),

(4.28)

$$\begin{aligned} &D_q[(q^{1+\lambda-\nu}e^{i\theta},q^{1+\lambda-\nu}e^{-i\theta};q)_{2\nu}V_n(x)]=-\frac{2(1-q^{n+2\nu})}{1-q}\,q^{1+\lambda-\nu}C_{n,r}\\ &\quad\cdot(q^{\lambda+3/2-\nu}e^{i\theta},q^{\lambda+3/2-\nu}e^{-i\theta};q)_{n+2\nu-1}\\ &\cdot{}_6\phi_5\left[\begin{matrix} q^{n-r},q^{n+r+2\nu+2\lambda+1},q^{n+\nu+1/2},-q^{n+\nu+1/2},q^{n+\nu+\lambda+1/2}e^{i\theta},q^{n+\nu+\lambda+1/2}e^{-i\theta}\\ q^{2n+2\nu+1},q^{n+\nu+\lambda+1},-q^{n+\nu+\lambda+1},-q^{n+\nu+\lambda+1/2},-q^{n+\nu+\lambda+1/2}\end{matrix};q,q\right]\le 0\end{aligned}$$

when $\nu>-1/2$, $\lambda\ge 0$ and $n=1,\dots,r$.

The q-extensions of de Branges' inequalities and differential equations contained in this paper suggest that it might be possible to extend some of the other parts of his proof of the Bieberbach, Robertson, and Milin conjectures. Besides (1.2) and (1.7), de Branges also used the fact that if $F(t,z)$ is a Löwner family of Riemann mapping functions, then

$$(4.29)\qquad t\frac{\partial}{\partial t}F(t,z)=\varphi(t,z)z\frac{\partial}{\partial z}F(t,z),$$

where $\varphi(t,z)$ is a power series with constant coefficient equal to 1, which represents a function with positive real part in the unit disk for every index t, and the coefficients of $\varphi(t,z)$ are measurable functions of t. q-Extensions of the Löwner [29] theory and of the coefficient estimates for Riemann mapping functions in [12] and [13, Thms. 1–4] are still open. In view of the definition of D_q in (4.11), a prospect for a q-extension of (4.29) is the equation

$$(4.30)\qquad (1-x)D_qG(x,z)=\Phi(x,z)z\frac{\partial}{\partial z}G(x,z)$$

or this equation with the partial derivative replaced by a difference operator. For an extremal function that is a q-extension of the Koebe function, the ${}_1F_0$ series representation for the Koebe function

$$(4.31)\qquad k(z)=\frac{z}{(1-z)^2}=z\;{}_1F_0\left[\begin{matrix}2\\ -\end{matrix};z\right]$$

suggests that a natural choice is the "q-Koebe" function

$$k_q(z) = z\ {}_1\phi_0 \left[\begin{matrix} q^2 \\ - \end{matrix} ; q, z \right] = \frac{z}{(1-z)(1-qz)}. \tag{4.32}$$

Note that, just as the Koebe function is starlike, $k_q(z)$ is a starlike function when $-1 < q < 1$, which can be shown by using [16, Thm. 2.10] and the positivity of the Poisson kernel for Fourier series.

REFERENCES

[1] G. E. Andrews, *On q-analogues of the Watson and Whipple summations*, SIAM J. Math. Anal., 7 (1976), pp. 332–336.

[2] G. E. Andrews and R. Askey, *Enumeration of partitions: The role of Eulerian series and q-orthogonal polynomials*, in Higher Combinatorics, M. Aigner and D. Reidel, eds., Dordrecht, Holland, 1977, pp. 3–26.

[3] R. Askey, *Orthogonal Polynomials and Special Functions*, Regional Conference Series in Applied Mathematics 21, Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1975.

[4] R. Askey and G. Gasper, *Positive Jacobi polynomial sums II*, Amer. J. Math., 98 (1976), pp. 709–737.

[5] ———, *Inequalities for polynomials*, in The Bieberbach Conjecture: Proc. of the Symposium on the Occasion of the Proof, A. Baernstein, D. Drasin, P. Duren, and A. Marden, eds., Math. Surveys Monographs 21, American Mathematical Society, Providence, R. I. 1986, pp. 7–32.

[6] R. Askey and M. Ismail, *A generalization of ultraspherical polynomials*, in Studies in Pure Mathematics, P. Erdös, ed., Birkhäuser, Basel, 1983, pp. 55–78.

[7] R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc., No. 319, 1985.

[8] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge University Press, London, 1935 (reprinted by Stechert-Hafner, New York, 1964).

[9] L. Bieberbach, *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, Sitzungsberichte Preuss. Akad. Wiss., Berlin, 1916, pp. 940–955.

[10] L. de Branges, *Quantum Cesàro operators*, in Topics in Functional Analysis, Advances in Mathematics Supplementary Studies, Vol. 3, I. Gohberg and M. Kac, eds., Academic Press, New York, 1978.

[11] ———, *A proof of the Bieberbach conjecture*, USSR Academy of Sciences, Steklov Math. Institute, LOMI, preprint E-5-84, Leningrad, 1984.

[12] ―――――, *A proof of the Bieberbach conjecture*, Acta Math., 154 (1985), pp. 137–152.

[13] ―――――, *Powers of Riemann mapping functions*, in The Bieberbach Conjecture: Proc. of the Symposium on the Occasion of the Proof, A. Baernstein, D. Drasin, P. Duren, and A. Marden, eds., Math. Surveys and Monographs, No. 21, 1986, American Mathematical Society, Providence, R. I., pp. 51–67.

[14] D. M. Bressoud, *Linearization and related formulas for q-ultraspherical polynomials*, SIAM J. Math. Anal., 12 (1981), pp. 161–168.

[15] T. Clausen, *Ueber die Fälle, wenn die Reihe von der Form … ein Quadrat von der Form … hat*, J. Reine Angew. Math., 3 (1828), pp. 89–91.

[16] P. L. Duren, *Univalent Functions*, Springer-Verlag, Berlin and New York, 1983.

[17] J. L. Fields and J. Wimp, *Expansions of hypergeometric functions in hypergometric functions*, Math. Comp., 15 (1961), pp. 390–395.

[18] G. Gasper, *Positivity and special functions*, in Theory and Applications of Special Functions, R. Askey, ed., Academic Press, New York, 1975, pp. 375–433.

[19] ―――――, *Positive sums of the classical orthogonal polynomials*, SIAM J. Math. Anal., 8 (1977), pp. 423–447.

[20] ―――――, *Rogers' linearization formula for the continuous q-ultraspherical polynomials and quadratic transformation formulas*, SIAM J. Math. Anal., 16 (1985), pp. 1061–1071.

[21] ―――――, *A short proof of an inequality used by de Branges in his proof of the Bieberbach, Robertson and Milin conjectures*, Complex Variables, Theory Appl., 7 (1986), pp. 45–50.

[22] ―――――, *Summation, transformation, and expansion formulas for bibasic series*, Trans. Amer. Math. Soc., to appear.

[23] ―――――, *Expansion formulas for basic hypergeometric series and nonnegative sums of Askey-Wilson polynomials*, to appear.

[24] G. Gasper and M. Rahman, *Product formulas of Watson, Bailey and Bateman types and positivity of the Poisson kernel for q-Racah polynomials*, SIAM J. Math. Anal., 15 (1984), pp. 768–789.

[25] ―――――, *Basic Hypergeometric Series*, Cambridge University Press, to appear.

[26] ―――――, *A nonterminating q-Clausen formula and some related product formulas*, SIAM J. Math. Anal., to appear.

[27] F. H. Jackson, *The q^θ equations whose solutions are products of solutions of q^θ equations of lower order*, Quart. J. Math. Oxford, 11 (1940), pp. 1–17.

[28] ―――――, *Certain q-identities*, Quart. J. Math. Oxford, 12 (1941), pp. 167–172.

[29] K. Löwner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. I*, Math. Ann., 89 (1923), pp. 103–121.

[30] I. M. Milin, *Univalent Functions and Orthogonal Systems*, Transl. Math. Monographs 49, American Mathematical Society, Providence, R. I., 1977.

[31] M. Rahman, *The linearization of the product of continuous q-Jacobi polynomials*, Canad. J. Math., 23 (1981), pp. 961–987.

[32] M. Rahman and A. Verma, *Product and addition formulas for the continuous q-ultraspherical polynomials*, SIAM J. Math. Anal., 17 (1986), pp. 1461–1474.

[33] M. S. Robertson, *A remark on the odd schlicht functions*, Bull. Amer. Math. Soc., 42 (1936), pp. 366–370.

[34] L. J. Rogers, *On the expansion of some infinite products*, Proc. London Math. Soc., 24 (1893), pp. 337–352.

[35] ―――――, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc., 25 (1894), pp. 318–343.

[36] ―――――, *Third memoir on the expansion of certain infinite products*, Proc. London Math. Soc., 26 (1895), pp. 15–32.

[37] D. B. Sears, *On the transformation theory of basic hypergeometric functions*, Proc. London Math. Soc. (2) 53 (1951), pp. 158–180.

[38] L. J. Slater, *Generalized Hypergometric Functions*, Cambridge University Press, London, 1966.

[39] A. Verma, *Certain expansions of the basic hypergeometric functions*, Math. Comp., 20 (1966), pp. 151–157.

[40] ―――――, *Some transformations of series with arbitrary terms*, Instituto Lombardo (Rend. Sc.) A 106 (1972), pp. 342–353.