An Indefinite Bibasic Summation Formula and Some Quadratic, Cubic, and Quartic Summation and Transformation Formulas

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Abstract. An indefinite bibasic summation formula containing four arbitrary parameters is derived and used to to derive a bilateral bibasic summation formula and some rather general quadratic, cubic, and quartic summation and transformation formulas. In particular, a cubic transformation formula with four parameters is derived which contains as a special case a q-extension of Gosper's previously unproven strange evaluation of a ${}_7F_6$ series.

1. Introduction. Let

$$(a;q)_n = \begin{cases} 1, & n=0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n=1,2,\dots, \\ [(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{n})]^{-1}, & n=-1,-2,\dots, \end{cases}$$

denote the q-shifted factorial,

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \qquad |q|<1,$$

and set

$$(a_1,a_2,\dots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,$$
$$(a_1,a_2,\dots,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

Recently, Gasper [7] showed that some bibasic summation formulas derived by Carlitz [5], Al-Salam and Verma [1], and Wm. Gosper could be extended to the indefinite bibasic summation formula

$$\sum_{k=0}^{n} \frac{(1-ap^kq^k)(1-bp^kq^{-k})}{(1-a)(1-b)} \frac{(a,b;p)_k(c,a/bc;q)_k}{(q,aq/b;q)_k(ap/c,bcp;p)_k} q^k \tag{1.1}$$
$$= \frac{(ap,bp;p)_n(cq,aq/bc;q)_n}{(q,aq/b;q)_n(ap/c,bcp;p)_n}, \qquad n=0,1,2,\dots,$$

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where p and q are independent bases and a, b, c are arbitrary parameters. He also showed that this formula could be used to derive a bibasic extension of Euler's transformation formula [7, (1.6)], a bibasic Lagrange inversion formula, a bibasic extension of Verma's q-analogue [13, p. 349] of the Fields and Wimp expansion formula [6, (1.3)], and some quadratic, cubic, and quartic summation formulas containing at least two parameters.

In particular, he derived the cubic summation formula

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{1-aq^{4k}}{1-a}\,\frac{(a,b;q)_k(q/b,q^2/b;q^2)_k(c,a^2b/c;q^3)_k}{(q^3,aq^3/b;q^3)_k(abq,ab;q^2)_k(aq/c,cq/ab;q)_k}\,q^k\\&+\frac{ab^2}{c}\,\frac{(aq,bq,1/b;q)_\infty(c,a^2bq^3/c^2;q^3)_\infty}{(ab,cq/ab,aq/c;q)_\infty(ab^2/c,aq^3/b;q^3)_\infty}\\&\cdot{}_2\phi_1\left[\begin{matrix}ab^2/c,a^2b/c\\a^2bq^3/c^2\end{matrix};q^3,q^3\right]\\&=\frac{(aq,ab/c;q)_\infty(ab^2,aq^3/bc;q^3)_\infty}{(ab,aq/c;q)_\infty(ab^2/c,aq^3/b;q^3)_\infty},\qquad |q|<1,\end{aligned}\tag{1.2}$$

and its hypergeometric limit case

$$\begin{aligned}&{}_7F_6\left[\begin{matrix}3a,1+3a/4,3b,(1-3b)/2,1-3b/2,c,2a+b-c\\3a/4,1+a-b,(1+3a+3b)/2,3(a+b)/2,1+3a-3c,1+3c-3a-3b\end{matrix};1\right]\\&=\frac{\Gamma(3a+3b)\Gamma(1+3a-3c)\Gamma(a+2b-c)\Gamma(1+a-b)}{\Gamma(3a+1)\Gamma(3a+3b-3c)\Gamma(a+2b)\Gamma(1+a-b-c)}\\&\cdot\left\{1+\frac{\sin 3\pi b\sin\pi c}{\sin 3\pi(a+b-c)\sin\pi(a+2b)}\right\}.\end{aligned}\tag{1.3}$$

The ${}_2\phi_1$ function in (1.2) is a special case of the ${}_r\phi_s$ basic hypergeometric series defined by

$${}_r\phi_s\left[\begin{matrix}a_1,\ldots,a_r\\b_1,\ldots,b_s\end{matrix};q,z\right]=\sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n}\,z^n[(-1)^nq^{\binom{n}{2}}]^{1+s-r}\tag{1.4}$$

with $\binom{n}{2}=n(n-1)/2$. In (1.4) and all formulas containing infinite sums or products it is assumed that $|q|<1$. An ${}_rF_s$ hypergeometric series is defined by

$${}_rF_s\left[\begin{matrix}a_1,\ldots,a_r\\b_1,\ldots,b_s\end{matrix};z\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_r)_n}{n!(b_1)_n\cdots(b_s)_n}\,z^n\tag{1.5}$$

where the shifted factorial $(a)_n$ is defined by

$$(a)_n=\prod_{k=0}^{n-1}(a+k).$$

In their paper [9] on strange evaluations of hypergeometric series, Gessel and Stanton were able to prove all of the "mysterious-looking evaluations" with more than one parameter that were stated by Wm. Gosper in a letter to R. Askey except for the evaluation

$$
\begin{aligned}
&{}_7F_6\left[\begin{matrix} a, a+1/2, b, 1-b, c, (2a+1)/3-c, a/2+1 \\ 1/2, (2a-b+3)/3, (2a+b+2)/3, 3c, 2a+1-3c, a/2 \end{matrix}; 1\right] \qquad (1.6)\\
&= \frac{2}{\sqrt{3}} \frac{\Gamma\left(c+\frac{1}{3}\right)\Gamma\left(c+\frac{2}{3}\right)\Gamma\left(\frac{2a-b+3}{3}\right)\Gamma\left(\frac{2a+b+2}{3}\right)}{\Gamma\left(\frac{2a+2}{3}\right)\Gamma\left(\frac{2a+3}{3}\right)\Gamma\left(\frac{3c+b+1}{3}\right)\Gamma\left(\frac{3c+2-b}{3}\right)}\\
&\cdot \frac{\Gamma\left(\frac{2+2a-3c}{3}\right)\Gamma\left(\frac{3+2a-3c}{3}\right)\sin\frac{\pi}{3}(b+1)}{\Gamma\left(\frac{2+2a+b-3c}{3}\right)\Gamma\left(\frac{3+2a-b-3c}{3}\right)}.
\end{aligned}
$$

Since the case $b = 2a$ of (1.6) is a special case of (1.3), this raised the question of whether (1.6) and a q-extension of it could be derived by extending the method in [7] and it led the authors to look for extensions of (1.1) and (1.2).

In this paper we show that (1.1) can be extended to

$$
\begin{aligned}
&\sum_{k=-m}^{n} \frac{(1-adp^kq^k)(1-bp^k/dq^k)}{(1-ad)(1-b/d)} \frac{(a,b;p)_k(c,ad^2/bc;q)_k}{(dq,adq/b;q)_k(adp/c,bcp/d;p)_k} q^k \qquad (1.7)\\
&= \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}\\
&\cdot \left\{ \frac{(ap,bp;p)_n(cq,ad^2q/bc;q)_n}{(dq,adq/b;q)_n(adp/c,bcp/d;p)_n} - \frac{(c/ad,d/bc;p)_{m+1}(1/d,b/ad;q)_{m+1}}{(1/c,bc/ad^2;q)_{m+1}(1/a,1/b;p)_{m+1}} \right\},
\end{aligned}
$$

where $n, m = 0, \pm1, \pm2, \ldots$, and then use this formula to derive the cubic summation formula

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{1-a^2q^{4k}}{1-a^2} \frac{(b,q/b;q)_k(a,aq;q^2)_k(c^3,a^2q/c^3;q^3)_k}{(a^2q^3/b,a^2bq^2;q^3)_k(q^2,q;q^2)_k(a^2q/c^3,c^3;q)_k} q^k \qquad (1.8)\\
&- \frac{(b,q/b,a^2q;q)_\infty(c^3,q^3,a^2q/c^3,c^6q^2/a^2;q^3)_\infty}{(q,c^3,a^2q/c^3;q)_\infty(bc^3/a^2,a^2bq^2,a^2q^3/b,c^3q/a^2b;q^3)_\infty} {}_2\phi_1\left[\begin{matrix} bc^3/a^2, c^3q/a^2b \\ c^6q^2/a^2 \end{matrix}; q^3, q^3\right]\\
&= \frac{(a^2q^2,a^2q^3,c^3/a^2,c^3q/a^2,bc^3q,bq,q^2/b,c^3q^2/b;q^3)_\infty}{(q,q^2,c^3q,c^3q^2,a^2q^3/b,a^2bq^2,bc^3/a^2,c^3q/a^2b;q^3)_\infty}, \qquad |q|<1,
\end{aligned}
$$

which contains Gosper's sum (1.6) as a limit case. It should be noted that in (1.7) and elsewhere we employ the standard convention of defining

$$
\sum_{k=m}^{n} a_k = \begin{cases} a_m + a_{m+1} + \cdots + a_n, & m \le n, \\ 0, & m = n+1, \\ -(a_{n+1} + a_{n+2} + \cdots + a_{m-1}), & m \ge n+2, \end{cases} \qquad (1.9)
$$

for $n, m = 0, \pm1, \pm2, \ldots$.

The bibasic sum (1.7) will also be used to derive a common generalization of (1.2) and (1.8) and some rather general quadratic, cubic, and quartic summation and transformation formulas.

2. Derivation of (1.7). Let

$$s_n = \frac{(ap, bp; p)_n (cq, ad^2q/bc; q)_n}{(dq, adq/b; q)_n (adp/c, bcp/d; p)_n} \tag{2.1}$$

for $n = 0, \pm 1, \pm 2, \ldots$, and define the difference operator Δ by

$$\Delta s_n = s_n - s_{n-1}.$$

Then

$$\sum_{k=-m}^{n} \Delta s_k = s_n - s_{-m-1} \tag{2.2}$$

and

$$\begin{aligned}\Delta s_k &= s_k - s_{k-1} \\ &= \frac{(ap, bp; p)_{k-1} (cq, ad^2q/bc; q)_{k-1}}{(dq, adq/b; q)_k (adp/c, bcp/d; p)_k} \\ &\quad \cdot \big\{(1-ap^k)(1-bp^k)(1-cq^k)(1-ad^2q^k/bc) \\ &\qquad - (1-dq^k)(1-adq^k/b)(1-adp^k/c)(1-bcp^k/d)\big\} \\ &= \frac{d(1-c/d)(1-ad/bc)(1-adp^kq^k)(1-bp^k/dq^k)}{(1-a)(1-b)(1-c)(1-ad^2/bc)} \, \frac{(a, b; p)_k (c, ad^2/bc; q)_k q^k}{(dq, adq/b; q)_k (adp/c, bcp/d; p)_k}\end{aligned} \tag{2.3}$$

for $k = 0, \pm 1, \pm 2, \ldots$. Hence

$$\begin{aligned}&\sum_{k=-m}^{n} \frac{(1-adp^kq^k)(1-bp^k/dq^k)}{(1-ad)(1-b/d)} \, \frac{(a, b; p)_k (c, ad^2/bc; q)_k}{(dq, adq/b; q)_k (adp/c, bcp/d; p)_k} \, q^k \\ &= \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)} \{s_n - s_{-m-1}\} \\ &= \frac{(1-a)(11-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)} \\ &\quad \cdot \left\{ \frac{(ap, bp; p)_n (cq, ad^2q/bc; q)_n}{(dq, adq/b; q)_n (adp/c, bcp/d; p)_n} - \frac{(c/ad, d/bc; p)_{m+1} (1/d, b/ad; q)_{m+1}}{(1/c, bc/ad^2; q)_{m+1} (1/a, 1/b; p)_{m+1}} \right\}\end{aligned} \tag{2.4}$$

for $n, m = 0, \pm 1, \pm 2, \ldots$, which completes the proof of (1.7). In (2.4) we used the fact that

$$(a; q)_{-n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a; q)_n}. \tag{2.5}$$

Observe that (1.1) is the case $d = 1$, $m = 0$ of (1.7) and that, just as in (1.1), the series in (1.7) is partially of well-poised type in the sense that $a(dq) = b(adq/b) = adq$ and $c(adp/c) = (ad^2/bc)(bcp/d) = adp$. It is these observations and the simplification that occurred on the right side of (2.3) which motivated the choice of s_n in (2.1).

If $|p| < 1$ and $|q| < 1$, then by letting n or m tend to infinity we find that (1.7) also holds with n or m replaced by ∞. In particular, this gives the following evaluation of a bilateral bibasic series

$$\sum_{k=-\infty}^{\infty} \frac{(1-adp^kq^k)(1-bp^k/dq^k)}{(1-ad)(1-b/d)} \frac{(a,b;p)_k(c,ad^2/bc;q)_k}{(dq,adq/b;q)_k(adp/c,bcp/d;p)_k} q^k \tag{2.6}$$
$$= \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}$$
$$\cdot \left\{ \frac{(ap,bp,p)_\infty(cq,ad^2q/bc;q)_\infty}{(dq,adq/b;q)_\infty(adp/c,bcp/d;p)_\infty} - \frac{(c/ad,d/bc;p)_\infty(1/d,b/ad;q)_\infty}{(1/c,bc/ad^2;q)_\infty(1/a,1/b;p)_\infty} \right\},$$

where $|p| < 1$ and $|q| < 1$.

In the following sections we shall use the $m = 0$ case of (1.7) in the form

$$\sum_{k=0}^{n} \frac{(1-adp^kq^k)(1-bp^k/dq^k)}{(1-ad)(1-b/d)} \frac{(a,b;p)_k(c,ad^2/bc;q)_k}{(dq,adq/b;q)_k(adp/c,bcp/d;p)_k} q^k \tag{2.7}$$
$$= \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)} \frac{(ap,bp;p)_n(cq,ad^2q/bc;q)_n}{(dq,adq/b;q)_n(adp/c,bcp/d;p)_n}$$
$$- \frac{(1-d)(1-ad/b)(1-ad/c)(1-bc/d)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}.$$

There is no loss in generality because, by setting $k = j - m$ in (1.7), it is easily seen that (1.7) is equivalent to (2.7) with n, a, b, c, d replaced by $n+m$, ap^{-m}, bp^{-m}, cq^{-m}, dq^{-m}, respectively. We shall also use the special case $c = q^{-n}$, $n = 0, 1, 2, \ldots$, of (2.7) in the form

$$\sum_{k=0}^{n} \frac{(1-adp^kq^k)(1-bp^k/dq^k)}{(1-ad)(1-b/d)} \frac{(a,b;p)_k(q^{-n},ad^2q^n/b;q)_k}{(dq,adq/b;q)_k(adpq^n,bp/dq^n;p)_k} q^k \tag{2.8}$$
$$= \frac{(1-d)(1-ad/b)(1-adq^n)(1-dq^n/b)}{(1-ad)(1-d/b)(1-dq^n)(1-adq^k/b)}, \qquad n = 0, 1, 2, \ldots.$$

3. q-Extensions of Gosper's sum (1.6). In view of our previous observation that the case $b = 2a$ of (1.6) is a special case of the sum (1.3), it is natural to try to use the bibasic sum (2.8) to extend the proof in [7] of the q-analogue (1.2) of (1.3) to derive a summation formula which contains both (1.3) and (1.6) as limit cases. Therefore, let us set $q = p^3$ in (2.8), replace d by c, and then change p to q to obtain the sum

$$\sum_{k=0}^{n} \frac{(1-acq^{4k})(1-b/cq^{2k})}{(1-ac)(1-b/c)} \frac{(a,b;q)_k(q^{-3n},ac^2q^{3n}/b;q^3)_k}{(cq^3,acq^3/b;q^3)0_k(acq^{3n+1},b/cq^{3n-1};q)_k} q^{3k} \tag{3.1}$$
$$= \frac{(1-c)(1-ac/b)(1-acq^{3n})(1-cq^{3n}/b)}{(1-ac)(1-c/b)(1-cq^{3n})(1-acq^{3n}/b)}, \qquad n = 0, 1, 2, \ldots,$$

which reduces to [7, (5.18)] when $c = 1$.

Multiply both sides of (3.1) by

$$\frac{(ac^2/b;q^3)_n(c/b;q)_{3n}}{(q^3;q^3)_n(acq;q)_{3n}}\,A_n$$

and sum over n to get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(ac^2/b;q^3)_n(cq/b;q)_{3n}}{(q^3;q^3)_n(ac;q)_{3n}}\,\frac{(1-c)(1-ac/b)}{(1-cq^{3n})(1-acq^{3n}/b)}\,A_n \qquad (3.2)\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(1-acq^{4k})(1-b/cq^{2k})}{(1-ac)(1-b/c)}\,\frac{(a,b;q)_k}{(cq^3,acq^3/b;q^3)_k}\\
&\cdot\frac{(ac^2/b;q^3)_{n+k}(c/b;q)_{3n-k}}{(q^3;q^3)_{n-k}(acq;q)_{3n+k}}\,q^{k(k+1)}\left(\frac{c}{b}\right)^k A_n\\
&=\sum_{k=0}^{\infty}\frac{(1-acq^{4k})(1-b/cq^{2k})}{(1-ac)(1-b/c)}\,\frac{(a,b;q)_k(ac^2/b;q^3)_{2k}(c/b;q)_{2k}}{(cq^3,acq^3/b;q^3)_k(acq;q)_{4k}}\,q^{k(k+1)}\left(\frac{c}{b}\right)^k\\
&\cdot\sum_{j=0}^{\infty}\frac{(ac^2q^{6k}/b;q^3)_j(cq^{2k}/b;q)_{3j}}{(q^3;q^3)_j(acq^{4k+1};q)_{3j}}\,A_{j+k}.
\end{aligned}$$

Now choose

$$A_n=\frac{1-ac^2q^{6n}/b}{1-ac^2/b}\,\frac{(d,e;q^3)_n}{(ac^2q^3/bd,ac^2q^3/be;q^3)_n}\,z^n$$

to obtain the expression

$$\begin{aligned}
&{}_{10}W_9(ac^2/b;c,d,e,ac/b,cq/b,cq^2/b,cq^3/b;q^3,z) \qquad (3.3)\\
&=\sum_{k=0}^{\infty}\frac{(1-acq^{4k})(1-b/cq^{2k})}{(1-ac)(1-b/c)}\,\frac{(a,b;q)_k(ac^2q^3/b;q^3)_{2k}(c/b;q)_{2k}}{(cq^3,acq^3/b;q^3)_k(acq;q)_{4k}}\\
&\cdot\frac{(d,e;q^3)_k}{(ac^2q^3/bd,ac^2q^3/bc;q^3)_k}\,q^{k(k+1)}\left(\frac{c}{b}\right)^k z^k\\
&\cdot{}_8W_7(ac^2q^{6k}/b;dq^{3k},eq^{3k},cq^{2k}/b,cq^{2k+1}/b,cq^{2k+2}/b;q^3,z),
\end{aligned}$$

where we let

$$\begin{aligned}
&{}_{r+1}W_r(a;b_1,b_2,\ldots,b_{r-2};q,z) \qquad (3.4)\\
&={}_{r+1}\phi_r\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},b_1,b_2,\ldots,b_{r-2}\\ \sqrt{a},-\sqrt{a},aq/b_1,aq/b_2,\ldots,aq/b_{r-2}\end{matrix};q,z\right].
\end{aligned}$$

Observe that if $e=a^2bcq^{3n}$, $z=q^3$, and $d=q^{-3n}$, $n=0,1,2,\ldots$, then the ${}_8W_7$ in (3.3) can be summed by Jackson's sum [12,(3.3.1.1)]

$$\begin{aligned}
&{}_8W_7(a;b,c,d,e,q^{-n};q,q) \qquad (3.5)\\
&=\frac{(aq,aq/bc,aq/cd,aq/bd;q)_n}{(aq/b,aq/c,aq/d,aq/bcd;q)_n},\qquad n=0,1,2,\ldots,
\end{aligned}$$

where $a^2q^{n+1} = bcde$, giving the formula

$$\sum_{k=0}^{n} \frac{1-acq^{4k}}{1-ac} \frac{(a,b;q)_k(cq/b;q)_{2k}(a^2bcq^{3n},q^{-3n};q^3)_k}{(cq^3,acq^3/b;q^3)_k(ab;q)_{2k}(1/abq^{3n-1},acq^{3n+1};q)_k} q^k \tag{3.6}$$
$$= \frac{(acq;q)_{3n}(ab^2/c;q^3)_n}{(ab;q)_{3n}(ac^2q^3/b;q^3)_n} {}_{10}W_9(ac^2/b;c,ac/b,cq/b,cq^2/b,cq^3/b,a^2bcq^{3n},q^{-3n};q^3,q^3)$$

for $n = 0,1,2,\ldots$.

If $b = q/a$, then the above ${}_{10}W_9$ reduces to a ${}_8W_7$ that can also be summed by (3.5), giving the cubic summation formula

$$\sum_{k=0}^{n} \frac{1-acq^{4k}}{1-ac} \frac{(a,q/a;q)_k(ac;q)_{2k}(acq^{3n+1},q^{-3n};q^3)_k}{(cq^3,a^2cq^2;q^3)_k(q;q)_{2k}(q^{-3n},acq^{3n+1};q)_k} q^k \tag{3.7}$$
$$= \frac{(aq,q^2/a,acq^2,acq^3;q^3)_n}{(q,q^2,a^2cq^2,cq^3;q^3)_n}, \qquad n = 0,1,2,\ldots,$$

which is a q-extension of the terminating case of Gosper's sum (1.6).

To derive a nonterminating extension of (3.7), we first set $e = a^2bc/d$ and $z = q^3$ in (3.3) to obtain

$${}_{10}W_9(ac^2/b;c,d,ac/b,a^2bc/d,cq/b,cq^2/b,cq^3/b;q^3,q^3) \tag{3.8}$$
$$= \sum_{k=0}^{\infty} \frac{(1-acq^{4k})(1-b/cq^{2k})}{(1-ac)(1-b/c)} \frac{(a,b;q)_k(ac^2q^3/b;q^3)_{2k}(c/b;q)_{2k}}{(cq^3,acq^3/b;q^3)_k(acq;q)_{4k}} q^{k(k+4)} \left(\frac{c}{b}\right)^k$$
$$\cdot \frac{(d,a^2bc/d;q^3)_k}{(ac^2q^3/bd,cdq^3/ab^2;q^3)_k} {}_8W_7(ac^2q^{6k}/b;dq^{3k},a^2bcq^{3k}/d,cq^{2k}/b,cq^{2k+1}/b,cq^{2k+2}/b;q^3,q^3).$$

Next we apply Bailey's nonterminating extension [12, (IV.15). p. 248] of (3.5)

$${}_8W_7(a;b,c,d,e,f;q,q) \tag{3.9}$$
$$= \frac{(aq,b/a,aq/de,aq/ce,aq/cd,aq/cf,aq/df,aq/ef;q)_\infty}{(aq/c,aq/d,aq/e,aq/f,bc/a,bd/a,be/a,bf/a;q)_\infty}$$
$$- \frac{(bq/c,bq/d,bq/e,bq/f,aq,c,d,e,f,b/a;q)_\infty}{(b^2q/a,bc/a,bd/a,be/a,bf/a,aq/c,aq/d,aq/e,aq/f,a/b;q)_\infty}$$
$$\cdot {}_8W_7(b^2/a;b,bc/a,bd/a,be/a,bf/a;q,q),$$

where $a^2q = bcdef$, to the ${}_8W_7$ in (3.8) to obtain

$$
\begin{aligned}
&{}_8W_7(ac^2q^{6k}/b; dq^{3k}, a^2bcq^{3k}/d, cq^{2k}/b, cq^{2k+1}/b, cq^{2k+2}/b; q^3, q^3) \qquad (3.10)\\
&= \frac{(ac^2q^{6k+3}/b, bd/ac^2q^{3k}, dq^{k+1}/ab, dq^{k+2}/ab, dq^{k+3}/ab, abq^{2k}, abq^{2k+1}, abq^{2k+2}; q^3)_\infty}{(cdq^{3k+3}/ab^2, acq^{4k+1}, acq^{4k+2}, acq^{4k+3}, ab^2/c, dq^{-k}/ac, dq^{1-k}/ac, dq^{2-k}/ac; q^3)_\infty}\\
&= \frac{bdq^{-3k}}{ac^2}\, \frac{(ac^2q^{6k+3}/b, a^2bcq^{3k}/d, cq^{2k}/b, cq^{2k+1}/b, cq^{2k+2}/b, bdq^{3-3k}/ac^2, d^2q^3/a^2bc; q^3)_\infty}{(ac^2q^{3k+3}/bd, cdq^{3k+3}/ab^2, acq^{4k+1}, acq^{4k+2}, acq^{4k+3}, ab^2/c, bd^2q^3/ac^2; q^3)_\infty}\\
&\cdot \frac{(bdq^{k+1}/c, bdq^{k+2}/c, bdq^{k+3}/c; q^3)_\infty}{(dq^{-k}/ac, dq^{1-k}/ac, dq^{2-k}/ac; q^3)_\infty}\\
&\cdot {}_8W_7(bd^2/ac^2; dq^{3k}, ab^2/c, dq^{-k}/ac, dq^{1-k}/ac, dq^{2-k}/ac; q^3, q^3)\\
&= \frac{(ac^2q^3/b, bd/ac^2; q^3)_\infty (dq/ab, ab; q)_\infty}{(ab^2/c, cdq^3/ab^2; q^3)_\infty (acq, d/ac; q)_\infty}\, \frac{(cdq^3/ab^2, ac^2q^3/bd; q^3)_k (acq; q)_{4k} q^{-k(k+1)}}{(ac^2q^3/b; q^3)_{2k} (ab; q)_{2k} (dq/ab, acq/d; q)_k} \left(\frac{b}{c}\right)^k\\
&+ \frac{bd}{ac^2}\, \frac{(ac^2q^3/b, a^2bc/d, d^2q^3/a^2bc, bdq^3/ac^2; q^3)_\infty (c/d, bdq/c; q)_\infty}{(ac^2q^3/bd, cdq^3/ab^2, ab^2/c, bd^2q^3/ac^2; q^3)_\infty (acq, d/ac; q)_\infty}\\
&\cdot \frac{(cdq^3/ab^2, ac^2q/bd, ac^2/bd; q^3)_k}{(ac^2q^3/b; q^3)_{2k} (a^2bc/d; q^3)_k}\, \frac{(acq; q)_{4k} q^{-k(k+1)} (b/c)^k}{(acq/d, bdq/c; q)_k (c/b; q)_{2k}}\\
&\cdot {}_8W_7(bd^2/ac^2; dq^{3k}, ab^2/c, dq^{-k}/ac, dq^{1-k}/ac, dq^{2-k}/ac; q^3, q^3).
\end{aligned}
$$

Using (3.10) in (3.8), we find that

$$
\begin{aligned}
&{}_{10}W_9(ac^2/b; c, d, ac/b, a^2bc/d, cq/b, cq^2/b, cq^3/b; q^3, q^3) \qquad (3.11)\\
&+ \frac{(ac^2q^3/b, a^2bc/d, d^2q^3/a^2bc, bd/ac^2; q^3)_\infty (c/b, bdq/c; q)_\infty}{(ac^2/bd, cdq^3/ab^2, ab^2/c, bd^2q^3/ac^2; q^3)_\infty (acq, d/ac; q)_\infty}\\
&\cdot \sum_{k=0}^{\infty} \frac{(1-acq^{4k})(1-b/cq^{2k})}{(1-ac)(1-b/c)}\, \frac{(a,b;q)_k (d, ac^2/bd; q^3)_k}{(cq^3, acq^3/b; q^3)_k (acq/d, bdq/c; q)_k}\, q^{3k}\\
&\cdot {}_8W_7(bd^2/ac; dq^{3k}, ab^2/c, dq^{-k}/ac, dq^{1-k}/ac, dq^{1-k}/ac; q^3, q^3)\\
&= \frac{(ac^2q^3/b, bd/ac^2; q^3)_\infty (ab, dq/ab; q)_\infty}{(ab^2/c, cdq^3/ab^2; q^3)_\infty (acq, d/ac; q)_\infty}\\
&\cdot \sum_{k=0}^{\infty} \frac{1-acq^{4k}}{1-ac}\, \frac{(a,b;q)_k (qc/b; q)_{2k} (d, a^2bc/d; q^3)_k}{(cq^3, acq^3/b; q^3)_k (ab; q)_{2k} (acq/d, dq/ab; q)_k}\, q^k.
\end{aligned}
$$

Now observe that the sum over k on the left side of (3.11) is

$$\sum_{k=0}^{\infty} \frac{(1-acq^{4k})(1-b/cq^{2k})}{(1-ac)(1-b/c)} \frac{(a,b;q)_k}{(cq^3, acq^3/b;q^3)_k} \tag{3.12}$$
$$\cdot \sum_{j=0}^{\infty} \frac{(bd^2/ac^2, ab^2/c;q^3)_j (1-bd^2q^{6j}/ac^2)(d;q^3)_{k+j}(ac^2/bd;q^3)_{k-j}}{(q^3, d^2q^3/a^2bc;q^3)_j (1-bd^2/ac^2)(acq/d;q)_{k-3j}(bdq/c;q)_{k+3j}} q^{3j^2+3k} \left(\frac{d^2}{a^2bc}\right)^j$$
$$= \sum_{j=0}^{\infty} \frac{(bd^2/ac^2, ab^2/c, d;q^3)_j (1-bd^2q^{6j}/ac^2)(d/ac;q)_{3j}}{(q^3, d^2q^3/a^2bc, bdq^3/ac^2;q^3)_j (1-bd^2/ac^2)(bdq/c;q)_{3j}} q^{3j}$$
$$\cdot \sum_{k=0}^{\infty} \frac{(1-acq^{4k})(1-b/cq^{2k})}{(1-ac)(1-b/c)} \frac{(a,b;q)_k (dq^{3j}, ac^2q^{-3j}/bd;q^3)_k}{(cq^3, acq^3/b;q^3)_k (acq^{1-3j}/d, bdq^{1+3j}/c;q)_k} q^{3k}$$

and that, by (2.7),

$$\sum_{k=0}^{\infty} \frac{(1-acq^{4k})(1-b/cq^{2k})}{(1-ac)(1-b/c)} \frac{(a,b;q)_k (dq^{3j}, ac^2q^{-3j}/bd;q^3)_k}{(cq^3, acq^3/b;q^3)_k (acq^{1-3j}/d, bdq^{1+3j}/c;q)_k} q^{3k} \tag{3.13}$$
$$= \frac{(1-c)(1-ac/b)(1-bdq^{3j}/c)(1-ac/dq^{3j})}{c(1-ac)(1-b/c)(1-dq^{3j}/c)(1-ac/bdq^{3j})} \frac{(a,b;q)_\infty (dq^{3j}, ac^2/bdq^{3j};q^3)_\infty}{(c, ac/b;q^3)_\infty (ac/dq^{3j}, bdq^{3j}/c;q)_\infty}$$
$$- \frac{(1-c)(1-ac/b)(1-bdq^{3j})(1-ac/dq^{3j})}{c(1-ac)(1-b/c)(1-dq^{3j}/c)(1-ac/bdq^{3j})}.$$

Hence the right side of (3.12) is

$$-\frac{bd(1-c)(1-ac/b)}{ac^2(1-ac)(1-b/c)} \sum_{j=0}^{\infty} \frac{(bd^2/ac^2, ab^2/c;q^3)_j (1-bd^2q^{6j}/ac^2)}{(q^3, d^2q^3/a^2bc;q^3)_j (1-bd^2/ac^2)} \tag{3.14}$$
$$\cdot \frac{(d^2q^3/a^2bc)^j q^{3j^2}}{(1-dq^{3j}/c)(1-bdq^{3j}/ac)} \frac{(a,b;q)_\infty (d, ac^2/bd;q^3)_\infty}{(c, ac/b;q^3)_\infty (acq/d, bdq/c;q)_\infty}$$
$$- \frac{b(1-c)(1-ac/b)(1-bd/c)(1-d/ac)}{c(1-ac)(1-b/c)(1-d/c)(1-bd/ac)}$$
$$\cdot {}_{10}W_9(bd^2/ac^2; ab^2/c, d/c, bd/ac, d, dq/ac, dq^2/ac, dq^3/ac; q^3, q^3)$$

and it follows from (3.11) that

$$
\begin{aligned}
&{}_{10}W_9(ac^2/b; d, c, a^2bc/d, ac/b, cq/b, cq^2/b, cq^3/b; q^3, q^3) \qquad (3.15)\\
&+ \frac{(1-c)(1-ac/b)}{(1-d/c)(1-bd/ac)} \frac{(cq/b, bd/c; q)_\infty (ac^2q^3/b, a^2bc/d, d^2q^3/a^2bc, bd/ac^2; q^3)_\infty}{(ac, dq/ac; q)_\infty (ac^2/bd, cdq^3/ab^2, ab^2/c, bd^2q^3/ac^2; q^3)_\infty}\\
&\cdot {}_{10}W_9(bd^2/ac^2; d, bd/ac, ab^2/c, d/c, dq/ac, dq^2/ac, dq^3/ac; q^3, q^3)\\
&+ \frac{d(1-c)(1-ac/b)}{ac(1-d/c)(1-bd/ac)} \frac{(a, b, cq/b; q)_\infty (d, ac^2q^3/b, a^2bc/d, d^2q^3/a^2bc, bd/ac^2; q^3)_\infty}{(ac, d/ac, acq/d; q)_\infty (c, ac/b, cdq^3/ab^2, ab^2/c, bd^2q^3/ac^2; q^3)_\infty}\\
&\cdot \sum_{j=0}^{\infty} \frac{(bd^2/ac^2; q^3)_j (1 - bd^2q^{6j}/ac^2)(ab^2/c, d/c, bd/ac; q^3)_j}{(q^3; q^3)_j (1 - bd^2/ac^2)(d^2q^3/a^2bc, bdq^3/ac, dq^3/c; q^3)_j} q^{3j^2} \left(\frac{d^2q^3}{a^2bc}\right)^j\\
&= \frac{(ab, dq/ab; q)_\infty (ac^2q^3/b, bd/ac^2; q^3)_\infty}{(acq, d/ac; q)_\infty (ab^2/c, cdq^3/ab^2; q^3)_\infty}\\
&\cdot \sum_{k=0}^{\infty} \frac{1 - acq^{4k}}{1 - ac} \frac{(a, b; q)_k (cq/b; q)_{2k} (d, a^2bc/d; q^3)_k}{(cq^3, acq^3/b; q^3)_k (ab; q)_{2k} (acq/d, dq/ab; q)_k} q^k.
\end{aligned}
$$

Using the transformation formula [7, (5.21)]

$$
\begin{aligned}
&\sum_{j=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, d, e, f; q)_j}{(q, \sqrt{a}, -\sqrt{a}, aq/d, aq/e, aq/f; q)_j} \left(\frac{a^2q^2}{def}\right)^j q^{j(j-1)} \qquad (3.16)\\
&= \frac{(aq, aq/ef; q)_\infty}{(aq/e, aq/f; q)_\infty} {}_2\phi_1 \left[\begin{matrix} e, f \\ aq/d \end{matrix}; q, \frac{aq}{ef} \right]
\end{aligned}
$$

and Heine's transformation formula [8, (1.4.5)]

$$
{}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix}; q, z \right] = \frac{(c/b, bz; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1 \left[\begin{matrix} abz/c, b \\ bz \end{matrix}; q, \frac{c}{b} \right], \qquad (3.17)
$$

we find that the sum over j in (3.15) equals

$$
\begin{aligned}
&\frac{(bd^2q^3/ac^2, dq^3/a^2b; q^3)_\infty}{(d^2q^3/a^2bc, bdq^3/ac; q^3)_\infty} {}_2\phi_1 \left[\begin{matrix} ab^2/c, d/c \\ dq^3/c \end{matrix}; q, \frac{dq^3}{a^2b} \right] \qquad (3.18)\\
&= \frac{(q^3, bd^2q^3/ac^2; q^3)_\infty}{(dq^3/c, bdq^3/ac; q^3)_\infty} {}_2\phi_1 \left[\begin{matrix} d/c, bd/ac \\ d^2q^3/a^2bc \end{matrix}; q^3, q^3 \right].
\end{aligned}
$$

Thus, we have the cubic transformation formula

$$
\begin{aligned}
&{}_{10}W_9(ac^2/b; d, c, a^2bc/d, ac/b, cq/b, cq^2/b, cq^3/b; q^3, q^3) \qquad (3.19)\\
&+ \frac{(1-c)(1-ac/b)}{(1-d/c)(1-bd/ac)} \frac{(cq/b, bd/c; q)_\infty (ac^2q^3/b, a^2bc/d, d^2q^3/a^2bc, bd/ac^2; q^3)_\infty}{(ac, dq/ac; q)_\infty (ac^2/bd, cdq^3/ab^2, ab^2/c, bd^2q^3/ac^2; q^3)_\infty}\\
&\cdot {}_{10}W_9(bd^2/ac^2; d, bd/ac, ab^2/c, d/c, dq/ac, dq^2/ac, dq^3/ac; q^3, q^3)\\
&- \frac{(a, b, cq/b; q)_\infty (q^3, d, ac^2q^3/b, a^2bc/d, d^2q^3/a^2bc, bd/ac^2; q^3)_\infty}{(ac, dq/ac, ac/d; q)_\infty (cq^3, acq^3/b, d/c, bd/ac, cdq^3/ab^2, ab^2/c; q^3)_\infty}\\
&\cdot {}_2\phi_1 \left[\begin{matrix} d/c, bd/ac \\ d^2q^3/a^2bc \end{matrix}; q, q \right]\\
&= \frac{(ab, dq/ab; q)_\infty (bd/ac^2, ac^2q^3/b; q^3)_\infty}{(acq, d/ac; q)_\infty (ab^2/c, cdq^3/ab^2; q^3)_\infty}\\
&\cdot \sum_{k=0}^{\infty} \frac{1-acq^{4k}}{1-ac} \frac{(a, b; q)_k (cq/b; q)_{2k} (d, a^2bc/d; q^3)_k}{(cq^3, acq^3/b; q^3)_k (ab; q)_{2k} (acq/d, dq/ab; q)_k} q^k.
\end{aligned}
$$

Formula (1.2) follows from (3.19) by setting $c = 1$ and then replacing d by a^2b/c. To derive the cubic summation formula (1.8) which contains Gosper's sum (1.6) as a limit case, set $b = q/a$ in (3.19) to get

$$
\begin{aligned}
&{}_8W_7(a^2c^2/q; d, c, acq/d, a^2c/q, acq; q^3, q^3) \qquad (3.20)\\
&+ \frac{(1-c)(1-a^2c/q)}{(1-d/c)(1-dq/a^2c)} \frac{(a^2c^2q^2, acq/d, d^2q^2/ac, dq/a^2c^2; q^3)_\infty}{(a^2c^2/dq, acdq, q^2/ac, d^2q^4/a^2c^2; q^3)_\infty}\\
&\cdot {}_8W_7(qd^2/a^2c^2; d, dq/a^2c, q^2/ac, d/c, dq^2/ac; q^3, q^3)\\
&- \frac{(a, q/a; q)_\infty (q^3, d, a^2c^2q^2, acq/d, d^2q^2/ac, dq/a^2c^2; q^3)_\infty}{(ac/d, dq/ac; q)_\infty (cq^3, a^2cq^2, d/c, dq/a^2c, acdq, q^2/ac; q^3)_\infty}\\
&\cdot {}_2\phi_1 \left[\begin{matrix} d/c, dq/a^2c \\ d^2q^2/ac \end{matrix}; q^3, q^3 \right]\\
&= \frac{(q, d; q)_\infty (dq/a^2c^2, a^2c^2q^2; q^3)_\infty}{(acq, d/ac; q)_\infty (q^2/ac, acdq; q^3)_\infty}\\
&\cdot \sum_{k=0}^{\infty} \frac{1-acq^{4k}}{1-ac} \frac{(a, q/a; q)_k (ac; q)_{2k} (d, acq/d; q^3)_k}{(cq^3, a^2cq^2; q^3)_k (q; q)_{2k} (acq/d, d; q)_k} q^k
\end{aligned}
$$

and then apply (3.9) to the above sum of two ${}_8W_7$ series to get

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{1-acq^{4k}}{1-ac} \frac{(a, q/a; q)_k (ac; q)_{2k} (d, acq/d; q^3)_k}{(cq^3, a^2cq^2; q^3)_k (q; q)_{2k} (acq/d, d; q)_k} q^k \qquad (3.21)\\
&= \frac{(acq^2, acq^3, d/ac, dq/ac, adq, aq, q^2/a, dq^2/a; q^3)_\infty}{(q, q^2, dq, dq^2, a^2cq^2, cq^3, dq/a^2c, d/c; q^3)_\infty}\\
&+ \frac{d(a, q/a, acq; q)_\infty (q^3, d, acq/d, d^2q^2/ac; q^3)_\infty}{ac(q, d, acq/d; q)_\infty (cq^3, a^2cq^2, d/c, dq/a^2c; q^3)_\infty}\\
&\cdot {}_2\phi_1 \left[\begin{matrix} d/c, dq/a^2c \\ d^2q^2/ac \end{matrix}; q^3, q^3 \right].
\end{aligned}
$$

Formula (1.8) follows from (3.21) by replacing a, c, d by b, a^2/b, c^3, respectively, and using the identity $(a;q)_{2k} = (a, aq; q^2)_k$.

To verify that Gosper's formula (1.6) is a limit case of (3.21), replace a, b, c in (3.21) by q^a, q^b, q^c, respectively, take the limit $q \uparrow 1$ and use the limits [2], [8, Chapter 1]

$$\lim_{q\uparrow 1} \frac{(q^x;q)_n}{(1-q)^n} = (x)_n, \quad \lim_{q\uparrow 1} \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty} = \Gamma(x), \tag{3.22}$$

Gauss' formula [12, (III.3), p. 243] and the identity $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$ to find that

$$\begin{aligned}
&{}_7F_8\left[\begin{matrix} a, a+1/2, b, 1-b, c, (2a+1)/3-c, a/2+1 \\ 1/2, (2a-b+3)/3, (2a+b+2)/3, 3c, 2a+1-3c, a/2 \end{matrix}; 1\right] \qquad (3.23)\\
&= \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(c+\frac{1}{3}\right)\Gamma\left(c+\frac{2}{3}\right)\Gamma\left(\frac{2a-b+3}{3}\right)\Gamma\left(\frac{2a+b+2}{3}\right)\Gamma\left(\frac{3c+b-2a}{3}\right)\Gamma\left(\frac{3c+1-2a-b}{3}\right)}{\Gamma\left(\frac{2a+2}{3}\right)\Gamma\left(\frac{2a+3}{3}\right)\Gamma\left(\frac{3c-2a}{3}\right)\Gamma\left(\frac{3c+1-2a}{3}\right)\Gamma\left(\frac{b+1+3c}{3}\right)\Gamma\left(\frac{b+1}{3}\right)\Gamma\left(\frac{2-b}{3}\right)\Gamma\left(\frac{3c+2-b}{3}\right)}\\
&+ \frac{\Gamma(3c)\Gamma(1+2a-3c)\Gamma\left(\frac{3c-2a+b}{3}\right)\Gamma\left(\frac{2a+b+2}{3}\right)\Gamma\left(\frac{2a-b+3}{3}\right)\Gamma\left(\frac{3c+1-2a-b}{3}\right)\Gamma\left(\frac{1+2a}{3}\right)}{\Gamma(b)\Gamma(1-b)\Gamma(1+2a)\Gamma(c)\Gamma\left(\frac{2a+1-3c}{3}\right)\Gamma\left(\frac{3c+2-b}{3}\right)\Gamma\left(\frac{3c+1+b}{3}\right)}\\
&= \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(c+\frac{1}{3}\right)\Gamma\left(c+\frac{2}{3}\right)\Gamma\left(\frac{2a-b+3}{3}\right)\Gamma\left(\frac{2a+b+2}{3}\right)\Gamma\left(\frac{3c-2a+b}{3}\right)\Gamma\left(\frac{3c-2a-b+1}{3}\right)}{\Gamma\left(\frac{2a+2}{3}\right)\Gamma\left(\frac{2a+3}{3}\right)\Gamma\left(\frac{b+1}{3}\right)\Gamma\left(\frac{2-b}{3}\right)\Gamma\left(\frac{3c+b+1}{3}\right)\Gamma\left(\frac{3c+2-b}{3}\right)\Gamma\left(\frac{3c-2a}{3}\right)\Gamma\left(\frac{3c-2a+1}{3}\right)} B
\end{aligned}$$

with

$$\begin{aligned}
B &= 1 + \frac{\Gamma\left(\frac{3c-2a}{3}\right)\Gamma\left(\frac{3c-2a+1}{3}\right)\Gamma\left(\frac{3+2a-2c}{3}\right)\Gamma\left(\frac{2+2a-3c}{3}\right)}{\Gamma\left(\frac{b}{3}\right)\Gamma\left(1-\frac{b}{3}\right)\Gamma\left(\frac{1-b}{3}\right)\Gamma\left(\frac{2+b}{3}\right)} \qquad (3.24)\\
&= 1 + \frac{\sin\frac{\pi b}{3}\sin\frac{\pi}{3}(1-b)}{\sin\frac{\pi}{3}(3c-2a)\sin\frac{\pi}{3}(3c-2a+1)}.
\end{aligned}$$

Now observe that by setting $\alpha = b/3$, and $\beta = (3c-2a)/3$, and using trigonometric identities we have

$$\begin{aligned}
&\sin\pi\beta\sin\left(\frac{\pi}{3}+\pi\beta\right) + \sin\pi\alpha\sin\left(\frac{\pi}{3}-\pi\alpha\right) \qquad (3.25)\\
&= \sin\pi(\alpha+\beta)\sin\frac{\pi}{3}(1+\beta-\alpha)\\
&= \sin\frac{\pi}{3}(3c-2a+b)\sin\frac{\pi}{3}(1+3c-2a-b).
\end{aligned}$$

Therefore

$$\begin{aligned}
B &= \frac{\sin\frac{\pi}{3}(3c-2a+b)\sin\frac{\pi}{3}(1+3c-2a-b)}{\sin\frac{\pi}{3}(3c-2a)\sin\frac{\pi}{3}(1+3c-2a)} \qquad (3.26)\\
&= \frac{\Gamma\left(\frac{3c-2a}{3}\right)\Gamma\left(\frac{3+2a-3c}{3}\right)\Gamma\left(\frac{1+3c-2a}{3}\right)\Gamma\left(\frac{2+2a-3c}{3}\right)}{\Gamma\left(\frac{3c-2a+b}{3}\right)\Gamma\left(\frac{3+2a-b-3c}{3}\right)\Gamma\left(\frac{1+3c-2a-b}{3}\right)\Gamma\left(\frac{2+2a+b-3c}{3}\right)}
\end{aligned}$$

and thus

$$
{}_7F_6\left[\begin{matrix} a, a+\frac{1}{2}, b, 1-b, c, (2a+1)/3-c, a/2+1 \\ 1/2, (2a-b+3)/3, (2a+b+2)/3, 3c, 3a+1-3c, a/2 \end{matrix}; 1\right] \tag{3.27}
$$

$$
= \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(c+\frac{1}{3}\right)\Gamma\left(c+\frac{2}{3}\right)\Gamma\left(\frac{2a-b+3}{3}\right)\Gamma\left(\frac{2a+b+2}{3}\right)\Gamma\left(\frac{2+2a-3c}{3}\right)\Gamma\left(\frac{3+2a-3c}{3}\right)}{\Gamma\left(\frac{b+1}{3}\right)\Gamma\left(\frac{2-b}{3}\right)\Gamma\left(\frac{2a+2}{3}\right)\Gamma\left(\frac{2a+3}{3}\right)\Gamma\left(\frac{3c+b+1}{3}\right)\Gamma\left(\frac{3c+2-b}{3}\right)\Gamma\left(\frac{2+2a+b-3c}{3}\right)\Gamma\left(\frac{3+2a-b-3c}{3}\right)}
$$

$$
= \frac{2}{\sqrt{3}}\,\frac{\Gamma\left(c+\frac{1}{3}\right)\Gamma\left(c+\frac{2}{3}\right)\Gamma\left(\frac{2a-b+3}{3}\right)\Gamma\left(\frac{2a+b+2}{3}\right)\Gamma\left(\frac{2+2a-3c}{3}\right)\Gamma\left(\frac{3+2a-3c}{3}\right)\sin\frac{\pi}{3}(b+1)}{\Gamma\left(\frac{2a+2}{3}\right)\Gamma\left(\frac{2a+3}{3}\right)\Gamma\left(\frac{3c+b+1}{3}\right)\Gamma\left(\frac{3c+2-b}{3}\right)\Gamma\left(\frac{2+2a+b-3c}{3}\right)\Gamma\left(\frac{3+2a-b-3c}{3}\right)},
$$

which gives (1.6).

4. Some consequences of formula (3.19). When $d = a^2bcq^{3n}$and $n = 0, 1, 2, \ldots$, reduces to the transformation formula

$$
\sum_{k=0}^{n} \frac{1-acq^{4k}}{1-ac}\,\frac{(a,b;q)_k(cq/b;q)_{2k}(a^2bcq^{3n},q^{-3n};q^3)_k}{(cq^3,acq^3/b;q^3)_k(ab;q)_{2k}(q^{1-3n}/ab,acq^{3n+1};q)_k}\,q^k \tag{4.1}
$$

$$
= \frac{(acq;q)_{3n}(ab^2/c;q^3)_n}{(ab;q)_{3n}(ac^2q^3/b;q^3)_n}
$$

$$
\cdot\, {}_{10}W_9(ac^2/b;c,ac/b,cq/b,cq^2/b,cq^3/b,a^2bcq^{3n},q^{-3n};q^3,q^3).
$$

If we use Bailey's [3, 8.5(1)] transformation

$$
{}_{10}W_9(a;c,d,e,f,g,h,q^{-n};q,q) \tag{4.2}
$$

$$
= \frac{(aq,aq/fq,aq/fh,aq/gh;q)_n}{(aq/f,aq/g,aq/h,aq/fgh;q)_n}
$$

$$
\cdot\, {}_{10}W_9(\sigma;\sigma c/a,\sigma d/a,\sigma e/a,f,g,h,q^{-n};q,q),
$$

where $\sigma = a^2q/cde$, $a^3q^2 = cdefghq^{-n}$ and $n = 0, 1, 2, \ldots$, to show that the ${}_{10}W_9$ in (4.1) equals

$$
\frac{(ac^2q^3/b,ab/q^2,q^{1-3n}/ab,q^{-3n}/ab;q^3)_n}{(ac,acq,cq^{3-3n}/ab^2,q^{-3n-2}/ac;q^3)_n}
$$

$$
\cdot\, {}_{10}W_9(acq^2;q^3,aq^2,bq^2,cq^2/b,cq^3/b,a^2bcq^{3n},q^{-3n};q^3,q^3),
$$

then we find that (4.1) can be written in the equivalent form

$$
\sum_{k=0}^{n} \frac{1-acq^{4k}}{1-ac}\,\frac{(a,b;q)_k(cq/b;q)_{2k}(a^2bcq^{3n},q^{-3n};q^3)_k}{(cq^3,acq^3/b;q^3)_k(ab;q)_{2k}(q^{1-3n}/ab,acq^{3n+1};q)_k}\,q^k \tag{4.3}
$$

$$
= \frac{(1-acq^2)(1-ab/q^2)(1-abq^{3n})(1-acq^{3n})}{(1-acq^{3n+2})(1-abq^{3n-2})(1-ab)(1-ac)}
$$

$$
\cdot \sum_{k=0}^{n} \frac{1-acq^{6k+2}}{1-acq^2}\,\frac{(aq^2,bq^2,cq^2/b,cq^3/b,a^2bcq^{3n},q^{-3n};q^3)_k}{(cq^3,acq^3/b,abq^3,abq^2,q^{5-3n}/ab,acq^{3n+5};q^3)_k}\,q^{3k},
$$

where $n = 0, 1, 2, \ldots$. Notice the rather striking similarities and differences between the two series in (4.3). The other equivalent forms of (4.1) which follow by applying (4.2) are not as interesting and so will be omitted.

When $b = q^2/a$, the first two terms on the left side of (3.19) give

$$\begin{aligned}
&{}_8W_7(a^2c^2/q^2; d, c, acq^2/d, a^2c/q^2, ac/q; q^3, q^3) \qquad (4.4)\\
&+ \frac{(1-c)(1-a^2c/q^2)}{(1-d/c))(1-dq^2/a^2c)} \frac{(ac/q, dq^2/ac; q)_\infty (a^2c^2q, acq^2/d, d^2q/ac, dq^2/a^2c^2; q^3)_\infty}{(ac, dq/ac; q)_\infty (a^2c^2/dq^2, acd/q, q^4/ac, d^2q^5/a^2c^2; q^3)_\infty}\\
&\cdot {}_8W_7(d^2q^2/a^2c^2; d, dq^2/a^2c, q^4/ac, d/c, dq/ac; q^3, q^3)\\
&= \frac{(a^2c^2q, dq^2/a^2c^2, ad/q, q^3, aq^2, dq/a, d, q^4/a; q^3)_\infty}{(a^2cq, acd/q, cq^3, acq^2, dq^2/a^2c, q^4/ac, d/c, dq/ac; q^3)_\infty}
\end{aligned}$$

by (3.9). Hence, we have the summation formula

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{1-acq^{4k}}{1-ac} \frac{(a, q^2/a; q)_k (ac/q; q)_{2k} (d, acq^2/d; q^3)_k}{(cq^3, a^2cq; q^3)_k (q^2; q)_{2k} (acq/d, d/q; q^3)_k} q^k \qquad (4.5)\\
&+ \frac{(a, aq, aq^2, q^2/a, q^3/a, q^4/a, ac/q, acq, acq^3, d/ac, d^2q/ac; q^3)_\infty}{(q^2, q^4, d/q, dq, ac/d, acq/d, dq^3/ac, dq^2/a^2c, cq^3, a^2cq, d/c; q^3)_\infty}\\
&\cdot {}_2\phi_1 \left[\begin{matrix} d/c, dq^2/a^2c \\ d^2q/ac \end{matrix} ; q^3, q^3 \right]\\
&= \frac{(aq^2, q^4/a, ad/q, dq/a, d/ac, dq^2/ac, acq, acq^3; q^3)_\infty}{(q^2, q^4, d/q, dq, d/c, cq^3, a^2cq, dq^2/a^2c; q^3)_\infty}.
\end{aligned}$$

Replace a, c, d by b, a^2/b, c^3, respectively to get

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{1-a^2q^{4k}}{1-a^2} \frac{(b, q^2/b; q)_k (a^2/q; q)_{2k} (c^3, a^2q^2/c^3; q^3)_k}{(a^2q^3/b, a^2bq; q^3)_k (q^2; q)_{2k} (a^2q/c^3, c^3/q; q)_k} q^k \qquad (4.6)\\
&= \frac{(bq^2, q^4/b, bc^3/q, c^3q/b, c^3/a^2, c^3q^2/a^2, a^2q, a^2q^3; q^3)_\infty}{(q^2, q^4, c^3/q, c^3q, bc^3/a^2, a^2q^3/b, a^2bq, c^3q^2/a^2b; q^3)_\infty}\\
&- \frac{(b, bq, bq^2, q^2/b, q^3/b, q^4/b, qa^2/q, a^2q, a^2q^3, c^3/a^2, c^6q/a^2; q^3)_\infty}{(q^2, q^4, c^3/q, c^3q, a^2/c^3, a^2q/c^3, c^3q^3/a^2, c^3q^2/a^2b, a^2q^3/b, a^2bq, bc^3/a^2; q^3)_\infty}\\
&\cdot {}_2\phi_1 \left[\begin{matrix} bc^3/a^2, c^3q^2/a^2b \\ c^6q/a^2 \end{matrix} ; q^3, q^3 \right].
\end{aligned}$$

As in the derivation of (3.27), it can be shown that (4.6) has the $q \uparrow 1$ limit case

$$\begin{aligned}
&{}_7F_6 \left[\begin{matrix} a-1/2, a, b, 2-b, c, (2a+2-3c)/3, a/2+1 \\ 3/2, (2a-b+3)/3, (2a+b+1)/3, 3c-1, 2a+1-3c, a/2 \end{matrix} ; 1 \right] \qquad (4.7)\\
&= \frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)\Gamma\left(c-\frac{1}{3}\right)\Gamma\left(c+\frac{1}{3}\right)\Gamma\left(\frac{2a-b+3}{3}\right)\Gamma\left(\frac{2a+b+1}{3}\right)\Gamma\left(\frac{2a-3c+3}{3}\right)\Gamma\left(\frac{2a-3c+1}{3}\right)}{\Gamma\left(\frac{b+2}{3}\right)\Gamma\left(\frac{4-b}{3}\right)\Gamma\left(\frac{b+3c-1}{3}\right)\Gamma\left(\frac{3c-b+1}{3}\right)\Gamma\left(\frac{2a+1}{3}\right)\Gamma\left(\frac{2a+3}{3}\right)\Gamma\left(\frac{2a-3c-b+3}{3}\right)\Gamma\left(\frac{2a-3c+b+1}{3}\right)}.
\end{aligned}$$

It should be noted that the evaluation

$$
{}_7F_6\left[\begin{matrix}1,(d-b+2)/2,(a+d+4)/4,a,b,c/3,(a+2d-b-c)/3\\(d-b)/2,(a+d)/4,(d+3)/3,(a+d+3-b)/3,a+d+1-c,b+c+1-d\end{matrix};1\right] \tag{4.8}
$$
$$
=\frac{abc(a+2a-b-c)}{(a+d)(b-d)(c-d)(a+d-b-c)}\frac{\Gamma(a+d+1-c)\Gamma(b+c+1-d)\Gamma\left(\frac{d+3}{3}\right)\Gamma\left(\frac{a+d+3-b}{3}\right)}{\Gamma\left(\frac{c+3}{3}\right)\Gamma\left(\frac{a+2d+3-b-c}{3}\right)\Gamma(a+1)\Gamma(b+1)}
$$
$$
+\frac{d(a+d-b)(a+d-c)(b+c-d)}{(a+d)(b-d)(d-c)(a+d-b-c)}
$$

and the corresponding truncated series formula are limit cases of the $q=p^3$ case of (2.7).

By reversing the order of summation of the series on the left side of (4.1) this series is converted to a series which is truncated at the $(n+1)^{\text{th}}$ term. Another transformation formula for such a series in terms of a ${}_{10}W_9$ series can be derived by proceeding as follows. Use (2.7) and (3.5) to obtain

$$
\sum_{k=0}^{n}\frac{(1-aq^{4k})(1-bq^{-2k})}{(1-b)}\frac{(a,b;q)_k(c,a/bc;q^3)_k}{(q^3,aq^3/b;q^3)_k(aq/c,bcq;q)_k}q^{3k} \tag{4.9}
$$
$$
=\frac{(aq,bq;q)_n(cq^3,aq^3/bc;q^3)_n}{(aq/c,bcq;q)_n(q^3,aq^3/b;q^3)_n},\qquad n=0,1,2,\ldots,
$$

and

$$
{}_8W_7(a/b;aq^k,aq^{k+1},aq^{k+2},1/ab^2,q^{-3k};q^3,q^3) \tag{4.10}
$$
$$
=\frac{(aq^3/b,q^{-2k}/ab,q^{1-2k}/ab,q^{2-2k}/ab;q^3)_k}{(q^{-3k}/a^2b,b^{-1}q^{1-k},b^{-1}q^{2-k},b^{-1}q^{3-k};q^3)_k}
$$
$$
=\frac{(aq^3/b;q^3)_k(1/ab;q)_k(abq;q)_{2k}}{(a^2bq^3;q^3)_k(b;q)_k(q/b;q)_{2k}},\qquad k=0,1,2,\ldots.
$$

Hence

$$\sum_{k=0}^{n}\frac{(1-aq^{4k})(1-bq^{-2k})}{(1-a)(1-b)}\,\frac{(a,1/ab;q)_k(abq;q)_{2k}(c,a/bc;q^3)_k}{(q^3,a^2bq^3;q^3)_k(q/b;q)_{2k}(aq/c,bcq;q)_k}\,q^{3k} \tag{4.11}$$
$$=\sum_{k=0}^{n}\frac{(1-aq^{4k})(1-bq^{-2k})}{(1-a)(1-b)}\,\frac{(c,a/bc;q^3)_k}{(aq/c,bcq;q)_k}\,q^{3k}$$
$$\cdot\sum_{j=0}^{k}\frac{(a/b;q^3)_j(1-aq^{3j}/b)(a;q)_{k+3j}(bq^{-3j};q)_k(1/ab^2;q^3)_j(-1)^jq^{3j(j+1)/2}}{(q^3;q^3)_{k-j}(q^3;q^3)_j(aq^3/b;q^3)_{k+j}(q/b;q)_{3j}(a^2bq^3;q^3)_j(1-a/b)}$$
$$=\sum_{j=0}^{n}\frac{(a/b;q^3)_j(1-aq^{3j}/b)(1/ab^2;q^3)_j(a;q)_{4j}(bq^{-3j};q)_j(c,a/bc;q^3)_j}{(q^3;q^3)_j(1-a/b)(a^2bq^3;q^3)_j(aq^3/b;q^3)_{2j}(q/b;q)_{3j}(aq/c,bcq;q)_j}$$
$$\cdot\frac{(1-aq^{4j})(1-bq^{-2j})}{(1-a)(1-b)}(-1)^jq^{3j(j+1)/2+3j}$$
$$\cdot\sum_{m=0}^{n-j}\frac{(1-aq^{4j+4m})(1-bq^{-2j-2m})}{(1-aq^{4j})(1-bq^{-2j})}\,\frac{(aq^{4j},bq^{-2j};q)_m(cq^{3j},aq^{3j}/bc;q^3)_m}{(q^3,aq^{6j+3}/b;q^3)_m(aq^{j+1}/c,bcq^{j+1};q)_m}\,q^{3m}$$
$$=\frac{(aq,bq;q)_n(cq^3,aq^3/bc;q^3)_n}{(aq/c,bcq;q)_n(q^3,aq^3/b;q^3)_n}$$
$$\cdot\sum_{j=0}^{n}\frac{(a/b,1/ab^2,aq^{n+1},aq^{n+2},aq^{n+3},q^{-3n};q^3)_j}{(q^3,a^2bq^3,q^{-n}/b,q^{1-n}/b,q^{2-n}/b,aq^{3n+3}/b;q^3)_j}$$
$$\cdot\frac{(1-aq^{3j}/b)(1-c)(1-a/bc)}{(1-a/b)(1-cq^{3j})(1-aq^{3j}/bc)}\,q^{3j}$$

by (2.7), which yields the transformation formula

$$\sum_{k=0}^{n}\frac{(1-aq^{4k})(1-bq^{-2k})}{(1-a)(1-b)}\,\frac{(a,1/ab;q)_k(abq;q)_{2k}(c,a/bc;q^3)_k}{(q^3,a^2bq^3;q^3)_k(q/b;q)_{2k}(aq/c,bcq;q)_k}\,q^{3k} \tag{4.12}$$
$$=\frac{(aq,bq;q)_n(cq^3,aq^3/bc;q^3)_n}{(aq/c,bcq;q)_n(q^3,aq^3/b;q^3)_n}$$
$$\cdot{}_{10}W_9(a/b;1/ab^2,c,a/bc,aq^{n+1},aq^{n+2},aq^{n+3},q^{-3n};q^3,q^3),$$

where $n=0,1,2,\ldots$.

5. A quadratic transformation formula with five parameters. In [7,(5.15)] and

[10,(4.7)] special cases of (2.8) were used to derive the quadratic summation formula

$$\sum_{k=0}^{\infty} \frac{1-aq^{3k}}{1-a} \frac{(a,b,q/b;q)_k (c,d,a^2q/cd;q^2)_k}{(q^2,aq^2/b,abq;q^2)_k (aq/c,aq/d,cd/a;q)_k} q^k \tag{5.1}$$
$$- \frac{aq(aq,b,q/b;q)_\infty (c,d,a^2q^3/cd^2,a^2q^3/c^2d;q^2)_\infty}{cd(aq/c,aq/d,cd/a;q)_\infty (abq,abq/cd,aq^2/b,aq^2/bcd;q^2)_\infty}$$
$$\cdot {}_3\phi_2 \left[\begin{matrix} abq/cd, a^2q/cd, aq^2/bcd \\ a^2q^3/cd^2, a^2q^3/c^2d \end{matrix} ; q^2, q^2 \right]$$
$$= \frac{(aq,aq/cd;q)_\infty (abq/c,abq/d,aq^2/bc,aq^2/bd;q^2)_\infty}{(aq/c,aq/d;q)_\infty (abq,abq/cd,aq^2/b,aq^2/bcd;q^2)_\infty}$$

and this formula was used to derive a quadratic summation formula [7, (5.1)] stated earlier by Gosper. Here we show how (2.8) can be used to extend (5.1) to a quadratic transformation formula containing five arbitrary parameters.

Take $q = p^2$ in (2.8) and then change p and d to q and c to get

$$\sum_{k=0}^{n} \frac{(1-acq^{3k})(1-b/cq^k)}{(1-ac)(1-b/c)} \frac{(a,b;q)_k (q^{-2n}, ac^2q^{2n}/b;q^2)_k}{(cq^2,acq^2/b;q^2)_k (acq^{2n+1}, bq^{1-2n}/c;q)_k} q^{2k} \tag{5.2}$$
$$= \frac{(1-c)(1-ac/b)(1-acq^{2n})(1-cq^{2n}/b)}{(1-ac)(1-c/b)(1-cq^{2n})(1-acq^{2n}/b)}, \qquad n = 0,1,2,\ldots.$$

Multiply both sides of (5.2) by

$$\frac{(ac^2/b;q^2)_n (c/b;q)_{2n}}{(q^2;q^2)_n (acq;q)_{2n}} C_n$$

and sum over n to obtain

$$\sum_{n=0}^{\infty} \frac{(ac^2/b;q^2)_n (cq/b;q)_{2n} (1-c)(1-ac/b)}{(q^2;q^2)_n (ac;q)_{2n} (1-cq^{3n})(1-acq^{2n}/b)} C_n \tag{5.3}$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(1-acq^{3k})(1-b/cq^k)}{(1-ac)(1-b/c)} \frac{(a,b;q)_k (ac^2/b;q^2)_{n+k} (c/b;q)_{2n-k}}{(cq^2,acq^2/b;q^2)_k (q^2;q^2)_{n-k} (acq;q)_{2n+k}} \left(\frac{cq}{b}\right)^k q^{\binom{k}{2}} C_n$$
$$= \sum_{k=0}^{\infty} \frac{(1-acq^{3k})(1-b/cq^k)}{(1-ac)(1-b/c)} \frac{(a,b;q)_k (ac^2/b;q^2)_{2k} (c/b;q)_k}{(cq^2,acq^2/b;q^2)_k (acq;q)_{3k}} \left(\frac{cq}{b}\right)^k q^{\binom{k}{2}}$$
$$\cdot \sum_{m=0}^{\infty} \frac{(ac^2q^{4k};q^2)_m (cq^k/b;q)_{2m}}{(q^2;q^2)_m (acq^{3k+1};q)_{2m}} C_{k+m}.$$

Choose

$$C_n = \frac{(1-ac^2q^{4n}/b)(d,e,f;q^2)_n (a^2q^3/def)^n}{(1-ac^2/b)(ac^2q^2/bd, ac^2q^2/be, ac^2q^2/bf;q^2)_n}. \tag{5.4}$$

Then (5.3) gives

$$
\begin{aligned}
&{}_{10}W_9(ac^2/b; ac/b, c, cq/b, cq^2/b, d, e, f; q^2, a^2c^2q^3/def) \qquad (5.5)\\
&= \sum_{k=0}^{\infty} \frac{(1-acq^{3k})(1-b/cq^k)}{(1-ac)(1-b/c)} \frac{(a, b, c/b; q)_k (ac^2q^2/b; q^2)_{2k}}{(cq^2, acq^2/b; q^2)_k (acq; q)_{3k}}\\
&\cdot \frac{(d, e, f; q^2)_k (a^2c^3q^4/bdef)^k}{(ac^2q^2/bd, ac^2q^2/be, ac^2q^2/bf; q^2)_k} q^{\binom{k}{2}}\\
&\cdot {}_8W_7(ac^2q^{4k}/b; cq^k/b, cq^{k+1}/b, dq^{2k}, eq^{2k}, fq^{2k}; q^2, a^2c^2q^3/def).
\end{aligned}
$$

Now assume that

$$a^2c^2q = def. \qquad (5.6)$$

Then we can apply (3.9) to the ${}_8W_7$ in (5.5) to get

$$
\begin{aligned}
&{}_8W_7(ac^2q^{4k}/b; cq^k/b, cq^{k+1}/b, dq^{2k}, eq^{2k}, fq^{2k}; q^2, q^2) \qquad (5.7)\\
&= \frac{(ac^2q^{4k+2}/b, bf/ac^2q^{2k}, abq^{2k+1}, acq^{k+2}/d, acq^{k+2}/e, acq^{k+1}/d, acq^{k+1}/e, ac^2q^2/bde; q^2)_\infty}{(acq^{3k+2}, acq^{3k+1}, ac^2q^{2k+2}/bd, ac^2q^{2k+2}/be, bef/ac^2, bdf/ac^2, f/acq^k, f/acq^{k-1}; q^2)_\infty}\\
&+ \frac{bfq^{-2k}(ac^2q^{4k+2}/b, cq^k/b, cq^{k+1}/b, dq^{2k}, eq^{2k}, fq^2/e, fq^2/d, bfq^{2-2k}/ac^2; q^2)_\infty}{ac^2(ac^2q^{2k+2}/bf, acq^{3k+2}, acq^{3k+1}, ac^2q^{2k+2}/bd, ac^2q^{2k+2}/be, bef/ac^2, bdf/ac^2; q^2)_\infty}\\
&\cdot \frac{(bfq^{k+1}/c, bfq^{k+2}/c; q^3)_\infty}{(f/acq^k, f/acq^{k-1}; q^3)_\infty} {}_8W_7(bf^2/ac^2; fq^{2k}, bef/ac^2, bdf/ac^2, f/acq^k, f/acq^{k-1}; q^2, q^2)
\end{aligned}
$$

which, combined with (5.5), gives

$$
\begin{aligned}
&{}_{10}W_9(ac^2/b; c, d, e, f, ac/b, cq/b, cq^2/b; q^2, q^2) \qquad (5.8)\\
&= \frac{(ac^2q^2/b, ac^2q^2/bde, abq, bf/ac^2; q^2)_\infty (acq/d, acq/e; q)_\infty}{(ac^2q^2/bd, ac^2q^2/be, bdf/ac^2, bef/ac^2; q^2)_\infty (acq, f/ac; q)_\infty}\\
&\cdot \sum_{k=0}^{\infty} \frac{(1-acq^{3k})(1-b/cq^k)}{(1-ac)(1-b/c)} \frac{(a, b, c/b; q)_k (d, e, f; q^2)_k}{(cq^2, acq^2/b, abq; q^2)_k (acq/d, acq/e, acq/f; q)_k} q^{2k}\\
&+ \frac{bf(ac^2q^2/b, d, e, fq^2/d, fq^2/e, bfq^2/ac^2; q^2)_\infty (bfq/c, c/b; q)_\infty}{ac^2(ac^2q^2/bf, ac^2q^2/be, ac^2q^2/bd, bdf/ac^2, bef/ac^2, bf^2q^2/ac^2; q^2)_\infty (acq, f/ac; q)_\infty}\\
&\cdot \sum_{k=0}^{\infty} \frac{(1-acq^{3k})(1-b/cq^k)}{(1-ac)(1-b/c)} \frac{(a, b; q)_k (ac^2/bf, f; q^2)_k}{(cq^2, acq^2/b; q^2)_k (bfq/c, acq/f; q)_k} q^{2k}\\
&\cdot {}_8W_7(bf^2/ac^2; fq^{2k}, bdf/ac^2, bef/ac^2, f/acq^k, f/acq^{k-1}; q^2, q^2).
\end{aligned}
$$

The last sum over k is

$$\sum_{j=0}^{\infty} \frac{(bf^2/ac^2, f, bdf/ac^2, bef/ac^2, f/ac, fq/ac; q^2)_j (1 - bf^2q^{4j}/ac^2)}{(q^2, bfq^2/ac^2, fq^2/d, fq^2/e, bfq^2/c, bfq/c; q^2)_j (1 - bf^2/ac^2)} q^{2j} \tag{5.9}$$

$$\cdot \sum_{k=0}^{\infty} \frac{(1 - acq^{3k})(1 - b/cq^k)}{(1 - ac)(1 - b/c)} \frac{(a, b; q)_k (ac^2/bfq^{2j}, fq^{2j}; q^2)_k}{(cq^2, acq^2/b; q^2)_k (bfq^{2j+1}/c, acq^{1-2j}/f; q)_k} q^{2k}$$

$$= \sum_{j=0}^{\infty} \frac{(bf^2/ac^2, f, bdf/ac^2, bef/ac^2, f/ac, fq/ac; q^2)_j (1 - bf^2q^{4j}/ac^2)}{(q^2, bfq^2/ac^2, fq^2/d, fq^2/e, bfq^2/c, bfq/c; q^2)_j (1 - bf^2/ac^2)} q^{2j}$$

$$\cdot \frac{(1 - c)(1 - ac/b)(1 - ac/fq^{2j})(1 - bfq^{2j}/c)}{c(1 - ac)(1 - b/c)(1 - fq^{2j}/c)(1 - ac/bfq^{2j})}$$

$$\cdot \left\{ \frac{(a, b; q)_\infty (fq^{2j}, ac^2/bfq^{2j}; q^2)_\infty}{(c, ac/b; q^2)_\infty (ac/fq^{2j}, bfq^{2j}/c; q)_\infty} - 1 \right\}$$

$$= \frac{(1 - c)(1 - ac/b)(1 - f/ac)(1 - bf/c)}{(1 - c/b)(1 - ac)(1 - f/c)(1 - bf/ac)}$$

$$\cdot {}_{10}W_9(bf^2/ac^2; f, bdf/ac^2, bef/ac^2, bf/ac, f/c, fq/ac, fq^2/ac; q^2, q^2)$$

$$+ \frac{f(a, b; q)_\infty (f, ac^2/bf; q^2)_\infty}{ac(1 - ac)(1 - c/b)(1 - f/c)(1 - bf/ac)(cq^2, acq^2/b; q^2)_\infty (acq/f, bfq/c; q)_\infty}$$

$$\cdot \sum_{j=0}^{\infty} \frac{(bf^2/ac^2, bdf/ac^2, bef/ac^2, f/c, bf/ac; q^2)_j (1 - bf^2q^{4j}/ac^2)}{(q^2, fq^2/d, fq^2/e, bfq^2/ac, fq^2/c; q^2)_j (1 - bf^2/ac^2)} \left(-\frac{fq^2}{ab} \right)^j q^{j^2}$$

by the $n = \infty$ case of (2.7). Hence,

$$ {}_{10}W_9(ac^2/b; c, d, e, f, ac/b, cq/b, cq^2/b; q^2, q^2) \tag{5.10}$$

$$- \frac{bf(1 - c)(1 - ac/b)}{ac^2(1 - f/c)(1 - bf/ac)} \frac{(ac^2q^2/b, d, e, fq^2/d, fq^2/e, bfq^2/ac^2; q^2)_\infty}{(bf^2q^2/ac^2, ac^2q^2/be, ac^2q^2/bd, bdf/ac^2, bef/ac^2, ac^2q^2/bf; q^2)_\infty}$$

$$\cdot \frac{(bf/c, cq/b; q)_\infty}{(ac, fq/ac; q)_\infty} {}_{10}W_9 \left(\frac{bf^2}{ac^2}; \frac{bdf}{ac^2}, \frac{bef}{ac^2}, \frac{f}{c}, \frac{bf}{ac}, \frac{fq}{ac} \frac{fq^2}{ac^2}; q^2, q^2 \right)$$

$$= \frac{(acq/d, acq/e; q)_\infty (ac^2q^2/b, abq, bf/ac^2, ac^2q^2/bde; q^2)_\infty}{(acq, f/ac; q)_\infty (ac^2q^2/bd, ac^2q^2/be, bdf/ac^2, bef/ac^2; q^2)_\infty}$$

$$\cdot \sum_{k=0}^{\infty} \frac{1 - acq^{3k}}{1 - ac} \frac{(a, b, cq/b; q)_k (d, e, f; q^2)_k}{(cq^2, acq^2/b, abq; q)_k (acq/d, acq/e, acq/f; q)_k} q^k$$

$$+ \frac{bf^2(a, b, cq/b; q)_\infty}{a^2c^3(1 - f/c)(1 - bf/ac)(ac, f/ac, acq/f; q)_\infty}$$

$$\cdot \frac{(f, ac^2/bf, ac^2q^2/b, d, e, fq^2/d, fq^2/e, bfq^2/ac^2; q^2)_\infty}{(cq^2, acq^2/b, ac^2q^2/bf, ac^2q^2/be, ac^2q^2/bd, bdf/ac^2, bef/ac^2, bf^2q^2/ac^2; q^2)_\infty}$$

$$\cdot \sum_{j=0}^{\infty} \frac{(bf^2/ac^2, bdf/ac^2, bef/ac^2, f/c, bf/ac; q^2)_j (1 - bf^2q^{4j}/ac^2)}{(q^2, fq^2/d, fq^2/e, bfq^2/ac, fq^2/c; q^2)_j (1 - bf^2/ac^2)} \left(-\frac{fq^2}{ab} \right)^j q^{j^2}$$

when (5.6) holds. By applying the transformation formula [7, (5.14)]

$$\sum_{j=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, c, e, d, f; q)_j}{(q, \sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e, aq/f; q)_j} \left(-\frac{a^2q^2}{cdef}\right)^j q^{\binom{j}{2}} = \frac{(aq, aq/ef; q)_\infty}{(aq/e, aq/f; q)_\infty} {}_3\phi_2\left[\begin{matrix} aq/cd, e, f \\ aq/c, aq/d \end{matrix}; q, \frac{aq}{ef}\right] \tag{5.11}$$

to the sum over j in (5.10), we find that it equals

$$\frac{(q^2, bf^2q^2/ac^2; q^2)_\infty}{(bfq^2/ac, fq^2/c; q^2)_\infty} {}_3\phi_2\left[\begin{matrix} f/c, bf/ac, ac^2q^2/bde \\ fq^2/d, fq^2/e \end{matrix}; q^2, q^2\right] \tag{5.12}$$

and hence, by setting $e = a^2c^2q/df$ in (5.10), we obtain the quadratic transformation formula

$$\begin{aligned} &{}_{10}W_9(ac^2/b; f, ac/b, c, cq/b, cq^2/b, d, a^2c^2q/df; q^2, q^2) \\ &+ \frac{(ac^2q^2/b, bf/ac^2, ac/b, c, cq/b, cq^2/b, bfq^2/ac; q^2)_\infty}{(bf^2q^2/ac^2, ac^2/bf, ac^2q^2/bd, dfq/ab, bdf/ac^2, abq/d, cq^2; q^2)_\infty} \\ &\cdot \frac{(fq^2/c, bf/c, bfq/c, fq^2/d, df^2q/a^2c^2, d, a^2c^2q/df; q^2)_\infty}{(acq^2/b, f/c, bf/ac, ac, acq, fq/ac, fq^2/ac; q^2)_\infty} \\ &\cdot {}_{10}W_9(bf^2/ac^2; f, bdf/ac^2, abq/d, f/c, bf/ac, fq/ac, fq^2/ac; q^2, q^2) \\ &- \frac{(a, b, cq/f; q)_\infty (f, d, a^2c^2q/df, bf/ac^2, ac^2q^2/b, fq^2/d, df^2q/a^2c^2, q^2; q^2)_\infty}{(ac, ac/f, fq/ac; q)_\infty (bf/ac, f/c, cq^2, acq^2/b, ac^2q^2/bf, dfq/ab, bdf/ac^2, abq/d; q^2)_\infty} \\ &\cdot {}_3\phi_2\left[\begin{matrix} f/c, bf/ac, fq/ab \\ fq^2/d, df^2q/a^2c^2 \end{matrix}; q^2, q^2\right] \\ &= \frac{(acq/d, df/ac; q)_\infty (ac^2q^2/b, abq, bf/ac^2, fq/ab; q^2)_\infty}{(acq, f/ac; q)_\infty (ac^2q^2/bd, dfq/ab, bdf/ac^2, abq/d; q^2)_\infty} \\ &\cdot \sum_{k=0}^{\infty} \frac{1 - acq^{3k}}{1 - ac} \frac{(a, b, cq/b; q)_k (d, f, a^2c^2q/df; q^2)_k}{(cq^2, acq^2/b, abq; q^2)_k (acq/d, acq/f, df/ac; q)_k} q^k. \end{aligned} \tag{5.13}$$

Formula (5.1) follows from the case $c = 1$ of (5.13) by applying Sears' transformation formula [11, (5.2)] to the ${}_3\phi_2$ series. If, as in the derivation of (3.21) from (3.19), we set $b = q/a$ in (5.13) and use (3.9), we obtain a quadratic summation formula which can also be derived by replacing a, b, c, d in (5.1) by ac, a, d, a^2c^2q/df, respectively. The quadratic transformation formula (3.12) in [10] can be derived by multiplying both sides of (5.13) by $(f/ac; q)_\infty$ and then setting $f = ac$.

It should be observed that the first two terms on the left side of (5.13) containing the ${}_{10}W_9$ series can be transformed to another pair of ${}_{10}W_9$ series by applying Bailey's four term transformation formula [4, (7.2)]. Also, since the ${}_3\phi_2$ series in (5.13) is balanced (Saalschützian) it can be summed by [12, (3.3.2.2)] whenever it terminates. In addition, since $(q^{-n};q)_\infty = 0$ for $n = 0, 1, 2, \ldots$, the case $d = q^{-2n}$ of (5.13) gives

$$\sum_{k=0}^{n} \frac{1-acq^{3k}}{1-ac} \frac{(a, b, cq/b; q)_k (f, a^2c^2q^{2n+1}/f, q^{-2n}; q^2)_k}{(cq^2, acq^2/b, abq; q^2)_k (acq/f, f/acq^{2n}, acq^{2n+1}; q)_k} q^k \tag{5.14}$$
$$= \frac{(acq;q)_{2n}(ac^2q^2/bf, abq/f; q^2)_n}{(acq/f;q)_{2n}(abq, ac^2q^2/b; q^2)_n}$$
$$\cdot {}_{10}W_9(ac^2/b; f, ac/b, c, cq/b, cq^2/b, a^2c^2q^{2n+1}/f, q^{-2n}; q^2, q^2)$$

and the case $b = cq^{n+1}$ gives

$$\sum_{k=0}^{n} \frac{1-acq^{3k}}{1-ac} \frac{(d, f, a^2c^2q/df; q^2)_k (a, cq^{n+1}, q^{-n}; q)_k}{(acq/d, acq/f, df/ac; q)_k (cq^2, aq^{1-n}, acq^{n+2}; q^2)_k} q^k \tag{5.15}$$
$$= \frac{(acq^2/d, q/ac, acq/f, dfq/ac; q^2)_n}{(acq^2, fq/ac, dq/ac, acq/df; q^2)_n}$$
$$\cdot {}_{10}W_9(acq^{-n-1}; c, d, f, a^2c^2q/df, aq^{-n-1}, q^{1-n}, q^{-n}; q^2, q^2),$$

where $n = 0, 1, 2, \ldots$.

6. A quartic transformation formula with three parameters. By starting with the special case

$$\sum_{k=0}^{n} \frac{(1-acq^{5k})(1-b/cq^{3k})}{(1-ac)(1-b/c)} \frac{(a, b; q)_k (q^{-4n}, ac^2q^{4k}/b; q^4)_k}{(cq^4, acq^4/b; q^4)_k (acq^{4n+1}, b/cq^{4n-1}; q)_k} q^{4k} \tag{6.1}$$
$$= \frac{(1-c)(1-ac/b)(1-acq^{4n})(1-cq^{4n}/b)}{(1-ac)(1-c/b)(1-cq^{4n})(1-acq^{4n}/b)}, \qquad n = 0, 1, 2, \ldots,$$

of (2.8), multiplying both sides of (6.1) by

$$\frac{(1-ac^2q^{8n}/b)(ac^2/b, a^2b^2/q^2; q^4)_n (c/b;q)_{4n}}{(1-ac^2/b)(q^4, c^2q^6/ab^3; q^4)_n (acq;q)_{4n}} q^{4n},$$

summing over n and changing the order of summation, we get the expansion formula

$${}_{10}W_9(ac^2/b; a^2b^2/q^2, ac/b, c, cq/b, cq^2/b, cq^3/b, cq^4/b; q^4, q^4) \tag{6.2}$$
$$= \sum_{k=0}^{\infty} \frac{(a, b; q)_k (a^2b^2/q^2; q^4)_k (ac^2q^4)_{2k} (cq/b;q)_{3k}}{(cq^4, acq^4/b, c^2q^6/ab^3; q^4)_k (ac;q)_{5k}} \left(\frac{c}{b}\right)^k q^{4k+3\binom{k}{2}}$$
$$\cdot {}_8W_7(ac^2q^{8k}/b; a^2b^2q^{4k}/q^2, cq^{3k}/b, cq^{3k+1}/b, cq^{3k+2}/b, cq^{3k+3}/b; q^4, q^4).$$

Since the above ${}_8W_7$ equals

$$\frac{(abq^{2k};q)_\infty (abq^{2k-1};q^2)_\infty (ac^2q^{8k+4}/b, ab^3/c^2q^{4k+2}; q^4)_\infty}{(acq^{5k+1}, ab^2/cq^{k+2}; q)_\infty} \tag{6.3}$$
$$- \frac{(cq^{3k}/b, a^2b^3q^{k-1}/c; q)_\infty (ac^2q^{8k+4}/b, ab^3/c^2q^{4k+2}; q^4)_\infty}{(acq^{5k+1}, ab^2/cq^{k+2}; q)_\infty (a^3b^5/c^2, c^2q^{4k+2}/ab^3; q^4)_\infty}$$
$$\cdot {}_8W_7(a^3b^5/c^2q^4; a^2b^2q^{4k-2}, ab^2/cq^{k+2}, ab^2/cq^{k+1}, ab^2/cq^k, ab^2/cq^{k-1}; q^4, q^4)$$

by (3.9), it follows that

$$
\begin{aligned}
&{}_{10}W_9(ac^2/b; a^2b^2/q^2, ac/b, c, cq/b, cq^2/b, cq^3/b, cq^4/b; q^4, q^4) \qquad (6.4)\\
&= \frac{(ab;q)_\infty (ab/q,q^2)_\infty (ac^2q^4/b, ab^3/c^2q^2; q^4)_\infty}{(acq, ab^2/cq^2; q)_\infty}\\
&\cdot \sum_{k=0}^{\infty} \frac{1-acq^{5k}}{1-ac} \frac{(a,b;q)_k (cq/b, cq^2/b, cq^3/b; q^3)_k (a^2b^2/q^2; q^4)_k}{(cq^4, acq^4/b; q^4)_k (abq, ab, ab/q; q^2)_k (cq^3/ab^2; q)_k} q^k\\
&- \frac{(a^2b^3/cq, c/b; q)_\infty (ac^2q^4/b, ab^3/c^2q^2; q^4)_\infty}{(acq, ab^2/cq^2; q)_\infty (a^3b^5/c^2, c^2q^2/ab^3; q^4)_\infty}\\
&\cdot \sum_{j=0}^{\infty} \frac{(1-a^3b^5q^{8j-4}/c^2)(a^3b^5/c^2q^4, a^2b^2/q^2; q^4)_j (ab^2/cq^2; q)_{4j}}{(1-a^3b^5/c^2q^4)(q^4, ab^3q^2/c^2; q^4)_j (a^2b^3/cq; q)_{4j}} q^{4j}\\
&\cdot \sum_{k=0}^{\infty} \frac{(1-acq^{5k})(1-b/cq^{3k})}{(1-ac)(1-b/c)} \frac{(a,b;q)_k (a^2b^2q^{4j-2}, c^2/ab^3q^{4j-2}; q^4)_k}{(cq^4, acq^4/b; q^4)_k (c/ab^2q^{4j-3}, a^2b^3q^{4j-1}/c; q)_k} q^{4k}.
\end{aligned}
$$

By (2.7), the above sum over k equals

$$
\begin{aligned}
&\frac{(1-c)(1-ac/b)(1-a^2b^3q^{4j-2}/c)(1-ab^2q^{4j-2}/c)}{(1-ac)(1-c/b)(1-a^2b^2q^{4j-2}/c)(1-ab^3q^{4j-2}/c)}\\
&- \frac{ab^3(a,b;q)_\infty (a^2b^2/q^2, c^2q^2/ab^3; q^4)_\infty (-a^3b^5/c^2)^j q^{12\binom{j}{2}}}{c^2q^2(1-ac)(1-b/c)(cq^3/ab^2, a^2b^3/cq; q)_\infty (cq^4, acq^4/b; q^4)_\infty}\\
&\cdot \frac{(ab^3q^2/c^2; q^4)_j (a^2b^3/cq; q)_{4j}}{(1-a^2b^2q^{4j-2}/c)(1-ab^3q^{4j-2}/c)(a^2b^2/q^2; q^4)_j (ab^2/cq^2; q)_{4j}}
\end{aligned}
$$

and so, from (6.4) and the transformation formula [7,(5.27)], we obtain the quartic transformation formula

$$
\begin{aligned}
&{}_{10}W_9(ac^2/b; a^2b^2/q^2, ac/b, c, cq/b, cq^2/b, cq^3/b, cq^4/b; q^4, q^4) \qquad (6.5)\\
&+ \frac{(1-c)(1-ac/b)(a^2b^3/cq^2, cq/b; q)_\infty (ac^2q^4/b, ab^3/c^2q^2; q^4)_\infty}{(1-a^2b^2/cq^2)(1-ab^3/cq^2)(ab^2/cq, ac; q)_\infty (a^3b^5/c^2, c^2q^2/ab^3; q^4)_\infty}\\
&\cdot {}_{10}W_9(a^3b^5/c^2q^4; a^2b^2/q^2, a^2b^2/cq^2, ab^3/cq^2, ab^2/cq, ab^2/c, ab^2q/c, ab^2q^2/c; q^4, q^4)\\
&+ \frac{ab^2(a,b,cq/b;q)_\infty (ac^2q^4/b, ab^3/c^2q^2, a^2b^2/q^2; q^4)_\infty}{cq^2(1-a^2b^2/cq^2)(ac, ab^2/cq^2, cq^3/ab^2; q)_\infty (ab^3/cq^2, cq^4, acq^4/b; q^4)_\infty}\\
&\cdot {}_1\phi_1\left[\begin{matrix} a^2b^2/cq^2 \\ a^2b^2q^2/c \end{matrix}; q^4, \frac{ab^3q^2}{c}\right]\\
&= \frac{(ab;q)_\infty (ab/q;q^2)_\infty (ac^2q^4/b, ab^3/c^2q^2; q^4)_\infty}{(acq, ab^2/cq^2; q)_\infty}\\
&\cdot \sum_{k=0}^{\infty} \frac{1-acq^{5k}}{1-ac} \frac{(a,b;q)_k (cq/b, cq^2/b, cq^3/b; q^3)_k (a^2b^2/q^2; q^4)_k}{(cq^4, acq^4/b; q^4)_k (abq, ab, ab/q; q^2)_k (cq^3/ab^2; q)_k} q^k.
\end{aligned}
$$

When $c = 1$, (6.5) reduces to the quartic summation formula [7, (5.28)]. When $b = q^2/a$, the sum of the two ${}_{10}W_9$ series on the left side of (6.5) reduces to a sum of two ${}_8W_7$ series which can be summed by (3.9) to give

$$\sum_{k=0}^{\infty} \frac{1 - acq^{5k}}{1 - ac} \frac{(a, q^2/a; q)_k (ac/q, ac, acq; q^3)_k (q^2; q^4)_k}{(cq^4, a^2cq^2; q^4)_k (q^3, q^2, q; q^2)_k (ac/q; q)_k} q^k \tag{6.6}$$
$$- \frac{q^2(a, q^2/a, acq; q)_\infty (q^2; q^4)_\infty}{ac(1 - q^2/c)(q^2, ac; q)_\infty (q; q^2)_\infty (cq^4, a^2cq^2, q^4/a^2c; q^4)_\infty}$$
$$\cdot {}_1\phi_1 \left[\begin{matrix} q^2/c \\ q^6/c \end{matrix} ; q^4, \frac{q^8}{a^2c} \right]$$
$$= \frac{(acq^2, q^2/ac, aq, q^3/a; q^2)_\infty}{(q, q^3; q^2)_\infty (cq^4, q^2/c, a^2cq^2, q^4/a^2c; q^4)_\infty}.$$

It should also be observed that by multiplying both sides of (6.5) by $(ab^2/cq^2; q)_\infty$ and then setting $c = ab^2/q^2$ we get the quartic transformation formula

$$\sum_{k=0}^{\infty} \frac{1 - a^2b^2q^{5k-2}}{1 - a^2b^2/q^2} \frac{(a, b; q)_k (ab/q, ab, abq; q^3)_k (a^2b^2/q^2; q^4)_k}{(ab^2q^2, a^2bq^2; q^4)_k (abq, ab, ab/q; q^2)_k (q; q)_k} q^k \tag{6.7}$$
$$= \frac{(aq, b; q)_\infty (a^2b^2q^2; q^4)_\infty}{(q; q)_\infty (abq; q^2)_\infty (b, ab^2q^2, a^2bq^2; q^4)_\infty} {}_1\phi_1 \left[\begin{matrix} a \\ aq^4 \end{matrix} ; q^4, bq^4 \right].$$

Additional summation, transformation, and expansion formulas will be considered elsewhere.

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