Global solutions of shock reflection by large-scale wedges for potential flow

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Abstract

When a plane shock hits a wedge head on, it experiences a reflection-diffraction process and then a self-similar reflected shock moves outward as the original shock moves forward in time. Experimental, computational, and asymptotic analysis has shown that various patterns of shock reflection may occur, including regular and Mach reflection. However, most of the fundamental issues for shock reflection have not been understood, including the global structure, stability, and transition of the different patterns of shock reflection. Therefore, it is essential to establish the global existence and structural stability of solutions of shock reflection in order to understand fully the phenomena of shock reflection. On the other hand, there has been no rigorous mathematical result on the global existence and structural stability of shock reflection, including the case of potential flow which is widely used in aerodynamics. Such problems involve several challenging difficulties in the analysis of nonlinear partial differential equations such as mixed equations of elliptic-hyperbolic type, free boundary problems, and corner singularity where an elliptic degenerate curve meets a free boundary. In this paper we develop a rigorous mathematical approach to overcome these difficulties involved and establish a global theory of existence and stability for shock reflection by large-angle wedges for potential flow. The techniques and ideas developed here will be useful for other nonlinear problems involving similar difficulties.

1. Introduction

We are concerned with the problems of shock reflection by wedges. These problems arise not only in many important physical situations but also are fundamental in the mathematical theory of multidimensional conservation laws since their solutions are building blocks and asymptotic attractors of general solutions to the multidimensional Euler equations for compressible fluids (for example, see
Courant-Friedrichs [16], von Neumann [49], and Glimm-Majda [22]; also see [4], [9], [21], [30], [44], [45], [48]). When a plane shock hits a wedge head on, it experiences a reflection-diffraction process and then a self-similar reflected shock moves outward as the original shock moves forward in time. The complexity of the reflection picture was first reported by Ernst Mach [41] in 1878, and experimental, computational, and asymptotic analysis has shown that various patterns of shock reflection may occur, including regular and Mach reflection (cf. [4], [19], [22], [25], [26], [27], [44], [48], [49]). However, most of the fundamental issues for shock reflection have not been understood, including the global structure, stability, and transition of the different patterns of shock reflection. Therefore, it is essential to establish the global existence and structural stability of solutions of shock reflection in order to understand fully the phenomena of shock reflection.

On the other hand, there has been no rigorous mathematical result on the global existence and structural stability of shock reflection, including the case of potential flow which is widely used in aerodynamics (cf. [5], [15], [22], [42], [44]). One of the main reasons is that the problems involve several challenging difficulties in the analysis of nonlinear partial differential equations such as mixed equations of elliptic-hyperbolic type, free boundary problems, and corner singularity where an elliptic degenerate curve meets a free boundary. In this paper we develop a rigorous mathematical approach to overcome these difficulties and establish a global theory of existence and stability for shock reflection by large-angle wedges for potential flow. The techniques and ideas developed here will be useful for other nonlinear problems involving similar difficulties.

The Euler equations for potential flow consist of the conservation law of mass and the Bernoulli law for the density \( \rho \) and velocity potential \( \Phi \):

\[
\begin{align*}
\partial_t \rho + \text{div}_x (\rho \nabla_x \Phi) &= 0, \\
\partial_t \Phi + \frac{1}{2} |\nabla_x \Phi|^2 + i(\rho) &= K,
\end{align*}
\]

where \( K \) is the Bernoulli constant determined by the incoming flow and/or boundary conditions, and

\[ i'(\rho) = p'(\rho)/\rho = c^2(\rho)/\rho \]

with \( c(\rho) \) being the sound speed. For polytropic gas,

\[ p(\rho) = \kappa \rho^\gamma, \quad c^2(\rho) = \kappa \gamma \rho^{\gamma-1}, \quad \gamma > 1, \quad \kappa > 0. \]

Without loss of generality, we choose \( \kappa = (\gamma - 1)/\gamma \) so that

\[ i(\rho) = \rho^{\gamma-1}, \quad c(\rho)^2 = (\gamma - 1) \rho^{\gamma-1}, \]

which can be achieved by the following scaling:

\[ (x, t, K) \rightarrow (\alpha x, \alpha^2 t, \alpha^{-2} K), \quad \alpha^2 = \kappa \gamma/(\gamma - 1). \]
Equations (1.1) and (1.2) can be written as the following nonlinear equation of second order:

$$\frac{3}{4} \partial_t \rho \left( K - \partial_x \Phi - \frac{1}{2} |\nabla_x \Phi|^2 \right) + \text{div}_x \left( \rho \left( K - \partial_x \Phi - \frac{1}{2} |\nabla_x \Phi|^2 \right) \nabla_x \Phi \right) = 0,$$

where \( \rho(s) = s^{1/(n-1)} = i^{-1}(s) \) for \( s \geq 0 \).

When a plane shock in the \((x, t)\)-coordinates, \( x = (x_1, x_2) \in \mathbb{R}^2 \), with left state \((\rho, \nabla_x \Phi) = (\rho_1, u_1, 0)\) and right state \((\rho_0, 0, 0), u_1 > 0, \rho_0 < \rho_1\), hits a symmetric wedge

\[ W := \{ |x_2| < x_1 \tan \theta_w, x_1 > 0 \} \]

head on, it experiences a reflection-diffraction process, and the reflection problem can be formulated as the following mathematical problem.

**Problem 1 (Initial-boundary value problem).** Seek a solution of systems (1.1) and (1.2) with \( K = \rho_0^{-1} \), the initial condition at \( t = 0 \):

$$\operatorname{div} \Phi |_{t=0} = \begin{cases} (\rho_0, 0) & \text{for } |x_2| > x_1 \tan \theta_w, x_1 > 0, \\ (\rho_1, u_1 x_1) & \text{for } x_1 < 0, \end{cases}$$

and the slip boundary condition along the wedge boundary \( \partial W \):

$$\nabla \Phi \cdot v |_{\partial W} = 0,$$

where \( v \) is the exterior unit normal to \( \partial W \) (see Fig. 1.1).

![Figure 1.1. Initial-boundary value problem](image-url)
Notice that the initial-boundary value problem (1.1)–(1.5) is invariant under the self-similar scaling:
\[
(x, t) \rightarrow (\alpha x, \alpha t), \quad (\rho, \Phi) \rightarrow (\rho/\alpha, \Phi/\alpha)
\quad \text{for} \quad \alpha \neq 0.
\]
Thus, we seek self-similar solutions with the form
\[
\rho(x, t) = \rho(\xi, \eta), \quad \Phi(x, t) = t \psi(\xi, \eta)
\quad \text{for} \quad (\xi, \eta) = x/t.
\]
Then the pseudo-potential function \(\varphi = \psi - \frac{1}{2}(\xi^2 + \eta^2)\) satisfies the following Euler equations for self-similar solutions:
\[
\begin{align*}
(1.6) & \quad \text{div} (\rho D\varphi) + 2\rho = 0, \\
(1.7) & \quad \frac{1}{2} |D\varphi|^2 + \varphi + \rho^{\gamma-1} = \rho_0^{\gamma-1},
\end{align*}
\]
where the divergence div and gradient D are with respect to the self-similar variables \((\xi, \eta)\). This implies that the pseudo-potential function \(\varphi(\xi, \eta)\) is governed by the following potential flow equation of second order:
\[
\begin{align*}
(1.8) & \quad \text{div} (\rho(|D\varphi|^2, \varphi) D\varphi) + 2\rho(|D\varphi|^2, \varphi) = 0 \\
(1.9) & \quad \rho(|D\varphi|^2, \varphi) = \beta \left(\rho_0^{\gamma-1} - \varphi - \frac{1}{2} |D\varphi|^2\right).
\end{align*}
\]
Then we have
\[
\begin{align*}
(1.10) & \quad c^2 = c^2(|D\varphi|^2, \varphi, \rho_0^{\gamma-1}) = (\gamma - 1) \left(\rho_0^{\gamma-1} - \frac{1}{2} |D\varphi|^2 - \varphi\right), \\
(1.11) & \quad |D\varphi| < c(|D\varphi|^2, \varphi, \rho_0^{\gamma-1}),
\end{align*}
\]
which is equivalent to
\[
|D\varphi| < c_*(\varphi, \rho_0, \gamma) := \sqrt{\frac{2(\gamma - 1)}{\gamma + 1} (\rho_0^{\gamma-1} - \varphi)}.
\]
Shocks are discontinuities in the pseudo-velocity \(D\varphi\). That is, if \(\Omega^+ := \Omega \setminus \Omega\) are two nonempty open subsets of \(\Omega \subset \mathbb{R}^2\) and \(S := \partial \Omega^+ \cap \Omega\) is a \(C^1\) curve where \(D\varphi\) has a jump, then \(\varphi \in W^{1,1}_{\text{loc}}(\Omega) \cap C^1(\Omega^+ \cup S) \cap C^2(\Omega^+)\) is a global weak solution of (1.8) in \(\Omega\) if and only if \(\varphi\) is in \(W^{1,\infty}_{\text{loc}}(\Omega)\) and satisfies (1.8) in \(\Omega^+\) and the Rankine-Hugoniot condition on \(S\):
\[
\begin{align*}
(1.12) & \quad \frac{1}{2} |D\varphi|^2, \varphi D\varphi \cdot n = 0.
\end{align*}
\]
containing the arc $P_1 P_4$ in Figure 1.3, so that $\varphi_2$ is the unique solution in the
domain $P_0 P_1 P_4$, as argued in [9] [45]. In the domain $\Omega$, the solution is expected
to be pseudo-subsonic, smooth, and $C^1$-smoothly matching with state (2) across
$P_1 P_4$ and to satisfy $\varphi_\eta = 0$ on $P_2 P_3$; the transonic shock curve $P_1 P_2$ matches up
to second-order with $P_0 P_1$ and is orthogonal to the \( \xi \)-axis at the point $P_3$ so that
the standard reflection about the \( \xi \)-axis yields a global solution in the whole plane.
Then the solution of Problem 2 can be shown to be the solution of Problem 1.

Main Theorem (See §9 for the proof): There exist $\theta_c = \theta_c(\rho_0, \rho_1, \gamma) \in
(0, \pi/2)$ and $\alpha = \alpha(\rho_0, \rho_1, \gamma) \in (0, 1/2)$ such that, when $\theta_w \in [\theta_c, \pi/2)$, there
exists a global self-similar solution

$$\Phi(x, t) = t \varphi \left( \frac{x}{t} \right) + \frac{|x|^2}{2t} \quad \text{for } -\infty < x < \infty, \quad t > 0$$

with

$$\varphi(x, t) = \left( \Phi_{\theta}^{-1} - \Phi_t - \frac{1}{2} |\nabla_x \Phi|^2 \right)^{\gamma/2}$$

of Problem 1 (equivalently, Problem 2) for shock reflection by the wedge, which
satisfies that, for $x/t = \xi/\eta$,

$$\varphi \in C^\infty(\Omega) \cap C^{1,\alpha}(\bar{\Omega}),$$

(1.20) $\varphi = \begin{cases} 
\varphi_0 & \text{for } \xi > \xi_0 \text{ and } \eta > \xi \tan \theta_w, \\
\varphi_1 & \text{for } \xi < \xi_0 \text{ and above the reflection shock } P_0 P_1 P_2, \\
\varphi_2 & \text{in } P_0 P_1 P_4,
\end{cases}$

as the right.
\[ \psi \] is \( C^{1,1} \) across the part \( P_1P_4 \) of the sonic circle including the endpoints \( P_1 \) and \( P_4 \), and the reflected shock \( P_0P_1P_2 \) is \( C^2 \) at \( P_1 \) and \( C^\infty \) except \( P_1 \). Moreover, the solution \( \psi \) is stable with respect to the wedge angle in \( W^{1,1}_{\text{loc}} \) and converges in \( W^{1,1}_{\text{loc}} \) to the solution of the normal reflection described in Section 3.1 as \( \theta_\infty \to \pi/2 \).

One of the main difficulties for the global existence is that the ellipticity condition (1.12) for (1.8) is hard to control, in comparison to our earlier work on steady flow [10], [12]. The second difficulty is that the ellipticity degenerates at the sonic circle \( P_1P_4 \) (the boundary of the pseudo-subsonic flow). The third difficulty is that, on \( P_1P_4 \), we need to match the solution in \( \Omega \) with \( \varphi_2 \) at least in \( C^1 \), that is, the two conditions on the fixed boundary \( P_1P_4 \) - the Dirichlet and conormal conditions, which are generically overdetermined for an elliptic equation since the conditions on the other parts of the boundary have been prescribed. Thus we have to prove that, if \( \varphi \) satisfies (1.8) in \( \Omega \), the Dirichlet continuity condition on the sonic circle, and the appropriate conditions on the other parts of \( \partial \Omega \) derived from Problem 2, then the normal derivative \( D\varphi \cdot n \) automatically matches with \( D\varphi_2 \cdot n \) along \( P_1P_4 \).

We show that, in fact, this follows from the structure of elliptic degeneracy of (1.8) on \( P_1P_4 \) for the solution \( \varphi \). Indeed, (1.8), written in terms of the function \( u = \varphi - \varphi_2 \) in the \((x,y)\)-coordinates defined near \( P_1P_4 \) such that \( P_1P_4 \) becomes a segment on \( \{x = 0\} \), has the form:

\[
(2x - (y + 1)u_x)u_{xx} + \frac{1}{c_2^2} u_{yy} - u_x = 0 \quad \text{in } x > 0 \text{ and near } x = 0,
\]

plus the "small" terms that are controlled by \( \pi/2 - \theta_\infty \) in appropriate norms. (1.21) is elliptic if \( u_x < 2x/(y + 1) \). Thus, we need to obtain the \( C^{1,1} \) estimates near \( P_1P_4 \) to ensure \( |u_x| < 2x/(y + 1) \) which in turn implies both the ellipticity of the equation in \( \Omega \) and the match of normal derivatives \( D\varphi \cdot n = D\varphi_2 \cdot n \) along \( P_1P_4 \).

Taking into account the "small" terms to be added to (1.21), we need to make the stronger estimate \( |u_x| \leq 4x/(3(y + 1)) \) and assume that \( \pi/2 - \theta_\infty \) is appropriately small to control these additional terms. Another issue is the non-variational structure and nonlinearity of this problem which makes it hard to apply directly the approaches of Caffarelli [6] and Alt-Caffarelli-Friedman [1], [2]. Moreover, the elliptic degeneracy and geometry of the problem makes it difficult to apply the hodograph transform approach in Kinderlehrer-Nirenberg [28] and Chen-Feldman [11] to fix the free boundary.

For these reasons, one of the new ingredients in our approach is to further develop the iteration scheme in [10], [12] to a partially modified equation. We modify (1.8) in \( \Omega \) by a proper cutoff that depends on the distance to the sonic circle, so that the original and modified equations coincide for \( \varphi \) satisfying \( |u_x| \leq 4x/(3(y + 1)) \), and the modified equation \( \mathcal{N}\varphi = 0 \) is elliptic in \( \Omega \) with elliptic degeneracy on \( P_1P_4 \). Then we solve a free boundary problem for this modified
second-order nonlinear equation of mixed type in a convenient form. In Section 5, we develop an iteration scheme, along with an elliptic cutoff technique, to solve the free boundary problem and set up the ten detailed steps of the iteration procedure. Finally, we complete the remaining steps in our iteration procedure in Sections 6–9: Step 2 for the existence of solutions of the boundary value problem to the degenerate elliptic equation via the vanishing viscosity approximation in Section 6; Steps 3–8 for the existence of the iteration map and its fixed point in Section 7; and Step 9 for the removal of the ellipticity cutoff in the iteration scheme by using appropriate comparison functions and deriving careful global estimates for some directional derivatives of the solution in Section 8. We complete the proof of Main Theorem in Section 9. Careful estimates of the solutions to both the “almost tangential derivative” and oblique derivative boundary value problems for elliptic equations are made in the appendix, which are applied in Sections 6 and 7.

2. Self-similar solutions of the potential flow equation

In this section we present the potential flow equation in self-similar coordinates and exhibit some basic properties of solutions of the potential flow equation (also see Morawetz [44]).

2.1. The potential flow equation for self-similar solutions. (1.8) is a mixed equation of elliptic-hyperbolic type. It is elliptic if and only if (1.12) holds. The hyperbolic-elliptic boundary is the pseudo-sonic curve: |Dφ| = c_*(φ, ρ_0, γ).

We first define the notion of weak solutions of (1.8) and (1.9). Essentially, we require the equation to be satisfied in the distributional sense.

Definition 2.1 (Weak solutions). A function φ ∈ W^{1,1}_{loc}(Λ) is called a weak solution of (1.8) and (1.9) in a self-similar domain Λ if

(i) ρ_0^γ - φ - \frac{1}{2} |Dφ|^2 ≥ 0 a.e. in Λ;
(ii) (ρ(|Dφ|^2, φ), ρ(|Dφ|^2, φ)|Dφ|) ∈ (L^1_{loc}(Λ))^2;
(iii) For every ξ ∈ C^{∞}_c(Λ),
\[ \int Λ (ρ(|Dφ|^2, φ) Dφ · Dξ - 2ρ(|Dφ|^2, φ)ξ) dξ dη = 0. \]

It is straightforward to verify the equivalence between time-dependent self-similar solutions and weak solutions of (1.8) defined in Definition 2.1 in the weak sense. It can also be verified that, if φ ∈ C^{1,1}(Λ) (and thus φ is twice differentiable a.e. in Λ), then φ is a weak solution of (1.8) in Λ if and only if φ satisfies (1.8) a.e. in Λ. Finally, it is easy to see that, if Λ^+ and Λ^- = Λ \setminus Λ^+ are two nonempty open subsets of Λ ⊂ R^2 and S = ∂Λ^+ ∩ Λ is a C^1 curve where Dφ has a jump, then φ ∈ W^{1,1}_{loc}(D) ∩ C^1(Λ ± ∪ S) ∩ C^{1,1}(Λ^±) is a weak solution of (1.8) in Λ if and
only if \( \varphi \) is in \( W_{loc}^{1,\infty}(\Lambda) \) and satisfies (1.8) a.e. in \( \Lambda^\pm \) and the Rankine-Hugoniot condition (1.13) on \( S \).

Note that, for \( \varphi \in C^1(\Lambda^\pm \cup S) \), the condition \( \varphi \in W_{loc}^{1,\infty}(\Lambda) \) implies

(2.1) \[ [\varphi]_S = 0. \]

Furthermore, the Rankine-Hugoniot conditions imply

(2.2) \[ [\varphi_\xi][\rho \varphi_\xi] - [\varphi_\eta][\rho \varphi_\eta] = 0 \text{ on } S \]

which is a useful identity.

A discontinuity of \( D\varphi \) satisfying the Rankine-Hugoniot conditions (2.1) and (1.13) is called a shock if it satisfies the physical entropy condition: The density function \( \rho \) increases across a shock in the pseudo-flow direction. The entropy condition indicates that the normal derivative function \( \varphi_\nu \) on a shock always decreases across the shock in the pseudo-flow direction.

2.2. The states with constant density. When the density \( \rho \) is constant, (1.8) and (1.9) imply that \( \varphi \) satisfies

\[ \Delta \varphi + 2 = \frac{1}{2} |D\varphi|^2 + \varphi = \text{const.} \]

This implies \( (\Delta \varphi)_\xi = 0 \), \( (\Delta \varphi)_\eta = 0 \), and \( (\varphi_\xi + 1)^2 + \varphi_\eta^2 = 0 \). Thus, we have

\[ \varphi_\xi = -1, \quad \varphi_\eta = 0, \quad \varphi_\xi_\eta = -1, \]

which yields

(2.3) \[ \varphi(\xi, \eta) = \frac{1}{2} (\xi^2 + \eta^2) + a \xi + b \eta + c, \]

where \( a, b, \) and \( c \) are constants.

2.3. Location of the incident shock. Consider state (0): \( (\rho_0, u_0, v_0) = (\rho_0, 0, 0) \) with \( \rho_0 > 0 \) and state (1): \( (\rho_1, u_1, v_1) = (\rho_1, u_1, 0) \) with \( \rho_1 > \rho_0 > 0 \) and \( u_1 > 0 \).

The plane incident shock solution with state (0) and state (1) corresponds to a continuous weak solution \( \varphi \) of (1.8) in the self-similar coordinates \( (\xi, \eta) \) with form (1.14) and (1.15) for state (0) and state (1) respectively, where \( \xi = \xi_0 > 0 \) is the location of the incident shock.

The unit normal to the shock line is \( \nu = (1, 0) \). Using (2.2), we have

\[ u_1 = \frac{\rho_1 - \rho_0}{\rho_1} \xi_0 > 0. \]

Then (1.9) implies

\[ \rho_1^{\gamma-1} - \rho_0^{\gamma-1} = -\frac{1}{2} |D\varphi_1|^2 - \varphi_1 = \frac{1}{2} \frac{\rho_1^2 - \rho_0^2}{\rho_1^2} \xi_0^2. \]
Indeed, for fixed $\gamma > 1$ and $\rho_1, u_1 > 0$ and for $F(\bar{\rho}_2)$ that is the right-hand side of (3.4), we have

$$\lim_{s \to \infty} F(s) = \rho_1^{\frac{1}{\gamma} - 1} + \frac{1}{2} u_1^2 > \rho_1^{\frac{1}{\gamma} - 1}, \quad \lim_{s \to \rho_1^+} F(s) = \infty.$$  

Thus there exists a unique $\bar{\rho}_2 \in (\rho_1, \infty)$ satisfying $\bar{\rho}_2^{\frac{1}{\gamma} - 1} = F(\bar{\rho}_2)$, i.e., (3.4). Then the position of the reflected shock $\xi = \bar{\xi} < 0$ is uniquely determined by (3.3).

Moreover, for the sonic speed $c_2 = \sqrt{(\gamma - 1)\bar{\rho}_2^{\frac{1}{\gamma} - 1}}$ of state (2), we have

$$\mathcal{L} c_2 = \frac{\bar{\xi}}{\bar{c}_2}.$$  

This can be seen as follows. First note that

$$\beta = (\gamma - 1)\bar{\rho}_2^{\frac{1}{\gamma} - 2} > 0 \text{ for some } \rho_* \in (\rho_1, \bar{\rho}_2).$$  

Case 1. $\gamma \geq 2$. Then

$$0 < (\gamma - 1)\rho_1^{\frac{1}{\gamma} - 2} \leq \beta \leq (\gamma - 1)\bar{\rho}_2^{\frac{1}{\gamma} - 2}.$$  

Since $\beta > 0$ and $\bar{\rho}_2 > \rho_1$, we use (3.4) and (3.6) to find

$$\bar{\rho}_2 = \rho_1 + \frac{u_1}{4\beta} \left( u_1 + \sqrt{u_1^2 + 16\beta \rho_1} \right).$$
and hence
\( \frac{1}{2} \)
\[ \xi = -\frac{4\beta \rho_1}{u_1 + \sqrt{u_1^2 + 16\beta \rho_1}}. \]

Then by (3.7) and (3.8), \( \tilde{\rho}_2 > \rho_1 > 0 \), and \( u_1 > 0 \) yields
\[ |\xi| = \frac{4\beta \rho_1}{u_1 + \sqrt{u_1^2 + 16\beta \rho_1}} < \sqrt{\beta \rho_1} \leq \sqrt{(\gamma - 1) \tilde{\rho}_2^{\gamma-2}} \tilde{\rho}_2 = \tilde{c}_2. \]

Case 2. \( 1 < \gamma < 2 \). Then, since \( \tilde{\rho}_2 > \rho_1 > 0 \),
\[ 0 < (\gamma - 1) \tilde{\rho}_2^{\gamma-2} \leq \beta \leq (\gamma - 1) \rho_1^{\gamma-2}. \]

Since \( \beta > 0 \), (3.8) holds by the calculation as in Case 1. Now we use (3.8) and (3.9), \( \tilde{\rho}_2 > \rho_1 > 0 \), \( u_1 > 0 \), and \( 1 < \gamma < 2 \) to find again
\[ |\xi| < \sqrt{\beta \rho_1} \leq \sqrt{(\gamma - 1) \rho_1^{\gamma-1}} \leq \sqrt{(\gamma - 1) \tilde{\rho}_2^{\gamma-1}} = \tilde{c}_2. \]

This shows that (3.5) holds in general.

3.2. The von Neumann criterion and local theory for regular reflection. In this subsection, we first follow the von Neumann criterion to derive the necessary condition for the existence of regular reflection and show that, when the wedge angle is large, there exists a unique state (2) with two-shock structure at the reflected point, which is close to the solution \( (\tilde{\rho}_2, \tilde{u}_2, \tilde{v}_2) = (\tilde{\rho}_2, 0, 0) \) of normal reflection for which \( \theta_w = \pi / 2 \) in §3.1.

For a possible two-shock configuration satisfying the corresponding boundary condition on the wedge \( \eta = \xi \tan \theta_w \), the three state functions \( \varphi_j \), \( j = 0, 1, 2 \), must be of form (1.14), (1.15), and (1.19) (cf. (2.3)).

Let \( P_0 = (\xi_0, \tilde{\rho}_0, \theta_w) \) be the reflection point (i.e., the intersection point of the incident shock with the wall), and let the reflected straight shock separating states (1) and (2) be the line that intersects with the axis \( \eta = 0 \) at the point \( (\xi, 0) \) with the angle \( \theta_w \) between the line and \( \eta = 0 \).

Note that \( \varphi_1 (\xi, \eta) \) is defined by (1.15). The continuity of \( \varphi \) at \( (\xi, 0) \) yields
\[ \varphi_2 (\xi, \eta) = \frac{1}{2} (\xi^2 + \eta^2) + u_2 \xi + v_2 \eta + (u_1 (\xi - \xi_0) - u_2 \xi). \]

Furthermore, \( \varphi_2 \) must satisfy the slip boundary condition at \( P_0 \):
\[ v_2 = u_2 \tan \theta_w. \]

Also we have
\[ \xi = \xi_0 - \xi_0 \frac{\tan \theta_w}{\tan \xi}. \]
4.1. Shifting coordinates. It is more convenient to change the coordinates in
the self-similar plane by shifting the origin to the center of the sonic circle of
state (2). Thus we define

\[(\xi, \eta)_{\text{new}} := (\xi, \eta) - (u_2, v_2)\].

For simplicity of notation, throughout this paper below, we will always work in the
new coordinates without changing the notation \((\xi, \eta)\), and we will not emphasize
this again later.

In the new shifted coordinates, the domain \(\Omega\) is expressed as

\[
(4.1) \quad \Omega = B_{c_2}(0) \cap \{\eta > -v_2\} \cap \{f(\eta) < \xi < \eta \cot \theta_w\}.
\]

where \(f\) is the position function of the free boundary, i.e., the curved part of the
reflected shock \(\Gamma_{\text{shock}} := \{\xi = f(\eta)\}\). The function \(f\) in (4.1) will be determined
below so that

\[
(4.2) \quad \|f - l\| \leq C\sigma
\]
in an appropriate norm, specified later. Here \(l = l(\eta)\) is the location of the reflected
shock of state (2) which is a straight line, that is,

\[
(4.3) \quad l(\eta) = \eta \cot \theta_s + \xi
\]

and

\[
(4.4) \quad \xi = \xi - u_2 + v_2 \cot \theta_s < 0,
\]

if \(\sigma = \pi/2 - \theta_w > 0\) is sufficiently small, since \(u_2\) and \(v_2\) are small and \(\xi < 0\) by
(3.3) in this case. Also note that, since \(u_2 = v_2 \cot \theta_w > 0\), it follows from (3.22)
that

\[
(4.5) \quad \xi > \xi.
\]

Another condition on \(f\) comes from the fact that the curved part and straight
part of the reflected shock should match at least up to first-order. Denote by \(P_1 =
(\xi_1, \eta_1)\) with \(\eta_1 > 0\) the intersection point of the line \(\xi = l(\eta)\) and the sonic circle
\(\xi^2 + \eta^2 = c_2^2\), i.e., \((\xi_1, \eta_1)\) is the unique point for small \(\sigma > 0\) satisfying

\[
(4.6) \quad l(\eta_1)^2 + \eta_1^2 = c_2^2, \quad \xi_1 = l(\eta_1), \quad \eta_1 > 0.
\]

The existence and uniqueness of such a point \((\xi_1, \eta_1)\) follows from \(-c_2 < \xi < 0,
which holds from (3.22), (3.25), (4.4), and the smallness of \(u_2\) and \(v_2\). Then \(f\)
satisfies

\[
(4.7) \quad f(\eta_1) = l(\eta_1), \quad f'(\eta_1) = l'(\eta_1) = \cot \theta_s.
\]
Note also that, for small \( \sigma > 0 \), we obtain from (3.25), (4.4), (4.5), and \( l'(\eta) = \cot \theta \tilde{\xi} > 0 \) that

\[
(4.8) \quad -c_2 < \tilde{\xi} < \tilde{\xi}_1 < 0, \quad c_2 - |\tilde{\xi}| \geq \frac{\tilde{c}_2 - |\tilde{\xi}|}{2} > 0.
\]

Furthermore, equations (1.8)-(1.9) and the Rankine-Hugoniot conditions (1.13) and (2.1) on \( \Gamma_{\text{shock}} \) do not change under the shift of coordinates. That is, we seek \( \varphi \) satisfying (1.8)-(1.9) in \( \Omega \), so that the equation is elliptic on \( \varphi \), and satisfying the following boundary conditions on \( \Gamma_{\text{shock}} \): the continuity of the pseudo-potential function across the shock:

\[
(4.9) \quad \varphi = \varphi_1 \quad \text{on} \quad \Gamma_{\text{shock}}
\]

and the gradient jump condition:

\[
(4.10) \quad \rho(|D\varphi|^2, \varphi)D\varphi \cdot n_\gamma = \rho_1 D\varphi_1 \cdot n_\gamma \quad \text{on} \quad \Gamma_{\text{shock}},
\]

where \( n_\gamma \) is the interior unit normal to \( \Omega \) on \( \Gamma_{\text{shock}} \).

The boundary conditions on the other parts of \( \partial \Omega \) are

\[
(4.11) \quad \varphi = \varphi_2 \quad \text{on} \quad \Gamma_{\text{sonic}} = \partial \Omega \cap \partial B_{c_2}(0),
\]

\[
(4.12) \quad \varphi_\nu = 0 \quad \text{on} \quad \Gamma_{\text{wedge}} = \partial \Omega \cap \{ \eta = \xi \tan \theta_w \},
\]

\[
(4.13) \quad \varphi_\nu = 0 \quad \text{on} \quad \partial \Omega \cap \{ \eta = -v_2 \}.
\]

Rewriting the background solutions in the shifted coordinates, we find

\[
(4.14) \quad \varphi_0(\tilde{\xi}, \eta) = -\frac{1}{2}(\tilde{\xi}^2 + \eta^2) - (u_2 \tilde{\xi} + v_2 \eta) - \frac{1}{2}q_2^2,
\]

\[
(4.15) \quad \varphi_1(\tilde{\xi}, \eta) = -\frac{1}{2}(\tilde{\xi}^2 + \eta^2) + (u_1 - u_2) \tilde{\xi} - v_2 \eta - \frac{1}{2}q_2^2 + u_1(u_2 - \tilde{\xi}_0),
\]

\[
(4.16) \quad \varphi_2(\tilde{\xi}, \eta) = -\frac{1}{2}(\tilde{\xi}^2 + \eta^2) - \frac{1}{2}q_2^2 + (u_1 - u_2) \tilde{\xi} + u_1(u_2 - \tilde{\xi}_0),
\]

where \( q_2^2 = u_2^2 + v_2^2 \).

Furthermore, substituting \( \tilde{\xi} \) in (4.4) into (3.17) and using (3.11) and (3.14), we find

\[
(4.17) \quad \rho_2 \tilde{\xi} = \rho_1 \left( \tilde{\xi} - \frac{(u_1 - u_2)^2 + v_2^2}{u_1 - u_2} \right),
\]

which expresses the Rankine-Hugoniot conditions on the reflected shock of state (2) in terms of \( \tilde{\xi} \). We use this equality below.

4.2. The equations and boundary conditions in terms of \( \psi = \varphi - \varphi_2 \). It is convenient to study the problem in terms of the difference between our solution
\[
\frac{1}{\rho_2^{\gamma - 1}} \left( \frac{\gamma - 1}{\gamma} \left( \rho_2^{\gamma - 1} - \psi + r \psi_r - \frac{1}{2} \left( \psi_r^2 + \frac{1}{r^2} \psi_\theta^2 \right) \right) \right).
\]

Also, from (4.11), (4.12) and (4.16)-(4.18), we obtain

\[
\begin{align*}
\psi &= 0 \quad \text{on } \Gamma_{\text{sane}} = \partial \Omega \cap \partial \partial B_{c_2}(0), \\
\psi_r &= 0 \quad \text{on } \Gamma_{\text{wedge}} = \partial \Omega \cap \{ \eta = \xi \tan \theta_w \}, \\
\psi_\eta &= -\psi_\xi \quad \text{on } \partial \Omega \cap \{ \eta = -\psi_2 \}.
\end{align*}
\]

Using (4.15)-(4.16), the Rankine-Hugoniot conditions in terms of \( \psi \) take the following form: The continuity of the pseudo-potential function across (4.9) is written as

\[
\begin{align*}
\psi - \frac{1}{2} \psi_\xi^2 + \psi_{\eta}(u_1 - u_2) + u_1(u_2 - \xi_0) \\
= \xi(u_1 - u_2) - \psi_{\eta} - \frac{1}{2} \psi_\xi^2 + u_1(u_2 - \xi_0) \quad \text{on } \Gamma_{\text{shock}},
\end{align*}
\]

that is,

\[
\xi = \frac{\psi(\xi, \eta) + \psi_{\eta} + \psi_\xi}{u_1 - u_2}.
\]

where \( \xi \) is defined by (4.4). The gradient jump condition (4.10) is

\[
\rho(D\psi, \psi)(D\psi - (\xi, \eta)) \cdot v_3 = \rho_1(u_1 - u_2 - \xi, -\psi_{\eta}) \cdot v_3 \quad \text{on } \Gamma_{\text{shock}},
\]

where \( \rho(D\psi, \psi) \) is defined by (4.20) and \( v_3 \) is the interior unit normal to \( \Omega \) on \( \Gamma_{\text{shock}} \). If \( |(u_2, v_2, D\psi)| < u_1/50 \), the unit normal \( v_3 \) can be expressed as

\[
\psi_{\eta} = \frac{D(\rho_1 - \varphi)}{|D(\rho_1 - \varphi)|} = \frac{(u_1 - u_2 - \psi_\xi, -\psi_{\eta} - \psi_\xi)}{\sqrt{(u_1 - u_2 - \psi_\xi)^2 + (\psi_{\eta} + \psi_\xi)^2}},
\]

where we have used (4.15), (4.16) and (4.18) to obtain the last expression.

Now we rewrite the jump condition (4.29) in a more convenient form for \( \psi \) satisfying (4.9) when \( \sigma > 0 \) and \( \| \psi \|_{C^1(\Omega)} \) are sufficiently small.

We first discuss the smallness assumptions for \( \sigma > 0 \) and \( \| \psi \|_{C^1(\Omega)} \). By (2.4), (3.20), and (3.24), it follows that, if \( \sigma \) is small depending only on the data, then

\[
\frac{5c_2}{6} \leq c_2 \leq \frac{6c_2}{5}, \quad \frac{5\rho_2}{6} \leq \rho_2 \leq \frac{6\rho_2}{5}, \quad \sqrt{u_2^2 + v_2^2} \leq \frac{u_1}{50}.
\]

We also require that \( \| \psi \|_{C^1(\Omega)} \) be sufficiently small so that, if (4.31) holds, the expressions (4.20) and (4.30) are well-defined in \( \Omega \), and \( \xi \) defined by the right-hand side of (4.28) satisfies \( \| \xi \| \leq \frac{7c_2}{5} \) for \( \eta \in (-\psi_2, \xi_2) \), which is the range of \( \eta \).
on $\Gamma_{\text{shock}}$. Since \((4.31)\) holds and $\Omega \subset B_{\tilde{c}_2}(0)$ by \((4.1)\), it suffices to assume

\[
\Vert \psi \Vert_{C^1(\tilde{\Omega})} \leq \min \left( \frac{\tilde{\rho}^{-1}}{50(1 + 4\tilde{c}_2)}, \min(1, \tilde{c}_2) \frac{u_1}{50} \right) =: \delta^*.
\]

For the rest of this section, we assume that \((4.31)\) and \((4.32)\) hold.

Under these conditions, we can substitute the right-hand side of \((4.30)\) for $u_\eta$ into \((4.29)\). Thus, we rewrite \((4.29)\) as

\[
F(D\psi, \psi, u_2, v_2, \xi, \eta) = 0 \quad \text{on } \Gamma_{\text{shock}},
\]

where, with $p = (p_1, p_2) \in \mathbb{R}^2$ and $z \in \mathbb{R}$,

\[
F(p, z, u_2, v_2, \xi, \eta) = (\tilde{\rho}(p - (z, \eta)) - p_1 (u_1 - u_2 - \xi, -v_2 - \eta)) \cdot \tilde{\nu}
\]

with $\tilde{\rho} := \tilde{\rho}(p, z, \xi, \eta)$ and $\tilde{\nu} := \tilde{\nu}(p, u_2, v_2)$ defined by

\[
\tilde{\rho}(p, z, \xi, \eta) = \left( \tilde{\rho}_2^{z,1} + \xi p_1 + \eta p_2 - \frac{|p|^2}{2} - z \right)^{z,1,1}
\]

\[
\tilde{\nu}(p, u_2, v_2) = \frac{(u_1 - u_2 - p_1, -v_2 - p_2)}{\sqrt{(u_1 - u_2 - p_1)^2 + (v_2 + p_2)^2}}.
\]

From the explicit definitions of $\tilde{\rho}$ and $\tilde{\nu}$, it follows from \((4.31)\) that

\[
\tilde{\rho} \in C^\infty(B_{3\delta}(0) \times (-\delta^*, \delta^*) \times B_{2\tilde{c}_2}(0)), \quad \tilde{\nu} \in C^\infty(B_{3\delta}(0) \times B_{u_1/50}(0)),
\]

where $B_R(0)$ denotes the ball in $\mathbb{R}^2$ with center 0 and radius $R$ and, for $k \in \mathbb{N}$ (the set of nonnegative integers), the $C^k$-norms of $\tilde{\rho}$ and $\tilde{\nu}$ over the regions specified above are bounded by the constants depending only on $\gamma, u_1, \tilde{\rho}_2, \tilde{c}_2$, and $k$, that is, by Section 3, the $C^k$-norms depend only on the data and $k$. Thus, \((4.37)\)

\[
F \in C^\infty(B_{3\delta}(0) \times (-\delta^*, \delta^*) \times B_{u_1/50}(0) \times B_{2\tilde{c}_2}(0)).
\]

Furthermore, since $\psi$ satisfies \((4.9)\) and hence \((4.28)\), we can substitute the right-hand side of \((4.28)\) for $\xi$ into \((4.33)\). Thus we rewrite \((4.29)\) as

\[
\Psi(D\psi, \psi, u_2, v_2, \eta) = 0 \quad \text{on } \Gamma_{\text{shock}},
\]

where

\[
\Psi(p, z, u_2, v_2, \eta) = F(p, z, u_2, v_2, (z + v_2\eta)/(u_1 - u_2) + \xi, \eta).
\]

If $\eta \in (-6\tilde{c}_2/5, 6\tilde{c}_2/5)$ and $|z| \leq \delta^*$, then, from \((4.8)\) and \((4.31)\), \((4.32)\), it follows that $|z + v_2\eta)/(u_1 - u_2) + \xi| \leq 7\tilde{c}_2/5$. That is, \((z + v_2\eta)/(u_1 - u_2) + \xi, \eta) \in B_{2\tilde{c}_2}(0)$ if $\eta \in (-6\tilde{c}_2/5, 6\tilde{c}_2/5)$ and $|z| \leq \delta^*$. Thus, from \((4.37)\) and \((4.39)\), $\Psi \in C^\infty(\tilde{\Omega})$ with $\Vert \Psi \Vert_{C^k(\tilde{\Omega})}$ depending only on the data and $k \in \mathbb{N}$, where $\tilde{\Omega} = B_{3\delta}(0) \times (-\delta^*, \delta^*) \times B_{u_1/50}(0) \times (-6\tilde{c}_2/5, 6\tilde{c}_2/5)$.\]
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(4.31) holds and $\psi \in C^1(\Omega)$ satisfies (4.32), then $\psi = \varphi - \varphi_2$ satisfies (4.9) on $\Gamma_{\text{shock}}$ if and only if $\psi$ satisfies conditions (4.28) on $\Gamma_{\text{shock}}$.

(4.42) $\rho_2^2(c_2^2 - \xi^2)\psi_\xi + \left(\frac{\rho_2^2 - \rho_1^2}{u_1} - \rho_2^2 \xi^2\right)(\eta \psi_\eta - \psi) + E_1(D\psi, \psi, \eta) \cdot D\psi + E_2(D\psi, \psi, \eta)\psi = 0,$

and the functions $E_i(p, z, \eta), i = 1, 2,$ are smooth on $B_\delta \times (-\delta^*, \delta^*) \times (-6\epsilon_2/5, 6\epsilon_2/5)$ and satisfy that, for all $(p, z, \eta) \in B_\delta \times (-\delta^*, \delta^*) \times (-6\epsilon_2/5, 6\epsilon_2/5),$

(4.43) $|E_i(p, z, \eta)| \leq C \left(|p| + |z| + \sigma \right)$

and, for all $(p, \eta) \in B_\delta \times (-\delta^*, \delta^*) \times (-6\epsilon_2/5, 6\epsilon_2/5),$

(4.44) $|(D_{(p,z,\eta)} E_i)\cdot D_{(p,z,\eta)} E_i)| \leq C.$

where we have used (3.24) in the derivation of (4.43) and $C$ depends only on the data.

Denote by $\nu_0$ the unit normal on the reflected shock to the region of state (2). Then $\nu_0 = (\sin \theta_2, -\cos \theta_2)$ from the definition of $\theta_2$. We compute

(4.45) $\left(\rho_2^2(c_2^2 - \xi^2), \left(\frac{\rho_2^2 - \rho_1}{u_1} - \rho_2^2 \xi^2\right)\eta\right) \cdot \nu_0$

$= \rho_2^2(c_2^2 - \xi^2) \sin \theta_2 - \left(\frac{\rho_2^2 - \rho_1^2}{u_1} - \rho_2^2 \xi^2\right)\eta \cos \theta_2 \geq \frac{1}{2}\rho_2^2(c_2^2 - \xi^2) > 0,$

if $\pi/2 - \theta_2$ is small and $\eta \in \text{Proj}_d(\Gamma_{\text{shock}})$. From (3.14) and (4.30), we obtain

$\|\nu_0 - \nu_0\|_{L^\infty(\Gamma_{\text{shock}})} \leq C \|D\psi\|_{C(\overline{\Omega})}$. Thus, if $\sigma > 0$ and $\|D\psi\|_{C(\overline{\Omega})}$ are small depending only on the data, then (4.42) is an oblique derivative condition on $\Gamma_{\text{shock}}$.

4.3. The equation and boundary conditions near the sonic circle. For the shock reflection solution, (1.8) is expected to be elliptic in the domain $\Omega$ and degenerate on the sonic circle of state (2) which is the curve $\Gamma_{\text{sonic}} = \partial \Omega \cap \partial B_{\epsilon_2}(0)$.

Thus we consider the subdomains:

(4.46) $\Omega' := \Omega \cap \{\xi, \eta : \text{dist}((\xi, \eta), \Gamma_{\text{sonic}}) < 2\epsilon\},$

$\Omega'' := \Omega \cap \{\xi, \eta : \text{dist}((\xi, \eta), \Gamma_{\text{sonic}}) > \epsilon\},$

(the small constant $\epsilon > 0$ will be chosen later). Obviously, $\Omega'$ and $\Omega''$ are open subsets of $\Omega$, and $\Omega = \Omega' \cup \Omega''$. (1.8) is expected to be degenerate elliptic in $\Omega'$ and uniformly elliptic in $\Omega''$ on the solution of the shock reflection problem.

In order to display the structure of the equation near the sonic circle where the ellipticity degenerates, we introduce the new coordinates in $\Omega'$ which flatten

\[\text{and}\]

\[\text{parentheses}\]
and rewrite (1.8) in these new coordinates. Specifically, denoting \((r, \theta)\) the polar coordinates in the \((\xi, \eta)\)-plane, i.e., \((\xi, \eta) = (r \cos \theta, r \sin \theta)\), we consider the coordinates:

\[
(4.47) \quad x = c_2 - r, \quad y = \theta - \theta_w \quad \text{on } \Omega'.
\]

By Section 3.2, the domain \(\Omega'\) does not contain the point \((\xi, \eta) = (0, 0)\) if \(\varepsilon\) is small. Thus, the change of coordinates \((\xi, \eta) \rightarrow (x, y)\) is smooth and smoothly invertible on \(\Omega'\). Moreover, it follows from the geometry of domain \(\Omega\) especially from (4.2)(4.7) that, if \(\sigma > 0\) is small, then, in the \((x, y)\)-coordinates,

\[
\Omega' = \{(x, y) : 0 < x < 2\varepsilon, \quad 0 < y < \pi + \arctan\left(\eta(x)/f(\eta(x))\right) - \theta_w\},
\]

where \(\eta(x)\) is the unique solution, close to \(\eta_1\), of the equation \(\eta^2 + f(\eta)^2 = (c_2 - x)^2\).

We write the equation for \(\psi\) in the \((x, y)\)-coordinates. As discussed in Section 4.2, \(\psi\) satisfies (4.22)(4.23) in the polar coordinates. Thus, in the \((x, y)\)-coordinates in \(\Omega'\), the equation for \(\psi\) is

\[
(2x - (y + 1))\psi_x + O_1 \psi_{xx} + O_2 \psi_{xy} + \left(\frac{1}{c_2} + O_3\right) \psi_{yy} - (1 + O_4) \psi_x + O_5 \psi_y = 0,
\]

where

\[
O_1(D\psi, \psi, x) = -\frac{x^2}{c_2} + \frac{y + 1}{2c_2} (2x - \psi_x) \psi_x - \frac{y - 1}{c_2} \left(\psi + \frac{1}{2(c_2 - x)^2} \psi_y^2\right),
\]

\[
O_2(D\psi, \psi, x) = -\frac{2}{c_2(c_2 - x)^2} (\psi_x + c_2 - x)^2 \psi_y,
\]

\[
O_3(D\psi, \psi, x) = \frac{1}{c_2(c_2 - x)^2} \left(x(2c_2 - x) - (y - 1) \left(\psi + (c_2 - x) \psi_x + \frac{1}{2} \psi_x^2\right)ight.
\]

\[
- \frac{y + 1}{2(c_2 - x)^2} \psi_y^2\right),
\]

\[
O_4(D\psi, \psi, x) = \frac{1}{c_2 - x} \left(\frac{x - y - 1}{c_2} \left(\psi + (c_2 - x) \psi_x + \frac{1}{2} \psi_x^2\right)
\]

\[
+ \frac{y + 1}{2(y - 1)(c_2 - x)^2} \psi_y^2\right),
\]

\[
O_5(D\psi, \psi, x) = -\frac{2}{c_2(c_2 - x)^3} \left(\psi_x + \frac{1}{2c_2 - 3x} \psi_y\right).
\]
The terms $O_k(D\psi, \psi, x)$ are small perturbations of the leading terms of (4.48) if the function $\psi$ is small in an appropriate norm considered below. In order to see this, we note the following properties: For any $(p, z, x) \in \mathbb{R}^2 \times \mathbb{R} \times (0, c_2/2)$ with $|p| < 1$,

\begin{align*}
|O_1(p, z, x)| &\leq C(|p|^2 + |z| + |x|^2), \\
|O_3(p, z, x)| + |O_4(p, z, x)| &\leq C(|p| + |z| + |x|), \\
|O_5(p, z, x)| + |O_6(p, z, x)| &\leq C(|p| + |z| + 1)|p|.
\end{align*}

(4.50)

In particular, dropping the terms $O_k$, $k = 1, \ldots, 5$, from (4.48), we obtain the transonic small disturbance equation (cf. [44]):

\begin{equation}
(2x - (y + 1)\psi_x)\psi_{xx} + \frac{1}{c_2} \psi_{yy} - \psi_x = 0.
\end{equation}

(4.51)

Now we write the boundary conditions on $\Gamma_{\text{sonic}}$, $\Gamma_{\text{shock}}$, and $\Gamma_{\text{wedge}}$ in the $(x, y)$-coordinates. Conditions (4.24) and (4.25) become

\begin{align*}
\psi &= 0 \quad \text{on } \Gamma_{\text{sonic}} = \partial\Omega \cap \{x = 0\}, \\
\psi_x &= 0 \quad \text{on } \Gamma_{\text{wedge}} = \partial\Omega \cap \{y = 0\}.
\end{align*}

(4.52) (4.53)

It remains to write condition (4.42) on $\Gamma_{\text{shock}}$ in the $(x, y)$-coordinates. Expressing $\psi_x$ and $\psi_y$ in the polar coordinates $(r, \theta)$ and using (4.47), we write (4.42) on $\Gamma_{\text{shock}} \cap \{x < 2\varepsilon\}$ in the form:

\begin{equation}
\left(-\rho_2'(c_2^2 - \xi^2)\cos(y + \theta_w) - \frac{\rho_2 - \rho_1}{u_1} - \rho_2' \xi \right) (c_2 - x) \sin^2(y + \theta_w) \right) \psi_x \\
+ \sin(y + \theta_w) \left( -\rho_2' \xi \xi^2 + \frac{\rho_2 - \rho_1}{u_1} \cos(y + \theta_w) \right) \psi_y \\
- \theta_w \frac{\rho_2 - \rho_1}{u_1} - \rho_2' \xi \psi + \tilde{E}_1(D_{(x,y)}\psi, \psi, x, y) + \tilde{E}_2(D_{(x,y)}\psi, \psi, x, y) \psi = 0,
\end{equation}

(4.54)

where $\tilde{E}_i(p, z, x, y), i = 1, 2,$ are smooth functions of $(p, z, x, y) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ satisfying

\begin{equation}
|\tilde{E}_i(p, z, x, y)| \leq C(|p| + |z| + \xi) \quad \text{for } |p| + |z| + \xi \leq \varepsilon_0(u_1, \varepsilon_2).
\end{equation}

(4.55)

We now rewrite (4.54) noting first that, in the $(\xi, \eta)$-coordinates, the point $P_1 = \Gamma_{\text{sonic}} \cap \Gamma_{\text{shock}}$ has the coordinates $(\xi_1, \eta_1)$ defined by (4.6). Using (3.20), (3.22), (4.3), and (4.6), we find

\begin{equation}
0 \leq |\tilde{\xi}| - |\xi_1| \leq C\sigma.
\end{equation}

(4.56)
In the \((x, y)\)-coordinates, the point \(P_1\) is \((0, y_1)\), where \(y_1\) satisfies
\[
\frac{1}{2} c_2 \cos(y_1 + \theta_w) = \xi_1, \quad \frac{1}{2} c_2 \sin(y_1 + \theta_w) = \eta_1.
\]
from (4.6) and (4.47). Using this and noting that the leading terms of the coefficients of (4.54) near \(P_1 = (0, y_1)\) are the coefficients at \((x, y) = (0, y_1)\), we rewrite (4.54) as follows:

\[
\begin{aligned}
\frac{\rho_2 - \rho_1}{u_1 c_2} \eta_1^2 \psi_x - \left( \rho_2' - \rho_1' \sqrt{1 - \xi_1^2} \right) \eta_1 \psi_y \\
- \left( \rho_2' - \rho_1' \right) \psi + \hat{E}_1(D(x, y) \psi, \psi, x, y) \hat{E}_2(D(x, y) \psi, \psi, x, y) \psi = 0 \text{ on } \Gamma_{\text{shock}} \cap \{x < 2 \varepsilon\},
\end{aligned}
\]
where the terms \(\hat{E}_i(p, z, x, y)\), \(i = 1, 2\), satisfy
\[
|\hat{E}_i(p, z, x, y)| \leq C \left( |p| + |z| + x +|y - y_1| + \sigma \right)
\]
for \((p, z, x, y) \in \mathcal{F} := \{(p, z, x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : |p| + |z| \leq \varepsilon_0(u_1, \rho_2')\} \) and
\[
\left\| (D_{(p, z, x, y)} \hat{E}_i, D^2_{(p, z, x, y)} \hat{E}_i) \right\|_{L^\infty(\mathcal{F})} \leq C.
\]

We note that the left-hand side of (4.56) is obtained by expressing the left-hand side of (4.42) on \(\Gamma_{\text{shock}} \cap \{c_2 - r < 2 \varepsilon\}\) in the \((x, y)\)-coordinates. Assume \(\varepsilon < \varepsilon_0 / 4\). In this case, transformation (4.47) is smooth on \(\{0 < c_2 - r < 2 \varepsilon\}\) and has nonzero Jacobian. Thus, condition (4.56) is equivalent to (4.42) and hence to (4.29) on \(\Gamma_{\text{shock}} \cap \{x < 2 \varepsilon\}\) if \(\sigma > 0\) is small so that (4.31) holds, and if \(\|\psi\|_{C^1(\overline{\Omega})}\) is small depending only on the data such that (4.32) is satisfied.

5. Iteration scheme

In this section, we develop an iteration scheme to solve the free boundary problem and set up the detailed steps of the iteration procedure in the shifted coordinates.

5.1. Iteration domains. Fix \(\theta_w < \pi/2\) close to \(\pi/2\). Since our problem is a free boundary problem, the elliptic domain \(\Omega\) of the solution is apriori unknown and thus we perform the iteration in a larger domain
\[
\mathcal{D} \equiv \mathcal{D}_{\theta_w} := B_{c_2}(0) \cap \{\eta > -v_2\} \cap \{l(\eta) < \xi < \eta \cos \theta_w\},
\]
where \(l(\eta)\) is defined by (4.3). We will construct a solution with \(\Omega \subset \mathcal{D}\). Moreover, the reflected shock for this solution coincides with \(\{\xi = l(\eta)\}\) outside the sonic circle, which implies \(\overline{\partial \mathcal{D}} \cap \partial B_{c_2}(0) = \partial \Omega \cap \partial B_{c_2}(0) =: \Gamma_{\text{sonic}}\). Then we decompose
\[ \partial_x (\varphi_1 - \varphi_2) = u_1 \sin y, \] and \[ \partial_y (\varphi_1 - \varphi_2) = -u_1 (c_2 - x) \cos y, \] which imply (5.3).

Now, (5.4) is true since \[ \xi = -c_2 \sin (f_{0,0}(0)) \] and thus
\[ \varphi_1 - \varphi_2 = u_1 (c_2 \sin (f_{0,0}(0)) - (c_2 - x) \sin y), \]
and (5.5) follows from (5.3) since \( (\varphi_1 - \varphi_2)(\kappa_0, f_{0,0}(\kappa_0)) = 0 \) and
\[ \left( f_{0,0}(\kappa_0) + \pi/2 \right)/2 - f_{0,0}(\kappa_0) \geq C^{-1}. \]

Now let \( \theta_w < \pi/2 \). Then, from (3.14)-(4.16) and (4.47), we have
\[ \varphi_1 - \varphi_2 = - (c_2 - x) \sin (y + \theta_w - \theta_s) \sqrt{(u_1 - u_2)^2 + v_2^2 - (u_1 - u_2) \xi}. \]

By Section 3.2, when \( \theta_w \to \pi/2 \), we know that \( (u_2, v_2) \to (0, 0) \), \( \theta_s \to \pi/2 \), \( \xi \to \xi \), and thus, by (4.4), we also have \( \xi \to \xi \). This shows that, if \( \sigma > 0 \) is small depending only on the data, then, for all \( \theta_w \in (\pi/2 - \sigma_0, \pi/2) \), estimates (5.3)-(5.5) hold with \( C \) which is equal to twice the constant \( C \) from the respective estimates (5.3)-(5.5) for \( \theta_w = \pi/2 \).

From (5.3)-(5.5) for \( \theta_w \in (\pi/2 - \sigma_0, \pi/2) \) and since
\[ \mathcal{B} \cap \{ c_2 - r < \kappa_0 \} = \{ \varphi_1 > \varphi_2 \} \cap \left\{ 0 \leq x \leq \kappa_0, 0 \leq y \leq \frac{f_{0,0}(\kappa_0) + \pi/2}{2} \right\}, \]
there exists \( \hat{f}_0 := \hat{f}_{0,\pi/2-\sigma_0} \in C_c(\mathbb{R}^+) \) such that
\[ \mathcal{B} \cap \{ c_2 - r < \kappa_0 \} = \{ 0 \leq x \leq \kappa_0, 0 \leq y < \hat{f}_0(x) \}, \]
\[ \hat{f}_0(0) = \sqrt{p_1}, \ C^{-1} \leq \hat{f}_0(x) \leq C \text{ on } [0, \kappa_0], \]
\[ \hat{f}_{0,0}(0)/2 \leq \hat{f}_0(0) < \hat{f}_0(\kappa_0) \leq (\hat{f}_{0,0}(\kappa_0) + \pi/2)/2. \]
In fact, the line \( y = \hat{f}_0(x) \) is the line \( \xi = l(\eta) \) expressed in the \( (x, y) \)-coordinates, and thus we obtain explicitly with the use of (3.14) that
\[ \hat{f}_0(x) = \arcsin \left( \frac{\left| \xi \right| \sin \theta_s}{(c_2 - x)} \right) - \theta_w + \theta_s \text{ on } [0, \kappa_0]. \]

5.2. Hölder norms in \( \Omega \). For the elliptic estimates, we need the Hölder norms in \( \Omega \) weighted by the distance to the corners \( P_2 = \Gamma_{\text{shock}} \cap \{ \eta = -v_2 \} \) and \( P_3 = (-u_2, -v_2) \), and with a “parabolic” scaling near the sonic circle.

More generally, we consider a subdomain \( \Omega \subset \mathcal{B} \) of the form \( \Omega := \mathcal{B} \cap \{ \xi \geq f(\eta) \} \) with \( f \in C^1(\mathbb{R}) \) and set the subdomains \( \Omega' := \Omega \cap \mathcal{B}' \) and \( \Omega'' := \Omega \cap \mathcal{B}'' \) defined by (4.46). Let \( \Sigma \subset \partial \Omega'' \) be closed. We now introduce the Hölder norms in \( \Omega'' \) weighted by the distance to \( \Sigma \). Denote by \( X = (\xi, \eta) \) the points of \( \Omega'' \) and set
\[ \delta_X := \text{dist}(X, \Sigma), \ \delta_{X,Y} := \min(\delta_X, \delta_Y) \] for \( X, Y \in \Omega'' \).
Then, for $k \in \mathbb{R}$, $\alpha \in (0, 1)$, and $m \in \mathbb{N}$, define

\begin{equation}
\|u\|_{m, \alpha, \Omega^\nu}^{(k, \Sigma)} := \sum_{\beta \leq m} \sup_{X \in \Omega^\nu} \left( \delta_{X,Y}^{\max(|\beta| + k, \nu)} |D^\beta u(X)| \right),
\end{equation}

where $D^\beta = \delta_{X}^{\beta_1} \delta_{Y}^{\beta_2}$, and $\beta = (\beta_1, \beta_2)$ is a multi-index with $\beta_j \in \mathbb{N}$ and $|\beta| = \beta_1 + \beta_2$. We denote by $C_{m, \alpha, \Omega^\nu}$ the space of functions with finite norm $\|u\|_{m, \alpha, \Omega^\nu}^{(k, \Sigma)}$.

Remark 5.1. If $m \geq -k \geq 1$ and $k$ is an integer, then any function $u \in C_{m, \alpha, \Omega^\nu}^{[k]}$ is $C^{k+1, 1}$ up to $\Sigma$, but not necessarily $C^{k, 1}$ up to $\Sigma$.

In $\Omega^\nu$, the equation is degenerate elliptic, for which the Hölder norms with parabolic scaling are natural. We define the norm $\|u\|_{2, \alpha, \Omega^\nu}^{(\text{par})}$ as follows: Denoting $z = (x, y)$ and $\tilde{z} = (\tilde{x}, \tilde{y})$ with $x, \tilde{x} \in (0, 2\varepsilon)$

\begin{equation}
\delta_{\alpha}^{(\text{par})}(z, \tilde{z}) := (|x - \tilde{x}|^2 + \min(x, \tilde{x})|y - \tilde{y}|^2)^{\alpha/2},
\end{equation}

then, for $u \in C^2(\Omega^\nu) \cap C^{1, 1}(\overline{\Omega}^\nu)$ written in the $(x, y)$-coordinates (4.47), we define

\begin{equation}
\|u\|_{2, \alpha, \Omega^\nu}^{(\text{par})} := \sum_{0 \leq k + l \leq 2} \sup_{z \in \Omega^\nu} \left( x^{k+l/2-2} |\partial_x^k \partial_y^l u(z)| \right),
\end{equation}

\begin{equation}
[u]_{2, \alpha, \Omega^\nu}^{(\text{par})} := \sup_{k+l=2, z, \tilde{z} \in \Omega^\nu, z \neq \tilde{z}} \left( \min(x, \tilde{x})^{\alpha/2} \frac{|\partial_x^k \partial_y^l u(z) - \partial_x^k \partial_y^l u(\tilde{z})|}{\delta_{\alpha}^{(\text{par})}(z, \tilde{z})} \right),
\end{equation}

\begin{equation}
\|u\|_{2, \alpha, \Omega^\nu}^{(\text{par})} := \|u\|_{2, \alpha, \Omega^\nu}^{(\text{par})} + [u]_{2, \alpha, \Omega^\nu}^{(\text{par})}.
\end{equation}

To motivate this definition, especially the parabolic scaling, we consider a scaled version of the function $u(x, y)$ in the parabolic rectangles:

\begin{equation}
R(x, y) = \left\{(s, t) : |s-x| < \frac{x}{4}, |t-y| < \frac{\sqrt{x}}{4} \right\} \cap \Omega^\nu \quad \text{for} \quad z = (x, y) \in \Omega^\nu.
\end{equation}

Denote $Q_1 := (-1, 1)^2$. Then the rescaled rectangle (5.12) is

\begin{equation}
Q_1^{(t)} := \left\{(S, T) \in Q_1 : (x + \frac{x}{4} S, y + \frac{\sqrt{x}}{4} T) \in \Omega \right\}.
\end{equation}
Denote by \( u^{(2)}(S, T) \) the following function in \( Q^{(2)}_1 \):

\[
(5.14) \quad u^{(2)}(S, T) := \frac{1}{x^2} u \left( x + \frac{x}{4} S, y + \frac{\sqrt{x}}{4} T \right) \quad \text{for } (S, T) \in Q^{(2)}_1.
\]

Then we have

\[
(5.15) \quad C^{-1} \sup_{\bar{\Omega} \cap \{ \eta < a \}} \| u^{(2)} \|_{C^{1,\alpha} \left( Q^{(2)}_2 \right)} \leq \| u \|_{C^{1,\alpha} \left( Q^{(2)}_1 \right)} \leq C \sup_{\bar{\Omega} \cap \{ \eta < a \}} \| u^{(2)} \|_{C^{1,\alpha} \left( Q^{(2)}_1 \right)},
\]

where \( C \) depends only on the domain \( \Omega \) and is independent of \( \varepsilon \in (0, \kappa_0/2) \).

5.3. Iteration set. We consider the wedge angle close to \( \pi/2 \), that is, \( \sigma = \frac{\pi}{2} - \theta_w > 0 \) is small which will be chosen below. Set \( \Sigma_0 := \partial \Omega \cap \{ \eta = -v_2 \} \).

Let \( \varepsilon, \sigma > 0 \) be the constants from (5.2) and (3.1). Let \( M_1, M_2 \geq 1 \). We define

\[
(5.16) \quad \mathcal{K} := \{ \phi \in C^{1,\alpha} \left( \partial \Omega \right) \cap C^2 \left( \bar{\Omega} \right) : ||\phi||_{C^{1,\alpha} \left( \partial \Omega \right)} \leq M_1, ||\phi||_{C^{1,\alpha} \left( \partial \Omega \right)} \leq M_2 \sigma, \phi \geq 0 \text{ in } \bar{\Omega} \}
\]

for \( \alpha \in (0, 1/2) \). Then \( \mathcal{K} \) is convex. Also, \( \phi \in \mathcal{K} \) implies that

\[
||\phi||_{C^{1,\alpha} \left( \partial \Omega \right)} \leq M_1, \quad ||\phi||_{C^{1,\alpha} \left( \partial \Omega \right)} \leq M_2 \sigma,
\]

so that \( \mathcal{K} \) is a bounded subset in \( C^{1,\alpha} \left( \partial \Omega \right) \). Thus, \( \mathcal{K} \) is a compact and convex subset of \( C^{1,\alpha} \left( \partial \Omega \right) \).

We note that the choice of constants \( M_1, M_2 \geq 1 \) and \( \varepsilon, \sigma > 0 \) below will guarantee the following property:

\[
(5.17) \quad \sigma \max(M_1, M_2) + \varepsilon^{1/4} M_1 + \sigma M_2 / \varepsilon^2 \leq \tilde{C}^{-1}
\]

for some sufficiently large \( \tilde{C} > 1 \) depending only on the data. In particular, (5.16) implies that \( \sigma \leq \tilde{C}^{-1} \) since \( \max(M_1, M_2) \geq 1 \), which implies \( \pi/2 - \theta_w \leq \tilde{C}^{-1} \) from (3.1). Thus, if we choose \( \tilde{C} \) large depending only on the data, then (4.31) holds. Also, for \( \psi \in \mathcal{K} \), we have

\[
||D\psi, \psi||_{3} \leq M_1 x^2 + M_1 x \quad \text{ in } \Omega', \quad ||\psi||_{C^{1,\alpha} \left( \partial \Omega \right)} \leq M_2 \sigma.
\]

Furthermore, \( 0 < x < 2\varepsilon \) in \( \mathcal{K} \) by (4.47) and (5.2). Now it follows from (5.16) that \( ||\psi||_{C^{1,\alpha} \left( \partial \Omega \right)} \leq 2 / \tilde{C} \). Then (4.32) holds if \( \tilde{C} \) is large depending only on the data.

Thus, in the rest of this paper, we always assume that (4.31) holds and that \( \psi \in \mathcal{K} \) implies (4.32). Therefore, (4.29) is equivalent to (4.43) and (4.44) for \( \psi \in \mathcal{K} \).

We also note the following fact.

**LEMMA 5.1.** There exist \( \tilde{C} \) and \( C \) depending only on the data such that, if \( \sigma, \varepsilon > 0 \) and \( M_1, M_2 \geq 1 \) in (5.15) satisfy (5.16), then, for every \( \phi \in \mathcal{K} \),

\[
(5.17) \quad ||\phi||_{C^{1,\alpha} \left( \partial \Omega \right)} \leq C (M_1 \varepsilon^{1-\alpha} + M_2 \sigma).
\]
Proof. In this proof, \( C \) denotes a universal constant depending only on the data. We use definitions (5.10)–(5.11) for the norms. We first show that

\[
\|\phi\|_{2,\sigma,\overline{\Omega}^c}^{(1-\alpha,\text{sonic})} \leq C M_1 e^{1-\alpha},
\]

where \( \delta(x,y) := \text{dist}((x,y), \Gamma_{\text{sonic}}) \) in (5.10). First we show (5.18) in the \((x,y)\)-coordinates. Using (5.6), we have \( \Omega' = \{ 0 < x < 2e, 0 < y < f_0(x) \} \) with \( \Gamma_{\text{sonic}} = \{ x = 0, 0 < y < f_0(x) \} \), where \( f_0 \in L^\infty((0,2e)) \) depends only on the data, and thus \( \text{dist}((x,y), \Gamma_{\text{sonic}}) \leq C x \) in \( \Omega' \). Then, since \( \|\phi\|_{2,\sigma,\Omega'}^{(\text{par})} \leq M_1 \), we obtain that, for \((x,y) \in \Omega' \),

\[
|\phi(x,y)| \leq M_1 x^2 \leq M_1 e^2, \quad |D\phi(x,y)| \leq M_1 x \leq M_1 e,
\]

\[
\delta^{1-\alpha}(x,y) |D^2\phi(x,y)| = x^{1-\alpha} |D^2\phi(x,y)| \leq e^{1-\alpha} M_1.
\]

Furthermore, from (5.16) with \( \hat{C} \geq 16 \), we obtain \( \epsilon \leq 1/2 \). Thus, denoting \( z = (x,y) \) and \( \hat{z} = (\hat{x}, \hat{y}) \) with \( x, \hat{x} \in (0,2e) \), we have

\[
\delta(\text{par})(z, \hat{z}) := \left( |x - \hat{x}|^2 + \text{min}(x, \hat{x}) |y - \hat{y}|^2 \right)^{\alpha/2}
\]

\[
\leq \left( |x - \hat{x}|^2 + 2\epsilon |y - \hat{y}|^2 \right)^{\alpha/2} \leq |z - \hat{z}|^\alpha,
\]

and \( \min(\delta_z, \delta_{\hat{z}}) = \min(x, \hat{x}) \), which implies

\[
\min(\delta_z, \delta_{\hat{z}}) \frac{|D^2\phi(z) - D^2\phi(\hat{z})|}{|z - \hat{z}|^{\alpha}} \leq C \epsilon^{1-\alpha} \min(x, \hat{x})^\alpha \frac{|D^2\phi(z) - D^2\phi(\hat{z})|}{\delta(\text{par})(z, \hat{z})}
\]

\[
\leq C \epsilon^{1-\alpha} M_1.
\]

Thus we have proved (5.18) in the \((x,y)\)-coordinates. By (4.31) and (5.16), we have \( \epsilon \leq c_2/50 \) if \( \hat{C} \) is large depending only on the data. Then the change \((\xi, \eta) \rightarrow (x,y) \) in \( \Omega' \) and its inverse have bounded \( C^3 \)-norms in terms of the data. Thus, (5.18) holds in the \((\xi, \eta)\)-coordinates.

Since \( \phi \in \mathcal{K} \), then \( \|\phi\|_{2,\sigma,\Omega' \cap \Sigma_0}^{(1-\alpha,\text{sonic})} \leq M_2 \sigma \). Thus, in order to complete the proof of Lemma 5.1, it suffices to estimate \( \min(\delta_z, \delta_{\hat{z}}) \frac{|D^2\phi(z) - D^2\phi(\hat{z})|}{|z - \hat{z}|^{\alpha}} \) in the case \( z \in \Omega' \setminus \Omega'' \) and \( \hat{z} \in \Omega'' \setminus \Omega' \) for \( \delta_z = \min(z, \Gamma_{\text{sonic}} \cup \Sigma_0) \). From \( z \in \Omega' \setminus \Omega'' \) and \( \hat{z} \in \Omega'' \setminus \Omega' \), we obtain \( 0 < c_2 - |z| < \epsilon/2 \) and \( c_2 - |\hat{z}| \geq 2\epsilon \), which implies that \( |z - \hat{z}| \geq 3\epsilon/2 \). We have \( c_2 - |z| \leq \text{dist}(z, \Gamma_{\text{sonic}}) \leq C(c_2 - |z|), \) where we have used (4.31) and (5.1). Thus, \( \min(\delta_z, \delta_{\hat{z}}) \leq C(c_2 - |z|) \leq C \epsilon \). Also we have \( |D^2\phi(z)| \leq M_1 \) by (5.11). If \( \delta_z \geq \delta_{\hat{z}} \), then \( \delta_z \geq \epsilon/2 \) and thus \( |D^2\phi(\hat{z})| \leq (\epsilon/2)^{-1+\alpha} M_2 \sigma \) by (5.10). Then we have

\[
\frac{\min(\delta_z, \delta_{\hat{z}}) |D^2\phi(z) - D^2\phi(\hat{z})|}{|z - \hat{z}|^{\alpha}} \leq C \epsilon \frac{M_1 + (2\epsilon)^{-1+\alpha} M_2 \sigma}{(3\epsilon/2)^{\alpha}} \leq C \left( \epsilon^{1-\alpha} M_1 + M_2 \sigma \right).
\]
If \( \delta_2 \leq \delta_1 \), then dist(\( \xi \), \( \Sigma_0 \)) \( \leq \) dist(\( \xi \), \( \Gamma_{\text{sonic}} \)), which implies by (4.8) that |\( z - \bar{z} \)| \( \geq \frac{1}{C} \varepsilon \) if \( \varepsilon \) is sufficiently small, depending only on the data. Then |\( D^2 \phi(\xi) \)| \( \leq \frac{\min(\delta_1, \delta_2)}{\delta_1^{\frac{1}{1+\alpha}} M_2 \sigma} \) and

\[
\min(\delta_1, \delta_2) \left| \frac{D^2 \phi(z) - D^2 \phi(\xi)}{|z - \xi|^{1+\alpha}} \right| \leq C \left( \delta_1 M_1 + \delta_2 \delta_2^{-1+\alpha} M_2 \sigma \right) \leq C (\varepsilon M_1 + M_2 \sigma).
\]

\[\square\]

5.4. Construction of the iteration scheme and choice of \( \alpha \). In this section, for simplicity of notation, the universal constant \( C \) depends only on the data and may differ at each occurrence.

By (3.24), it follows that, if \( \sigma \) is sufficiently small depending on the data, then

\[
q_2 \leq u_1 / 10,
\]

where \( q_2 = \sqrt{u_2^2 + v_2^2} \). Let \( \phi \in \mathcal{K} \). From (4.15), (4.16) and (5.19), it follows that

\[
(\phi_1 - \phi_2 - \phi)(\xi, \eta) \geq \frac{u_1}{2} > 0 \quad \text{in} \quad \mathcal{D}.
\]

Since \( \phi_1 - \phi_2 = 0 \) on \( \{\xi = l(\eta)\} \) and \( \phi \geq 0 \) in \( \mathcal{D} \), we have \( \phi \geq \phi_1 - \phi_2 \) on \( \{\xi = l(\eta)\} \cap \partial \mathcal{D} \), where \( l(\eta) \) is defined by (4.3). Then there exists \( f_\phi \in C^{1,\alpha}(\mathcal{R}) \) such that

\[
(5.21) \quad \{\phi = \phi_1 - \phi_2\} \cap \mathcal{D} = \{(f_\phi(\eta), \eta) : \eta \in (-v_2, 2)\}.
\]

It follows that \( f_\phi(\eta) \geq l(\eta) \) for all \( \eta \in (-v_2, 2) \) and

\[
\Omega^+(\phi) := \{\xi > f_\phi(\eta)\} \cap \mathcal{D} = \{\phi < \phi_1 - \phi_2\} \cap \mathcal{D}.
\]

Moreover, \( \partial\Omega^+(\phi) = \Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}} \cup \Gamma_{\text{wedge}} \cup \Sigma_0 \), where

\[
(5.23) \quad \Gamma_{\text{shock}}(\phi) := \{\xi = f_\phi(\eta)\} \cap \partial\mathcal{D} = \{\phi = \phi_1 - \phi_2\} \cap \partial\mathcal{D},
\]

\[
\Gamma_{\text{sonic}} := \partial\mathcal{D} \cap \partial B_{v_2}(0),
\]

\[
\Gamma_{\text{wedge}} := \partial\mathcal{D} \cap \{\eta = \xi \tan \theta_w\},
\]

\[
\Sigma_0 := \partial\Omega^+(\phi) \cap \{\eta = -v_2\},
\]

where \( (\xi_1, \eta_1) \) is determined by (4.6).

We denote by \( P_j, 1 \leq j \leq 4 \), the corner points of \( \Omega^+(\phi) \). Specifically, \( P_2 = \Gamma_{\text{shock}}(\phi) \cap \Sigma_0(\phi) \) and \( P_3 = (-v_2, -v_2) \) are the corners on the symmetry line \( \{\eta = -v_2\} \), and \( P_1 = \Gamma_{\text{sonic}} \cap \Gamma_{\text{shock}}(\phi) \) and \( P_4 = \Gamma_{\text{sonic}} \cap \Gamma_{\text{wedge}} \) are the corners on the sonic circle. Note that, since \( \phi \in \mathcal{K} \) implies \( \phi = 0 \) on \( \Gamma_{\text{sonic}} \), it follows that \( P_1 \) is the intersection point \( (\xi_1, \eta_1) \) of the line \( \xi = l(\eta) \) and the sonic circle \( \xi^2 + \eta^2 = v_2^2 \), where \( (\xi_1, \eta_1) \) is determined by (4.6).

We also note that \( f_0 \leq l \) for \( 0 \in \mathcal{K} \). From \( \phi \in \mathcal{K} \) and Lemma 5.1 with \( \alpha \in (0, 1/2) \), we obtain the following estimate of \( f_\phi \) on the interval \( (-v_2, \eta_1) \):

\[
(5.24) \quad \|f_\phi - l\|_{2,\alpha((-v_2, \eta_1))} \leq C \left( M_1 e^{1/2} + M_2 \sigma \right) \leq e^{1/4},
\]

where the second inequality in (5.24) follows from (5.16) with sufficiently large \( \tilde{C} \).
We also work in the $(x, y)$-coordinates. Denote $\kappa := \kappa_0/2$. Choosing $\hat{C}$ in 
(5.16) large depending only on the data, we conclude from (5.3)–(5.5) that, for 
every $\phi \in \mathcal{H}$, there exists a function $\hat{f} \equiv \hat{f}_\phi \in C_{2,\alpha, \phi}(0, \kappa)$ such that 
(5.25) \(\Omega^+(\phi) \cap \{c_2 - r < \kappa\} = \{0 < x < \kappa, \ 0 < y < \hat{f}_\phi(x)\}\),
with 
(5.26) \(\hat{f}_\phi(0) = \hat{f}_0(0) > 0, \ \hat{f}_\phi(\kappa) > 0 \text{ on } (0, \kappa), \ \|\hat{f}_\phi - \hat{f}_0\|_{2,\alpha, \phi}(0, \kappa) \leq C (M_1 \varepsilon^{1-\alpha} + M_2 \sigma),\)
where we have used Lemma 5.1. More precisely,
(5.27) \[
\sum_{k=0}^2 \sup_{x \in (0, 2\varepsilon)} (x^{k-2} |D^k (\hat{f}_\phi - \hat{f}_0)(x)|) 
+ \sup_{x_1 \neq x_2 \in (0, 2\varepsilon)} \left( (\min(x_1, x_2))^\alpha \frac{|(\hat{f}''_\phi - \hat{f}''_0)(x_1) - (\hat{f}''_\phi - \hat{f}''_0)(x_2)|}{|x_1 - x_2|^\alpha} \right) \leq CM_1,
\]
with \(\|\hat{f}_\phi - \hat{f}_0\|_{2,\alpha, \phi}(\varepsilon/2, \kappa) \leq CM_2 \sigma\).
Note that, in the $(\xi, \eta)$-coordinates, the angles $\theta_{P_2}$ and $\theta_{P_3}$ at the corners $P_2$ and $P_3$ of $\Omega^+(\phi)$ respectively satisfy 
(5.28) \[|\theta_{P_i} - \frac{\pi}{2}| \leq \frac{\pi}{16} \quad \text{for } i = 2, 3.\]
Indeed, $\theta_{P_3} = \pi/2 - \theta_0$. The estimate for $\theta_{P_2}$ follows from (5.24) with (5.16) for 
large $\hat{C}$.

We now consider the problem in the domain $\Omega^+(\phi)$:
(5.29) \[N(\psi) := A_{11} \xi \eta + 2A_{12} \xi \eta + A_{12} \xi \eta = 0 \quad \text{in } \Omega^+(\phi).\]
(5.30) \[M(\psi) := \rho_2(c_2^2 - \hat{\xi}^2) \xi \psi + \left( \frac{\rho_2 - \rho_1}{u_1} \right) \eta \psi \eta - \psi \]
\[+ E^0_1(\xi, \eta) \cdot D \psi + E^0_2(\xi, \eta) \psi = 0 \quad \text{on } \Gamma_{\text{shock}}(\phi).\]

(5.31) \[\psi = 0 \quad \text{on } \Gamma_{\text{sonic}}.\]
(5.32) \[\psi_v = 0 \quad \text{on } \Gamma_{\text{wedge}}.\]
(5.33) \[\psi_\eta = -v_2 \quad \text{on } \partial \Omega^+(\phi) \cap \{\eta = -v_2\},\]
where $A_{ij} = A_{ij}(D \psi, \xi, \eta)$ (which will be defined below), and (5.30) is obtained 
from (4.42) by substituting $\phi$ into $E_i, i = 1, 2$, i.e.,
(5.34) \[E^0_1(\xi, \eta) = E_i(D \phi(\xi, \eta), \phi(\xi, \eta), \eta).\]
Note that, for $\phi \in \mathcal{C}$ and $(\xi, \eta) \in \mathcal{D}$, we have $(D\phi(\xi, \eta), \phi(\xi, \eta), \eta) \in B_{\delta_*}(0) \times (-\delta^*, \delta^*) \times (-6\delta^2/5, 6\delta^2/5)$ by (4.31)-(4.32). Thus, the right-hand side of (5.34) is well-defined.

Also, we now fix $\alpha$ in the definition of $\mathcal{C}$. Note that the angles $\theta_{P_2}$ and $\theta_{P_3}$ at the corners $P_2$ and $P_3$ of $\Omega^+(\phi)$ satisfy (5.28). Near these corners, (5.29) is linear and its ellipticity constants near the corners are uniformly bounded in terms of the data. Moreover, the directions in the oblique derivative conditions on the arcs meeting at the corner $P_3$ (resp. $P_2$) are at the angles within the range $(7\pi/16, 9\pi/16)$, since (5.30) can be written in the form $\psi_\eta + e\psi_\eta - d\psi = 0$, where $|e| \leq C\sigma$ near $P_2$ from $e(P_2) = -v_2$, (3.24), (4.43), (4.44), and (5.16). Then, by [35], there exists $\alpha_0 \in (0, 1)$ such that, for any $\alpha \in (0, \alpha_0)$, the solution of (5.29)-(5.33) is in $C^{1, \alpha}$ near and up to $P_2$ and $P_3$ if the arcs are in $C^{1, \alpha}$ and the coefficients of the equation and the boundary conditions are in the appropriate Hölder spaces with exponent $\alpha$. We used $\alpha = \alpha_0/2$ in the definition of $\mathcal{C}$ for $\alpha_0 = \alpha_0(\pi/16, 1/2)$, where $\alpha_0(\theta_0, \epsilon)$ is defined as in [35, Lemma 1.3]. Note that $\alpha \in (0, 1/2)$ since $\alpha_0 \in (0, 1)$.

5.5. An elliptic cutoff and the equation for the iteration. In this subsection, we fix $\phi \in \mathcal{C}$ and define (5.29) such that

(i) It is strictly elliptic inside the domain $\Omega^+(\phi)$ with elliptic degeneracy at the sonic circle $\Gamma_{\text{sonic}} = \partial \Omega^+(\phi) \cap \partial B_{\delta_*}(0)$;

(ii) For a fixed point $\psi = \phi$ satisfying an appropriate smallness condition of $|D\phi|$, (5.29) coincides with the original (4.19).

We define the coefficients $A_{ij}$ of (5.29) in the larger domain $\mathcal{D}$. More precisely, we define the coefficients separately in the domains $\mathcal{D}'$ and $\mathcal{D}''$ and then combine them.

In $\mathcal{D}''$, we define the coefficients of (5.29) by substituting $\phi$ into the coefficients of (4.19), i.e.,

(5.35) $A_{11}^1(\xi, \eta) = c_2^2(D\phi, \phi, \xi, \eta) - (\phi_\xi - \xi)^2$,

$A_{22}^1(\xi, \eta) = c_2^2(D\phi, \phi, \xi, \eta) - (\phi_\eta - \eta)^2$,

$A_{12}^1(\xi, \eta) = A_{21}^1(\xi, \eta) = -(\phi_\xi - \xi)(\phi_\eta - \eta)$,

where $\phi$, $\phi_\xi$, and $\phi_\eta$ are evaluated at $(\xi, \eta)$. Thus, (5.29) in $\Omega^+(\phi) \cap \mathcal{D}''$ is a linear equation

(5.37) $A_{11}^1 \psi_{\xi \xi} + 2A_{12}^1 \psi_{\xi \eta} + A_{22}^1 \psi_{\eta \eta} = 0$ in $\Omega^+(\phi) \cap \mathcal{D}''$.

From the definition of $\mathcal{D}''$, it follows that $\sqrt{\xi^2 + \eta^2} \leq c_2 - \epsilon$ in $\mathcal{D}''$. Then calculating explicitly the eigenvalues of matrix $(A_{ij})_{1 \leq i, j \leq 2}$ defined by (5.35) and using (4.31)
Note that, in the polar coordinates, $I_1, \ldots, I_4$ have the following expressions:

\[ I_1 = \left( c_2^2 - r^2 + (\gamma - 1) \left( r \psi_r - \frac{1}{2} |D\psi|^2 - \psi \right) \right) \Delta \psi. \]

\[ I_2 = \psi_{\theta\theta} + r \psi_r. \]

\[ I_3 = r (|D\psi|^2)_r = 2 r \psi_r \psi_{rr} + \frac{2}{r} \psi_\theta \psi_{r\theta} - \frac{2}{r^2} \psi_\theta^2. \]

\[ I_4 = -\frac{1}{2} \left( \psi_r (|D\psi|^2)_r + \frac{1}{r} \psi_\theta (|D\psi|^2)_\theta \right) \]

with $|D\psi|^2 = \psi_r^2 + \frac{1}{r^2} \psi_\theta^2$ and $\Delta \psi = \psi_{rr} + \frac{1}{r^2} \psi_{r\theta}^2 = \frac{1}{r} \psi_r$.

From this, by (4.47), we see that the dominating terms of (4.48) come only from $I_1, I_2$, and the term $2r \psi_r \psi_{rr}$ of $I_3$, i.e., the remaining terms of $I_3$ and $I_4$ affect only the terms $O_1, \ldots, O_5$ in (4.48). Moreover, the term $(\gamma + 1) \psi_{xx}$ in the coefficient of $\psi_{xx}$ in (4.48) is obtained as the leading term in the sum of the coefficient $(\gamma - 1) r \psi_r$ of $\psi_{rr}$ in $I_1$ and the coefficient $2 r \psi_r$ of $\psi_{rr}$ in $I_3$. Thus we modify the terms $I_1$ and $I_3$ by cutting off the $\psi_r$-component of first derivatives in the coefficients of second-order terms as follows. Let $\xi_1 \in C^\infty(\mathbb{R})$ satisfy

\[ \xi_1(s) = \begin{cases} s, & \text{if } |s| < 4/(3(\gamma + 1)), \\ 5 \text{sign}(s)/3(\gamma + 1)^2, & \text{if } |s| > 2/(\gamma + 1), \end{cases} \]

so that

\[ \xi_1(s) \geq 0, \quad \xi_1(-s) = -\xi_1(s) \text{ on } \mathbb{R}; \]

\[ \xi_1''(s) \leq 0 \text{ on } |s| > 0. \]

Obviously, such a smooth function $\xi_1 \in C^\infty(\mathbb{R})$ exists. Property (5.39) will be used only in Proposition 8.1. Now we note that $\psi_\xi = \frac{\xi}{r} \psi_r - \frac{\eta}{r^2} \psi_\theta$ and $\psi_\eta = \frac{\eta}{r} \psi_r + \frac{\xi}{r^2} \psi_\theta$, and define

\[ \hat{I}_1 := \left( \psi_\xi^2 - r^2 + (\gamma - 1)r(c_2 - r) \xi_1 \left( \frac{\xi \psi_r + \eta \psi_\theta}{r(c_2 - r)} \right) - (\gamma - 1) \left( \frac{1}{2} |D\psi|^2 + \psi \right) \right) \Delta \psi, \]

\[ \hat{I}_3 := 2 \left( \frac{\xi}{r} (c_2 - r) \xi_1 \left( \frac{\xi \psi_r + \eta \psi_\theta}{r(c_2 - r)} \right) - \frac{\eta}{r^2} (\xi \psi_\theta - \eta \psi_\xi) \right) \left( \xi \psi_\xi + \eta \psi_\eta \right) \]

\[ + 2 \left( \frac{\eta}{r} (c_2 - r) \xi_1 \left( \frac{\xi \psi_r + \eta \psi_\theta}{r(c_2 - r)} \right) + \frac{\xi}{r^2} (\xi \psi_\theta - \eta \psi_\xi) \right) \left( \xi \psi_\xi + \eta \psi_\eta \right). \]

The modified equation in the domain $\Omega_{4g}$ is

\[ \hat{I}_1 + \hat{I}_2 + \hat{I}_3 + \hat{I}_4 = 0. \]

By (5.37), the modified (5.40) coincides with the original (4.19) if

\[ \left| \frac{\xi}{r} \psi_\xi + \frac{\eta}{r} \psi_\eta \right| < \frac{4(c_2 - r)}{3(\gamma + 1)}. \]
i.e., if \(|\psi_x| < 4x/(3(\gamma + 1))\) in the \((x, y)\)-coordinates. Also, (5.40) is of form (5.29) in the \((\xi, \eta)\)-coordinates.

Now we define (5.29) in \(\mathcal{A}_{4e}\) by substituting \(\phi\) into the coefficients of (5.40) except for the terms involving \(\xi_1 (\frac{\xi \psi_x + \eta \psi_y}{r(c_2 - r)})\). Thus, we obtain an equation of form (5.29) with the coefficients:

\begin{align*}
A_{11}^2 (D\psi, \xi, \eta) &= c_2^2 - (\gamma - 1) \left( r(c_2 - r) \xi_1 \left( \frac{\xi \psi_x + \eta \psi_y}{r(c_2 - r)} \right) + \frac{1}{2} D\phi^2 + \phi \right) \\
&\quad + \left( \psi_x^2 + \eta^2 \right) + 2\xi \left( \frac{\xi}{r(c_2 - r)} \xi_1 \left( \frac{\xi \psi_x + \eta \psi_y}{r(c_2 - r)} \right) - \frac{\eta}{r_2} (\xi \phi_\eta - \eta \phi_\xi) \right). \\
A_{22}^2 (D\psi, \xi, \eta) &= c_2^2 - (\gamma - 1) \left( r(c_2 - r) \xi_1 \left( \frac{\xi \psi_x + \eta \psi_y}{r(c_2 - r)} \right) + \frac{1}{2} D\phi^2 + \phi \right) \\
&\quad + \left( \psi_y^2 + \eta^2 \right) + 2\eta \left( \frac{\eta}{r(c_2 - r)} \xi_1 \left( \frac{\xi \psi_x + \eta \psi_y}{r(c_2 - r)} \right) + \frac{\xi}{r_2} (\xi \phi_\eta - \eta \phi_\xi) \right). \\
A_{12}^2 (D\psi, \xi, \eta) &= \left( \psi_x \phi_\chi + \eta \eta_\xi + \frac{\eta}{r(c_2 - r)} \xi_1 \left( \frac{\xi \psi_x + \eta \psi_y}{r(c_2 - r)} \right) \right) + \frac{\psi_x^2 - \eta^2}{r^2} (\xi \phi_\eta - \eta \phi_\xi). \\
A_{21}^2 (D\psi, \xi, \eta) &= A_{12}^2 (D\psi, \xi, \eta),
\end{align*}

where \(\phi, \phi_\xi,\text{ and } \phi_\eta\) are evaluated at \((\xi, \eta)\).

Now we write (5.40) in the \((x, y)\)-coordinates. By calculation, the terms \(\tilde{I}_1\) and \(\tilde{I}_3\) in the polar coordinates are
\begin{align*}
\tilde{I}_1 &= \left( c_2 - r^2 + (\gamma - 1) \left( r(c_2 - r) \xi_1 \left( \frac{\psi_x}{r(c_2 - r)} \right) - \frac{1}{2} D\psi^2 - \psi \right) \right) \Delta \psi, \\
\tilde{I}_3 &= 2r(c_2 - r) \xi_1 \left( \frac{\psi_y}{r(c_2 - r)} \right) \psi_{rr} + 2r \psi_\theta \psi_{r\theta} - \frac{2}{r^2} \psi_\theta^2.
\end{align*}
Thus, (5.40) in the \((x, y)\)-coordinates in \(\mathcal{A}_{4e}\) has the form

\begin{equation*}
(5.42) \quad 2x - (\gamma + 1) x \xi_1 \left( \frac{\psi_x}{x} \right) + O_1^\phi \psi_x + O_2^\phi \psi_y + \left( \frac{1}{c_2} + O_3^\phi \right) \psi_{yy} - (1 + O_4^\phi) \psi_x + O_5^\phi \psi_y = 0,
\end{equation*}

with \(O_k^\phi (p, x, y)\) defined by
\begin{equation*}
(5.43) \quad \frac{\partial_x^k \phi}{c_2} = \frac{x^2}{c_2} + \frac{\gamma + 1}{2c_2} \left( 2x^2 \xi_1 \left( \frac{\psi_x}{x} \right) - \phi_x \right) - \frac{\gamma - 1}{c_2} \left( \phi + \frac{1}{2(c_2 - x)^2} \phi_x \right),
\end{equation*}
delete "tilde"
where \( p = (p_1, p_2) \), and \((D\phi, \phi)\) are evaluated at \((x, y)\). The estimates in (4.50), the definition of the cutoff function \( \xi_1 \), and \( \phi \in \mathcal{X} \) with (5.16) imply

\[
|\partial_k^2 f(x, y)| \leq C|x|^{3/2}, \quad |\partial_k^3 f(x, y)| \leq C|x| \quad \text{for } k = 2, \ldots, 5.
\]

for all \( p \in \mathbb{R}^2 \) and \((x, y) \in \mathcal{D}_4 \). Indeed, using that \( \phi \in \mathcal{X} \) implies \( \|\phi\|_{2,2,\mathcal{D}_4} \leq M_1 \), we find that, for all \( p \in \mathbb{R}^2 \) and \((x, y) \in \mathcal{D}_4 \),

\[
|\partial_k^2 f(x, y)| \leq C(M_1^2 + \rho)|x|^2 \leq C|x|^{3/2},
\]

\[
|\partial_k^3 f(x, y)| \leq C(1 + M_1|x|)M_1|x| \leq C|x| \quad \text{for } k = 2, 5,
\]

\[
|\partial_k^2 f(x, y)| \leq C(|x| + M_1^2|x|^2) \leq C|x| \quad \text{for } k = 3, 4.
\]

In order to obtain the corresponding estimates in the domain \( \mathcal{D}_4 \setminus \mathcal{D}_e \), we note that \( \mathcal{D}_4 \setminus \mathcal{D}_e \subset \mathcal{D}_4 \). Since \( 2e \leq x \leq 4e \) in \( \mathcal{D}_4 \setminus \mathcal{D}_e \) and \( \phi \in \mathcal{X} \) implies \( \|\phi\|_{2,2,\mathcal{D}_4} \leq M_2 \), we find that, for any \( p \in \mathbb{R}^2 \) and \((x, y) \in \mathcal{D}_4 \setminus \mathcal{D}_e \),

\[
|\partial_k^2 f(x, y)| \leq C(1 + M_2^2|x|^2 + M_2\sigma\xi^2) \leq C\rho^2 \leq C|x|^2, \quad \text{for } k = 2, 5,
\]

\[
|\partial_k^3 f(x, y)| \leq C(1 + M_2\sigma\xi^2)M_2\sigma \leq C\rho^2 \leq C|x|^2 \quad \text{for } k = 3, 4.
\]

Estimates (5.45) and (5.46) imply (5.44).

The estimates in (5.44) imply that, if \( \phi \in \mathcal{X} \) and \( \varepsilon \) is sufficiently small depending only on the data (which are guaranteed by (5.16) with sufficiently large \( C \)), (5.42) is nonuniformly elliptic in \( \mathcal{D}_4 \). First, in the \((x, y)\)-coordinates, writing (5.42) as

\[
a_{11}\psi_{xx} + 2a_{12}\psi_{xy} + a_{22}\psi_{yy} + a_{11}\psi_x + a_2\psi_y = 0,
\]

and

\[
\partial_k^2 \psi = \sum_{\xi} \langle \partial_k^2 \phi, \xi \rangle \psi_{\xi} + \sum_{\xi} \langle \partial_k^2 \phi, \xi \rangle \psi_{\xi},
\]

where \( \xi_1 \) is a cutoff function, we find that, for all \( p \in \mathbb{R}^2 \) and \((x, y) \in \mathcal{D}_4 \),
Proof. Property (i) follows from (5.36) and (5.47), (5.48). Properties (ii) and (iii) follow from the explicit expressions (5.35) and (5.41) with \( \phi \in \mathcal{H} \). In estimating these expressions in property (iii), we use that \( |\xi_1'(y)| \leq C \) which follows from the smoothness of \( \xi_1 \) and (5.37).

Also, (5.29) coincides with (5.42) in the domain \( \mathcal{B}' \). Assume that \( \varepsilon < \kappa_0/24 \), which can be achieved by choosing \( \hat{C} \) large in (5.16). Then, in the larger domain \( \mathcal{B} \cap \{ c_2 - r < 12\varepsilon \} \), (5.29) written in the \( (x, y) \)-coordinates has form (5.42) with the only difference that the term \( x \xi_1(y) \) in the coefficient of \( \psi_{xx} \) of (5.42) and in the terms \( \tilde{O}_1^\phi, \tilde{O}_3^\phi, \) and \( \tilde{O}_4^\phi \) given by (5.43) is replaced by

\[
x \left( \xi_2(y) \xi_1 \left( \frac{\psi_y}{\psi} \right) + (1 - \xi_2(y)) \xi_1 \left( \frac{\psi}{\psi} \right) \right).
\]

From this, we have

**Lemma 5.3.** There exist \( C \) and \( \hat{C} \) depending only on the data such that the following holds. Assume that \( M_1, M_2, \varepsilon, \) and \( \sigma \) satisfy (5.16). Let \( \phi \in \mathcal{H} \). Then equation (5.29) written in the \( (x, y) \)-coordinates in \( \mathcal{B} \cap \{ c_2 - r < 12\varepsilon \} \) has the form

\[
\hat{A}_{11} \psi_{xx} + 2 \hat{A}_{12} \psi_{xy} + \hat{A}_{22} \psi_{yy} + \hat{A}_1 \psi_x + \hat{A}_2 \psi_y = 0,
\]

where \( \hat{A}_{ij} = \hat{A}_{ij}(\psi, x, y) \), \( \hat{A}_i = \hat{A}_i(\psi, x, y) \), and \( \hat{A}_{12} = \hat{A}_{12} \). Moreover, the coefficients \( \hat{A}_{ij}(p, x, y) \) and \( \hat{A}_i(p, x, y) \) with \( p = (p_1, p_2) \in \mathbb{R}^2 \) satisfy

(i) For any \( (x, y) \in \mathcal{B} \cap \{ x < 12\varepsilon \} \) and \( p, \mu \in \mathbb{R}^2 \),

\[
\frac{x}{6} |\mu|^2 \leq \sum_{i,j=1}^{2} \hat{A}_{ij}(p, x, y) \mu_i \mu_j \leq \frac{2}{c_2} |\mu|^2;
\]

(ii) For any \( (x, y) \in \mathcal{B} \cap \{ x < 12\varepsilon \} \) and \( p \in \mathbb{R}^2 \),

\[
|\langle \hat{A}_{ij}, D(p, x, y) \hat{A}_{ij} \rangle| + |\langle \hat{A}_i, D(p, x, y) \hat{A}_i \rangle| \leq C;
\]

(iii) \( \hat{A}_{11}, \hat{A}_{22}, \) and \( \hat{A}_1 \) are independent of \( p_2 \); and

(iv) \( \hat{A}_{12}, \hat{A}_{21}, \) and \( \hat{A}_2 \) are independent of \( p \), and

\[
|\langle \hat{A}_{12}, \hat{A}_{21}, \hat{A}_2(x, y) \rangle| \leq C \langle x \rangle, \quad |D(\hat{A}_{12}, \hat{A}_{21}, \hat{A}_2)(x, y)\rangle| \leq C \langle x \rangle^{1/2}.
\]

The last inequality in Lemma 5.3(iv) is proved as follows. Note that

\[
(\hat{A}_{12}, \hat{A}_2)(x, y) = (O_2, O_5)(D\phi(x, y), \phi(x, y), \phi(x, y), x),
\]

where \( O_2 \) and \( O_5 \) are given by (4.50). Then, by \( \phi \in \mathcal{H} \) and (5.16), we find that, for \( (x, y) \in \mathcal{B}' \), i.e., \( x \in (0, 2\varepsilon) \),
1^{1/2} \leq C(1 + M_1\varepsilon)|D\phi_{x}(x, y)| + (1 + M_1)|\phi_{y}(x, y)| \leq C(1 + M_1\varepsilon)M_1x^{1/2} + C(1 + M_1)M_1x^{3/2} \leq Cx^{1/2},

and, for \((x, y) \in \mathbb{D} \cap \{\varepsilon \leq x \leq 12\varepsilon\} \subset \mathbb{D}',\) we have \text{dist}(x, \Sigma_0) \geq c_2/2 \geq \ell_2/4 so that

\[|D(\hat{A}_{12}, \hat{A}_{21}, \hat{A}_2)(x, y)| \leq C(1 + M_2\sigma)M_2\sigma \leq C\varepsilon \leq Cx.\]

The next lemma follows directly from both (5.37) and the definition of \(A_{ij}.\)

**Lemma 5.4.** Let \(\Omega \subset \mathbb{D}, \psi \in C^2(\Omega),\) and \(\psi\) satisfy equation (5.29) with \(\phi = \psi\) in \(\Omega.\) Assume also that \(\psi, written in the (x, y)-coordinates, satisfies |\psi_{x}| \leq 4x/(3y + 1)\) in \(\Omega' := \Omega \cap \{c_2 - r < 4\varepsilon\}.\) Then \(\psi\) satisfies (4.19) in \(\Omega.\)

5.6. The iteration procedure and choice of the constants. With the previous analysis, our iteration procedure will consist of the following ten steps, in which Steps 2–9 will be carried out in detail in Sections 6–8 and the main theorem is completed in Section 9.

**Step 1.** Fix \(\phi \in \mathcal{H}.\) This determines the domain \(\Omega^+(\phi), (5.29),\) and condition (5.30) on \(\Gamma_{shock}(\phi),\) as described in 5.4 and 5.5 above.

**Step 2.** In Section 6, using the vanishing viscosity approximation of (5.29) via a uniformly elliptic equation

\[N(\psi) + \delta\Delta\psi = 0 \quad \text{for } \delta \in (0, 1),\]

and sending \(\delta \to 0,\) we establish the existence of a solution \(\psi \in C(\Omega^+(\phi)) \cap C^1(\Omega^+(\phi) \setminus \overline{\Omega}_{\text{conv}}) \cap C^2(\Omega^+(\phi))\) to problem (5.29)–(5.33). This solution satisfies

\[0 \leq \psi \leq C\sigma \quad \text{in } \Omega^+(\phi),\]

where \(C\) depends only on the data.

**Step 3.** For every \(s \in (0, c_2/2),\) set \(\Omega'' := \Omega^+(\phi) \cap \{c_2 - r > s\}.\) By Lemma 5.2, if (5.16) holds with sufficiently large \(\tilde{C}\) depending only on the data, then (5.29) is uniformly elliptic in \(\Omega''\) for every \(s \in (0, c_2/2),\) the ellipticity constant depends only on the data and \(s,\) and the bounds of coefficients in the corresponding Hölder norms also depend only on the data and \(s.\) Furthermore, (5.29) is linear on \(\{c_2 - r > 4\varepsilon\},\) which implies that it is also linear near the corners \(P_2\) and \(P_3.\)

Then, by the standard elliptic estimates in the interior and in the smooth parts of \(\partial\Omega^+(\phi) \cap \overline{\Omega''}\) and using Lieberman's estimates [35] for linear equations with the oblique derivative conditions near the corners \((-u_2, -v_2)\) and \(\Gamma_{shock}(\phi) \cap \{\eta = -v_2\},\) we have

\[\|\psi\|_{L^\infty(\overline{\Omega'})} = C(|\psi|_{L^\infty(\overline{\Omega'})} + |v_2|).\]
if $\|\psi\|_{L^\infty(\bar{\Omega}')} + |v_2| \leq 1$, where the second term on the right-hand side comes from the boundary condition (5.33), and the constant $C(s)$ depends only on the ellipticity constants, the angles at the corners $P_2 = \Gamma_{\text{shock}}(\phi) \cap \{\eta = -v_2\}$ and $P_3 = (-u_2, -v_2)$, the norm of $\Gamma_{\text{shock}}(\phi)$ in $C^{1,\alpha}$, and $s$, which implies that $C(s)$ depends only on the data and $s$.

Now, using (5.51) and (3.24), we obtain $\|\psi\|_{L^\infty(\bar{\Omega}')} + |v_2| \leq 1$ if $\sigma$ is sufficiently small, which is achieved by choosing $\tilde{C}$ in (5.16) sufficiently large. Then, from (5.52), we obtain

$$\|\psi\|_{2,\alpha, \Omega''_s} \leq C(s)\sigma$$

for every $s \in (0, c_2/2)$, where $C$ depends only on the data and $s$.

Step 4. Estimates of $\psi$ in $\bar{\Omega}'(\phi) := \Omega^+(\phi) \cap \{c_2 - r < \varepsilon\}$. We work in the $(x, y)$-coordinates, and then (5.29) is (5.42) in $\Omega''$.

Step 4.1. $L^\infty$ estimates of $\psi$ in $\Omega^+(\phi) \cap \Omega''$. Since $\phi \in \mathcal{K}$, the estimates in (5.44) hold for large $\tilde{C}$ in (5.16) depending only on the data. We also rewrite the boundary condition (5.30) in the $(x, y)$-coordinates and obtain (4.56) with $\tilde{\Phi}$ replaced by $\tilde{\Phi}_i(x, y) := \tilde{\Phi}_i(D\phi(x, y), \phi(x, y), x, y)$. Using $\phi \in \mathcal{K}$, (4.57), (4.58), and (5.27) with $\tilde{f}\phi(0) = \tilde{f}_0(0) = y_1$, we obtain

$$|\tilde{\Phi}_i(x, y)| \leq C(M_1\varepsilon + M_2\sigma) \leq C/\tilde{C}, \quad i = 1, 2,$$

for $(x, y) \in \Gamma_{\text{shock}}(\phi) \cap \{0 < x < 2\varepsilon\}$. Then, if $\tilde{C}$ in (5.16) is large, we find that the function

$$w(x, y) = \frac{3x^2}{5(y + 1)}$$

is a supersolution of (5.42) in $\Omega'(\phi)$ with the boundary condition (5.30) on $\Gamma_{\text{shock}}(\phi) \cap \{0 < x < 2\varepsilon\}$. That is, the right-hand sides of (5.30) and (5.42) are negative on $w(x, y)$ in the domains given above. Also, $w(x, y)$ satisfies the boundary conditions (5.31), (5.32) within $\Omega'(\phi)$. Thus,

$$0 \leq \psi(x, y) \leq \frac{3x^2}{5(y + 1)} \quad \text{in} \quad \Omega'(\phi),$$

if $w \geq \psi$ on $x = \varepsilon$. By (5.51), $w \geq \psi$ on $x = \varepsilon$ if

$$C\sigma \leq \varepsilon^2,$$

where $C$ is a large constant depending only on the data, i.e., if (5.16) is satisfied with large $\tilde{C}$. The details of the argument of Step 4.1 are in Lemma 7.3.

Step 4.2. Estimates of the norm $\|\psi\|_{2,\alpha, \Omega''(\phi)}$. We use the parabolic rescaling in the rectangle $R_{\varepsilon}$ defined by (5.12) in which $\Omega'$ is replaced by $\Omega''(\phi)$. Note that
$R_z \subset \Omega'$ for every $z = (x, y) \in \hat{\Omega}'(\phi)$. Thus, $\psi$ satisfies (5.42) in $R_z$. For every $z \in \hat{\Omega}'(\phi)$, we define the functions $\psi^{(z)}$ and $\phi^{(z)}$ by (5.14) in the domain $Q_1^{(z)}$ defined by (5.13). Then (5.42) for $\psi$ yields the following equation for $\psi^{(z)}(S, T)$ in $Q_1^{(z)}$:

\[
\begin{align*}
&\left(1 + \frac{S}{4}\right)\left(2 - (\gamma + 1)\xi_1\right)\left(4\psi_S^{(z)} \pm \frac{1}{S/4}\right) + xO_1^{(z, \phi)} \psi_{SS}^{(z)} + xO_2^{(z, \phi)} \psi_{ST}^{(z)} + \\
&+ \frac{1}{C_2} + xO_3^{(z, \phi)} \psi_T^{(z)} - \frac{1}{C_2} + xO_4^{(z, \phi)} \psi_T^{(z)} + x^2O_5^{(z, \phi)} \psi_T^{(z)} = 0,
\end{align*}
\]

where the terms $O_k^{(z, \phi)}(S, T, \rho), k = 1, \ldots, 5$, satisfy

\[
\|O_k^{(z, \phi)}\|_{C^{1,\alpha}(Q_1^{(z)} \times R^3)} \leq C(1 + M_1^2).
\]

Estimate (5.57) follows from the explicit expressions of $O_k^{(z, \phi)}$ obtained from both (5.43) by rescaling and the fact that

\[
\|\phi^{(z)}\|_{C^{2,\alpha}(Q_1^{(z)})} \leq CM_1,
\]

which is true since $\|\phi\|_{C^{2,\alpha}(Q_1^{(z)})} \leq M_1$. Now, since every term $O_k^{(z, \phi)}$ in (5.56) is multiplied by $x^{\beta_k}$ with $\beta_k \geq 1$ and $x \in (0, e)$, condition (5.16) (possibly after increasing $C$ depending only on the data) implies that (5.56) is uniformly elliptic in $Q_1^{(z)}$ and has the $C^{1,\alpha}$ bounds on the coefficients by a constant depending only on the data.

Now, if the rectangle $R_z$ does not intersect $\partial\Omega^+(\phi)$, then $Q_1^{(z)} = Q_1$, where $Q_1 = (-s, s)^2$ for $s > 0$. Thus, the interior elliptic estimates in Theorem A.1 in the appendix imply

\[
\|\psi^{(z)}\|_{C^{2,\alpha}(Q_1^{(z)})} \leq C,
\]

where $C$ depends only on the data and $\|\psi^{(z)}\|_{L^\infty(Q_1^{(z)})}$. From (5.55), we have

\[
\|\psi^{(z)}\|_{L^\infty(Q_1^{(z)})} \leq 1/(\gamma + 1).
\]

Therefore, we obtain (5.58) with $C$ depending only on the data.

Now consider the case when the rectangle $R_z$ intersects $\partial\Omega^+(\phi)$. From its definition, $R_z$ does not intersect $\Gamma_{\text{ionic}}$. Thus, $R_z$ intersects either $\Gamma_{\text{shock}}$ or the wedge boundary $\Gamma_{\text{wedge}}$. On these boundaries, we have the homogeneous oblique derivative conditions (5.30) and (5.32). In the case when $R_z$ intersects $\Gamma_{\text{wedge}}$, the rescaled condition (5.32) remains the same form, thus oblique, and we use the
Step 7. With the constants $\sigma, \varepsilon, M_1,$ and $M_2$ chosen in Step 6, estimates (5.61)

\[
\frac{1}{2} \leq \frac{1}{2}, \quad \frac{1}{3} \leq \frac{1}{2}, \quad \frac{1}{3} \leq \frac{1}{2},
\]

imply

\[
\|\phi\|_{2,1,2} \leq M_1, \quad \|\psi\|_{1,1,2} \leq M_2 \varepsilon.
\]

Thus, $\psi \in \mathcal{H}(\sigma, \varepsilon, M_1, M_2).$ Then the iteration map $J : \mathcal{H} \to \mathcal{H}$ is defined.

Step 8. In Lemma 7.5 and Proposition 7.1, by the argument similar to [10] and the fact that $\mathcal{H}$ is a compact and convex subset of $C^{1,\alpha/2}(\Omega)$, we show that the iteration map $J$ is continuous, by uniqueness of the solution $\psi \in C^{1,\alpha}(\Omega) \cap C^{2,\alpha}(\Omega)$ of (5.29)--(5.33). Then, by the Schauder Fixed Point Theorem, there exists a fixed point $\psi \in \mathcal{H}$. This is a solution of the free-boundary problem.

Step 9. Removal of the cutoff. By Lemma 5.4, a fixed point $\psi = \phi$ satisfies the original (4.19) in $\Omega^+(\psi)$ if $|\psi| \leq 4 \varepsilon/(3(\psi + 1))$ in $\Omega^+(\psi) \cap \{c_2 - r < 4 \varepsilon\}$. We prove this estimate in Section 8 by choosing $\lambda$ sufficiently large depending only on the data.

Step 10. Since the fixed point $\psi \in \mathcal{H}$ of the iteration map $J$ is a solution of (5.29)--(5.33) for $\phi = \psi$, we conclude

(i) $\psi \in C^{1,\alpha}(\Omega^+(\psi)) \cap C^{2,\alpha}(\Omega^+(\psi))$;

(ii) $\psi = 0$ on $\Gamma_{\text{sonic}}$ by (5.31), and $\psi$ satisfies the original (4.19) in $\Omega^+(\psi)$ by Step 9;

(iii) $D\psi = 0$ on $\Gamma_{\text{sonic}}$ since $\|\phi\|_{2,1,2} \leq M_1$;

(iv) $\psi = \phi_1 - \phi_2$ on $\Gamma_{\text{shock}}(\psi)$ by (5.21)--(5.23) since $\phi = \psi$;

(v) The Rankine-Hugoniot gradient jump condition (4.29) holds on $\Gamma_{\text{shock}}(\psi)$. Indeed, as we showed in (iv) above, the function $\psi = \phi + \phi_2$ satisfies (4.9) on $\Gamma_{\text{shock}}(\psi)$. Since $\psi \in \mathcal{H}$, it follows that $\psi$ satisfies (4.28). Also, $\psi$ on $\Gamma_{\text{shock}}(\psi)$ satisfies (5.30) with $\phi = \psi$, which is (4.42). Since $\psi \in \mathcal{H}$ satisfies (4.28) and (4.42), it has been shown in Section 4.2 that $\phi$ satisfies (4.10) on $\Gamma_{\text{shock}}(\psi)$, i.e., $\psi$ satisfies (4.29).

Extend the function $\phi = \psi + \phi_1$ from $\Omega := \Omega^+(\psi)$ to the whole domain $\Lambda$ by using (1.20) to define $\phi$ in $\Lambda \setminus \Omega$. Denote $\Lambda_0 := \{\xi > \xi_0\} \cap \Lambda$, $\Lambda_1$ the domain with $\xi < \xi_0$ and above the reflection shock $P_0 P_1$, and $\Lambda_2 := \Lambda \setminus (\Lambda_0 \cup \Lambda_1)$. Set $S_0 := \{\xi = \xi_0\} \cap \Lambda$ the incident shock and $S_1 := P_0 P_1 \cap \Lambda$ the reflected shock. We show in Section 9 that $S_1$ is a $C^2$-curve. Then we conclude that the domains $\Lambda_0, \Lambda_1,$ and $\Lambda_2$ are disjoint, $\partial \Lambda_0 \cap \Lambda = S_0$, $\partial \Lambda_1 \cap \Lambda = S_0 \cup S_1$, and $\partial \Lambda_2 \cap \Lambda = S_1$. Properties (i)--(v) above and the fact that $\psi$ satisfies (4.19) in $\Omega$ imply that

\[
\psi \in W^{1,\infty}(\Lambda), \quad \phi \in C^1(\Lambda_i) \cap C^{1,1}(\Lambda_i) \quad \text{for } i = 0, 1, 2.
\]
\( \varphi \) satisfies (1.8) a.e. in \( \Lambda \) and the Rankine-Hugoniot condition (1.13) on the \( C^2 \)-curves \( S_0 \) and \( S_1 \), which intersect only at \( P_0 \in \partial \Lambda \) and are transversal at the intersection point. Using this, Definition 2.1, and the remarks after Definition 2.1, we conclude that \( \varphi \) is a weak solution of Problem 2, thus of Problem 1. Note that the solution is obtained for every \( \sigma \in (0, \sigma_0] \), i.e., for every \( \theta_w \in [\pi/2 - \sigma_0, \pi/2] \) by (3.1), and that \( \sigma_0 \) depends only on the data since \( \hat{C} \) is fixed in Step 9.

6. Vanishing viscosity approximation and existence of solutions of problem (5.29)–(5.33)

In this section we perform Step 2 of the iteration procedure described in Section 5.6. Through this section, we keep \( \phi \in \mathcal{H} \) fixed, denote by \( \mathcal{P} := \{P_1, P_2, P_3, P_4\} \) the set of the corner points of \( \Omega^+ (\phi) \), and use \( \alpha \in (0, 1/2) \) as defined in Section 5.4.

We regularize (5.29) by the vanishing viscosity approximation via the uniformly elliptic equations

\[
\mathcal{N}(\psi) + \delta \Delta \psi = 0 \quad \text{for } \delta \in (0, 1).
\]

That is, we consider the equation

\[
(6.1) \quad \mathcal{N}_\delta(\psi) := (A_{11} + \delta)\psi_{\xi \xi} + 2A_{12}\psi_{\xi \eta} + (A_{22} + \delta)\psi_{\eta \eta} = 0 \quad \text{in } \Omega^+ (\phi).
\]

In the domain \( \Omega' \) in the \( (x, y) \)-coordinates defined by (4.47), this equation has the form

\[
(6.2) \quad \left( \delta + 2x - (\gamma + 1)x_1 \right) \psi_{xx} + \left( \frac{1}{c_2} + \frac{\delta}{(c_2 - x)^2} + O_5^\phi \right) \psi_{yy} - \left( 1 - \frac{\delta}{c_2 - x} + O_4^\phi \right) \psi_x + O_5^\phi \psi_y = 0
\]

by use of (5.42) and with the Laplacian operator \( \Delta \) in the \( (x, y) \)-coordinates. This is easily derived from the form of \( \Delta \) in the polar coordinates. The terms \( O_5^\phi \) in (6.2) are defined by (5.43).

We now study (6.1) in \( \Omega^+ (\phi) \) with the boundary conditions (5.30)–(5.33).

We first note some properties of the boundary condition (5.30). Using Lemma 5.1 with \( \alpha \in (0, 1/2) \) and (5.16), we find \( \| \phi \|_{2, \alpha, \infty} \leq C \), where \( C \) depends only on the data. Then, writing (5.30) as

\[
(6.3) \quad \mathcal{M}(\eta)(\xi, \eta) := h_1(\xi, \eta)\psi_{\xi} + h_2(\xi, \eta)\psi_{\eta} + h_3(\xi, \eta)\psi = 0 \quad \text{on } \Gamma_{\text{shock}}(\phi)
\]

and using (4.43)–(4.45), we obtain

\[
(6.4) \quad \| h_i \|_{1, \alpha, \Gamma_{\text{shock}}(\phi)} \leq C \quad \text{for } i = 1, 2, 3,
\]

where \( C \) depends only on the data.
Lemma 6.1. There exists \( \tilde{C} > 0 \) depending only on the data such that, if 
\( \sigma, \epsilon > 0 \) and \( M_1, M_2 \geq 1 \) in (5.15) satisfy (5.16), and \( \delta \in (0, 1) \), then, for any 
\( \psi \in C^{1, \alpha/2}(\Omega^+(\phi)) \), equation (6.13) is uniformly elliptic in \( \Omega^+(\phi) \):

\[
\beta |\mu|^2 \leq \sum_{i, j = 1}^{2} a_{ij}(\xi, \eta) \mu_i \mu_j \leq 2\lambda^{-1}|\mu|^2 \quad \text{for} \quad (\xi, \eta) \in \Omega^+(\phi), \mu \in \mathbb{R}^2.
\]

where \( \lambda \) is from Lemma 5.2. Moreover, for any \( s \in (0, c_2/2) \), the ellipticity constants depend only on the data and are independent of \( \delta \) in \( \Omega^+_r(\phi) = \Omega^+(\phi) \cap \{ c_2 - r > s \} \):

\[
\lambda(c_2 - s)|\mu|^2 \leq \sum_{i, j = 1}^{2} a_{ij}(\xi, \eta) \mu_i \mu_j \leq 2\lambda^{-1}|\mu|^2 \quad \text{for} \quad z = (\xi, \eta) \in \Omega^+_r(\phi), \mu \in \mathbb{R}^2.
\]

Furthermore,

\[
a_{ij} \in C^{\alpha/2}(\Omega^+(\phi)).
\]

\[1^{1/2} 2^{1/2}\]

**Proof.** Facts (6.15), (6.16) directly follow from the definition of \( a_{ij} \) and both the definition and properties of \( A_{ij} \) in Section 5.5 and Lemma 5.2.

Since \( A_{ij}(p, \xi, \eta) \) are independent of \( p \) in \( \Omega^+(\phi) \cap \{ c_2 - r > 4\epsilon \} \), it follows from (5.35), (5.41), and \( \phi \in \mathcal{P} \) that \( a_{ij} \in C^{(-\alpha, \Sigma_0)}_{1, \alpha/2, \Omega^+(\phi)^c \cap \mathbb{R}^+} \subset C^{\alpha}(\Omega^+(\phi) \cap \mathbb{R}^+) \).

To show \( a_{ij} \in C^{\alpha/2}(\Omega^+(\phi)) \), it remains to prove that \( A_{ij} \) are the only terms.

To achieve this, we note that the nonlinear terms in the coefficients \( A_{ij}(p, \xi, \eta) \) are smaller.

Since \( \xi_1 \) is a bounded and \( C^\infty \)-smooth function on \( \mathbb{R} \), and \( \xi_1 \) has compact support, there exists \( C > 0 \) such that, for any \( s > 0, q \in \mathbb{R} \),

\[
|s\xi_1\left(\frac{q}{s}\right)| \leq \left( \sup_{t \in \mathbb{R}} |\xi_1(t)| \right) s, \quad |D_{(q, s)}(s\xi_1\left(\frac{q}{s}\right))| \leq C.
\]

Then it follows that the function

\[
F(p, \xi, \eta) = (c_2 - r)\xi_1\left(\frac{s\xi_1 + \eta\xi_1}{r(c_2 - r)}\right)
\]

satisfies \( |F(p, \xi, \eta)| \leq ||\xi_1||_{L^\infty(\mathbb{R})}(c_2 - r) \) for any \( (p, \xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}' \), and \( |D_{(p, \xi, \eta)}F| \) is bounded on compact subsets of \( \mathbb{R}^2 \times \mathbb{R}' \). From this and \( \psi \in C^{1, \alpha/2}(\Omega^+(\phi)) \), we have \( a_{ij} \in C^{\alpha/2}(\Omega^+(\phi)) \).

Now we state some properties of (6.13) written in the \( (x, y) \)-coordinates.
LEMMA 6.3 (Comparison Principle). There exists $\hat{C} > 0$ depending only on the data such that, if $\alpha, \epsilon > 0$ and $M_1, M_2 \geq 1$ in (5.15) satisfy (5.16), and $\delta \in (0, 1)$, the following comparison principle holds: Let

$$\psi \in C(\Omega^+(\phi)) \cap C^1(\Omega^+(\phi) \setminus \Gamma_{\text{sonic}}) \cap C^2(\Omega^+(\phi)),$$

let the left-hand sides of (6.13), (5.30), and (5.32)–(5.33) be nonpositive for $\psi$, and let $\psi \geq 0$ on $\Gamma_{\text{sonic}}$. Then

$$\psi \geq 0 \quad \text{in} \quad \Omega^+(\phi).$$

Proof. We assume that $\hat{C}$ is large so that (5.19)–(5.22) hold. We first note that the boundary condition (5.30) on $\Gamma_{\text{shock}}(\phi)$, written as (6.3), satisfies

$$(b_1, b_2) \cdot v > 0, \quad b_3 < 0 \quad \text{on} \quad \Gamma_{\text{shock}}(\phi),$$

by (6.5) combined with $\hat{C} < 0$ and $\rho_2 > \rho_1$. Thus, if $\psi$ is not a constant in $\Omega^+(\phi)$, a negative minimum of $\psi$ over $\Omega^+(\phi)$ cannot be achieved:

(i) In the interior of $\Omega^+(\phi)$, by the strong maximum principle for linear elliptic equations;

(ii) In the relative interiors of $\Gamma_{\text{shock}}(\phi)$, $\Gamma_{\text{wedge}}$, and $\partial \Omega^+(\phi) \setminus \{\eta = -1\}$, by Hopf's Lemma and the oblique derivative conditions (5.30) and (5.32)–(5.33);

(iii) In the corners $P_2$ and $P_3$, by the result in Lieberman [33, Lemma 2.2], via a standard argument as in [20, Th. 8.19]. Note that we have to flatten the curve $\Gamma_{\text{shock}}$ in order to apply [33, Lemma 2.2] near $P_2$, and this flattening can be done by using the $C^1, \sigma$ regularity of $\Gamma_{\text{shock}}$.

Using that $\psi \geq 0$ on $\Gamma_{\text{sonic}}$, we conclude the proof.

LEMMA 6.4. There exists $\hat{C} > 0$ depending only on the data such that, if $\alpha, \epsilon > 0$ and $M_1, M_2 \geq 1$ in (5.15) satisfy (5.16), and $\delta \in (0, 1)$, then any solution

$$\psi \in C(\Omega^+(\phi)) \cap C^1(\Omega^+(\phi) \setminus \Gamma_{\text{sonic}}) \cap C^2(\Omega^+(\phi))$$

satisfies (6.9)–(6.10) with the constant $C$ depending only on the data.

Proof. First we note that, since $\Omega^+(\phi) \subset \{\eta < c_2\}$, the function

$$u(\xi, \eta) = -v_2(\eta - c_2)$$

and
for any \( s \in (0, c_2/2) \), where the constant \( C(s, \hat{\psi}) \) depends only on the data, \( s \), and 
\[
\|\hat{\psi}\|_{C^{1,\alpha/2}(\Omega^+(\phi))}\cdot\|\hat{\psi}\|_{C^{1,\alpha/2}(\Omega^+(\phi))}\cdot
\]
\[
\frac{1}{2}
\]
Proof. From (5.22), (5.24), (6.4), (6.5), (6.16), (6.17), and the choice of \( \alpha \) in 
Section 5.4, it follows by [35, Lemma 1.3] that 
(6.24) \[
\|\hat{\psi}\|_{2,\alpha/2}(\Omega^+(\phi))\cdot\|\hat{\psi}\|_{C^{1,\alpha/2}(\Omega^+(\phi))}\cdot\|\hat{\psi}\|_{C^{1,\alpha/2}(\Omega^+(\phi))}\cdot
\]
\[
\frac{7}{0}
\]
\[
\|\hat{\psi}\|_{\Omega^+(\phi)} \leq C(s, \hat{\psi})\|\hat{\psi}\|_{C^{1}(\Omega^+(\phi))} + |v_2| \leq C(s, \hat{\psi})\alpha,
\]
where we have used (3.24) and Lemma 6.4 in the second inequality. 
In deriving (6.24), we have used (5.24) and (6.4) only to infer that \( \Gamma_{\text{shock}}(\phi) \) 
is a \( C^{1,\alpha} \)-curve and \( b_i \in C^{\alpha}(\Gamma_{\text{shock}}(\phi)) \). To improve (6.24) to (6.23), we use 
the higher regularity of \( \Gamma_{\text{shock}}(\phi) \) and \( b_i \), given by (5.24) and (6.4) (and a similar 
regularity for the boundary conditions (5.32)–(5.33), which are given on the flat 
segments and have constant coefficients), combined with rescaling from the balls 
\( B_{\alpha/2}(z) \bigcap \Omega^+(\phi) \) for any \( z \in \Omega^+(\phi) \setminus \{P_2, P_3\} \) (with \( d = \text{dist}(z, \{P_2, P_3\} \cup \Sigma_0) \)) 
into the unit ball and the standard estimates for the oblique derivative problems for 
linear elliptic equations. \( \square \)

Now we show that the solution \( \psi \) is \( C^{2,\alpha/2} \) near the corner \( P_4 = \Gamma_{\text{sonic}} \cap 
\Gamma_{\text{wedge}}(\phi) \). We work in \( \mathcal{Y}' \) in the \((x, y)\)-coordinates.

**Lemma 6.6.** There exists \( \hat{C} > 0 \) depending only on the data such that, if 
\( \sigma, \epsilon > 0 \) and \( M_1, M_2 \geq 1 \) in (5.15) satisfy (5.16), and \( \delta \in (0, 1) \), any solution 
\( \psi \in C(\overline{\Omega^+(\phi)}) \cap C(\Omega^+(\phi) \setminus \Gamma_{\text{sonic}}) \cap C^2(\Omega^+(\phi)) \) of (6.13) and (5.30)–(5.33) is 
in \( C^{2,\alpha/2}(B_{\epsilon}(P_4) \cap \Omega^+(\phi)) \) for sufficiently small \( \epsilon > 0 \). 

For \( i, j = 1, 2, \ldots, C \), \( \epsilon \) is independent of \( \epsilon \).

Step 1. We work in the \((x, y)\)-coordinates. Then \( P_4 = (0, 0) \) and \( \Omega^+(\phi) \cap 
\Gamma_{\text{sonic}} \cap B_{2\epsilon} \) for \( \epsilon \in (0, \delta) \). Denote 
\( B_{2\epsilon} = \{x > 0, y > 0\} \cap B_{2\epsilon} \) for \( \epsilon \in (0, \delta) \). Denote 
\( B_{2\epsilon}^+ := B_0(0) \cap \{x > 0\}, \quad B_{2\epsilon}^{++} := B_0(0) \cap \{x > 0, y > 0\}. \)

Then \( \psi \) satisfies (6.19) in \( B_{2\epsilon}^{++} \) and 
(6.25) \[
\psi = 0 \quad \text{on} \quad \Gamma_{\text{sonic}} \cap B_{2\epsilon}^+ = B_0(0) \cap \{x = 0, y > 0\}.
\]
(6.26) \[
\psi_y = \psi_{yy} = 0 \quad \text{on} \quad \Gamma_{\text{wedge}} \cap B_{2\epsilon}^+ = B_0(0) \cap \{y = 0, x > 0\}.
\]
Rescale \( \psi \) by 
\[
\nu(z) = \psi(qz) \quad \text{for} \quad z = (x, y) \in B_{2\epsilon}^{++}.
\]
Then \( \nu \in C(B_{2\epsilon}^{++}) \cap C^1(B_{2\epsilon}^{++} \setminus \{x = 0\}) \cap C^2(B_{2\epsilon}^{++}) \) satisfies 
(6.27) \[
\|\nu\|_{L^\infty(B_{2\epsilon}^{++})} = \|\psi\|_{L^\infty(B_{2\epsilon}^{++})}.
\]
and $v$ is a solution of
\begin{equation}
\frac{1}{2} \delta_{ij} v_{xx} + 2 \delta_{ij} v_{xy} + \delta_{ij} v_{yy} + \delta_{ij} v_x + \delta_{ij} v_y = 0 \quad \text{in} \quad B_2^{++},
\end{equation}
\begin{equation}
v = 0 \quad \text{on} \quad \partial B_2^{++} \cap \{ x = 0 \},
\end{equation}
\begin{equation}
v_x = v_y = 0 \quad \text{on} \quad \partial B_2^{++} \cap \{ y = 0 \},
\end{equation}
where
\begin{equation}
\delta_{ij}^{(a)}(x, y) = \delta_{ij}(a_1 x, a_2 y), \quad \delta_{ij}^{(a)}(x, y) = a_i \delta_{ij}(a_1 x, a_2 y)
\end{equation}
for $(x, y) \in B_2^{++}, i, j = 1, 2.$

Thus, $\delta_{ij}^{(a)}$ satisfy (6.21) with the unchanged constant $\lambda > 0$ and, since $\rho \leq 1,$
\begin{equation}
\| (\delta_{ij}^{(a)}, \delta_{ij}^{(a)}) \|_{C^{\alpha/2}(B_2^{++})} \leq \| (\delta_{ij}, \delta_{ij}) \|_{C^{\alpha/2}(\Omega \setminus (\phi))} \quad \text{for} \quad i, j = 1, 2.
\end{equation}

Denote $Q := \{ z \in B_2^{++} : \text{dist}(z, \partial B_2^{++}) > 1/50 \}$. The interior estimates for the elliptic (6.28) imply $\| v \|_{C^{2, \alpha/2}(Q)} \leq C \| v \|_{L^\infty(B_2^{++})}$. The local estimates for the Dirichlet problem (6.28) (6.29) imply
\begin{equation}
\| v \|_{C^{2, \alpha/2}(B_1^{++} \setminus B_1^{++})} \leq C \| v \|_{L^\infty(B_2^{++})}
\end{equation}
for every $z = (x, y) \in \{ x = 0, 1/2 \leq y \leq 3/2 \}$. The local estimates for the oblique derivative problem (6.28) and (6.30) imply (6.33) for every $z \in \{ 1/2 \leq x \leq 3/2, y = 0 \}$.

Then we have
\begin{equation}
\| v \|_{C^{2, \alpha/2}(B_1^{++})} \leq C \| v \|_{L^\infty(B_2^{++})}.
\end{equation}

**Step 2.** We modify the domain $B_1^{++}$ by mollifying the corner at $(0, 1)$ and denote the resulting domain by $D^{++}$. That is, $D^{++}$ denotes an open domain satisfying
\[ D^{++} \subset B_1^{++}, \quad D^{++} \setminus B_{1/10}(0, 1) = B_1^{++} \setminus B_{1/10}(0, 1). \]

and
\[ \partial D^{++} \cap B_{1/5}(0, 1) \quad \text{is a} \quad C^{2, \alpha/2}\text{-curve}. \]

Then we prove the following fact: For any $g \in C^{\alpha/2}(\overline{D^{++}})$, there exists a unique solution $w \in C^{2, \alpha/2}(\overline{D^{++}})$ of the problem:
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(6.35) \[ a_{11}^{(q)} w_{xx} + a_{22}^{(q)} w_{yy} + a_{1}^{(q)} w_{x} = g \quad \text{in} \quad D^{++}, \]
\[ w = 0 \quad \text{on} \quad \partial D^{++} \cap \{ x = 0, \ y > 0 \}, \]
\[ w_{y} = 0 \quad \text{on} \quad \partial D^{++} \cap \{ x > 0, \ y = 0 \}, \]
\[ w = 0 \quad \text{on} \quad \partial D^{++} \cap \{ x > 0, \ y > 0 \}, \]

with

(6.36) \[ \| w \|_{C^{2, \alpha/2}(D^{++})} \leq C \| v \|_{L^{\infty}(B^{+}_1)} + \| g \|_{C^{0, \alpha/2}(D^{++})}. \]

This can be seen as follows. Denote by \( D^+ \) the even extension of \( D^{++} \) from \( \{ x, \ y > 0 \} \) into \( \{ x > 0 \} \), i.e.,

\[ D^+ := D^{++} \cup \{ (x,0) : x \in (0,1) \} \cup D^{+-}. \]

where \( D^{+-} := \{ (x,y) : (x,-y) \in D^{++} \} \). Then \( B^{+}_1 \subset D^+ \subset B^{+}_1 \) and \( \partial D^+ \) is a \( C^{2, \alpha/2} \)-curve. Extend \( F = (v, g, \hat{a}_{11}^{(q)}, \hat{a}_{22}^{(q)}, \hat{a}_{1}^{(q)}) \) from \( B^{+}_2 \) to \( B^{+}_2 \) by setting

\[ F(x,-y) = F(x,y) \quad \text{for} \quad (x,y) \in B^{+}_2. \]

Then it follows from (6.29), (6.30) and (6.34) that, denoting by \( \hat{v} \) the restriction of (extended) \( v \) to \( \partial D^+ \), we have \( \hat{v} \in C^{2, \alpha/2}(\partial D^+) \) with

(6.37) \[ \| \hat{v} \|_{C^{2, \alpha/2}(\partial D^+)} \leq C \| v \|_{L^{\infty}(B^{+}_1)}. \]

Also, the extended \( g \) satisfies \( g \in C^{0, \alpha/2}(D^+) \) with \( \| g \|_{C^{0, \alpha/2}(D^+)} = \| g \|_{C^{0, \alpha/2}(D^{++})}. \)

The extended \( (\hat{a}_{11}^{(q)}, \hat{a}_{22}^{(q)}, \hat{a}_{1}^{(q)}) \) satisfy (6.21) and

\[ \| (\hat{a}_{11}^{(q)}, \hat{a}_{22}^{(q)}, \hat{a}_{1}^{(q)}) \|_{C^{0, \alpha/2}(\overline{B^{+}_2})} = \| (a_{11}^{(q)}, a_{22}^{(q)}, a_{1}^{(q)}) \|_{C^{0, \alpha/2}(\overline{B^{+}_2})} \]
\[ \leq \sum_{i,j=1}^{2} \| \hat{a}_{ij} \|_{C^{0, \alpha/2}(\overline{B^{+}_2})}. \]

Then, by [20, Th. 6.8], there exists a unique solution \( w \in C^{2, \alpha/2}(D^+) \) of the Dirichlet problem

(6.38) \[ a_{11}^{(q)} w_{xx} + a_{22}^{(q)} w_{yy} + a_{1}^{(q)} w_{x} = g \quad \text{in} \quad D^+, \]

(6.39) \[ w = \hat{v} \quad \text{on} \quad \partial D^+, \]

and \( w \) satisfies

(6.40) \[ \| w \|_{C^{2, \alpha/2}(D^+)} \leq C (\| \hat{v} \|_{C^{2, \alpha/2}(\partial D^+)} + \| g \|_{C^{0, \alpha/2}(D^+)}. \]

From the structure of (6.38) and the symmetry of the domain and the coefficients and right-hand sides obtained by the even extension, it follows that \( \hat{w} \), defined by
\[ \dot{w}(x, y) = w(x, -y) \quad \text{in} \quad D^+. \]

Thus, \( w \) restricted to \( D^{++} \) is a solution of (6.35), where we use (6.29) to see that \( w = 0 \) on \( \partial D^{++} \cap \{x = 0, y > 0\} \). Moreover, (6.37) and (6.40) imply (6.36).

The uniqueness of the solution \( w \in C^{2, \alpha/2}(D^{++}) \) of (6.35) follows from the Comparison Principle (Lemma 6.3).

Step 3. Now we prove the existence of a solution \( w \in C^{2, \alpha/2}(D^{++}) \) of the problem:

\[ \begin{align*}
\hat{\alpha}_{11} w_{xx} + 2\hat{\alpha}_{12} w_{xy} + \hat{\alpha}_{22} w_{yy} + \hat{\alpha}_1 w_x + \hat{\alpha}_2 w_y &= 0 \quad \text{in} \quad D^{++}, \\
\hat{\alpha}_1 w_x + \hat{\alpha}_2 w_y &= 0 \quad \text{on} \quad \partial D^{++} \cap \{x = 0, y > 0\}, \\
\hat{\alpha}_1 w_y + \hat{\alpha}_2 w_x &= 0 \quad \text{on} \quad \partial D^{++} \cap \{y = 0, x > 0\}, \\
\hat{\alpha}_2 w_x &= \hat{\alpha}_2 w_y = 0 \quad \text{on} \quad \partial D^{++} \cap \{x > 0, y > 0\}. 
\end{align*} \]

Moreover, we prove that \( w \) satisfies

\[ \|w\|_{C^{2, \alpha/2}(D^{++})} \leq C \|v\|_{L^\infty(B_2^{++})}. \]

We obtain such \( w \) as a fixed point of map \( K : C^{2, \alpha/2}(D^{++}) \to C^{2, \alpha/2}(D^{++}) \) defined as follows. Let \( W \in C^{2, \alpha/2}(D^{++}) \). Define

\[ \begin{align*}
\hat{\alpha}_{12} w_{xy} &= \hat{\alpha}_2 (\hat{\alpha}_2 W_x - \hat{\alpha}_1 W_y), \\
\hat{\alpha}_1 W_x + \hat{\alpha}_2 W_y &= 0, \\
\hat{\alpha}_2 W_x - \hat{\alpha}_1 W_y &= 0. 
\end{align*} \]

By (6.22) and (6.31) with \( \rho \in (0, 1) \), we find

\[ \|\hat{\alpha}_{12}(\hat{\alpha}_2)\|_{C^{\alpha/2}(D^{++})} \leq C \rho^{1/2}, \]

which implies

\[ g \in C^{\alpha/2}(D^{++}). \]

Then, by the results of Step 2, there exists a unique solution \( w \in C^{2, \alpha/2}(D^{++}) \) of (6.35) with \( g \) defined by (6.43). We set \( K[W] = w \).

Now we prove that, if \( g > 0 \) is sufficiently small, the map \( K \) is a contraction map. Let \( W^{(i)} \in C^{2, \alpha/2}(D^{++}) \) and \( w^{(i)} := K[W^{(i)}] \) for \( i = 1, 2 \). Then \( w := w^{(1)} - w^{(2)} \) is a solution of (6.35) with

\[ g = -2\hat{\alpha}_{12} (W^{(1)} - W^{(2)}) - \hat{\alpha}_2 (W^{(1)} - W^{(2)}), \]

\[ v \equiv 0. \]
Then $g \in C^{\alpha/2}(\mathbb{D}^{++})$ and, by (6.44),
\[
\| \delta \|_{C^{\alpha/2}(\mathbb{D}^{++})} \leq C \phi^{1/2} \| W^{(1)} - W^{(2)} \|_{C^{\alpha/2}(\mathbb{D}^{++})}.
\]

Since $\nu \equiv 0$ satisfies (6.29)–(6.30), we can apply both (6.36) and the results of Step 2 to obtain
\[
\| w^{(1)} - w^{(2)} \|_{C^{\alpha/2}(\mathbb{D}^{++})} \leq C \phi^{1/2} \| W^{(1)} - W^{(2)} \|_{C^{\alpha/2}(\mathbb{D}^{++})}
\]
\[
\leq \frac{1}{2} \| W^{(1)} - W^{(3)} \|_{C^{\alpha/2}(\mathbb{D}^{++})},
\]
where the last inequality holds if $\phi > 0$ is sufficiently small. We fix such $\phi$. Then the map $K$ has a fixed point $w \in C^{\alpha/2}(\mathbb{D}^{++})$ which is a solution of (6.4.1).

**Step 4.** Since $\nu$ satisfies (6.28)–(6.30), it follows from the uniqueness of solutions in $C(\mathbb{D}^{++}) \cap C^{1}(\mathbb{D}^{++} \setminus \{x = 0\}) \cap C^{2}(\mathbb{D}^{++})$ of problem (6.4.1) that $w = \nu$ in $\mathbb{D}^{++}$. Thus $\nu \in C^{\alpha/2}(\mathbb{D}^{++})$ so that $\psi \in C^{\alpha/2}(\mathbb{B}_{\phi/2}(P_1) \cap \Omega^{+}(\phi))$. \(\square\)

Now we prove that the solution $\psi$ is $C^{1, \alpha}$ near the corner $P_1 = \Gamma_{\text{sonic}} \cap \Gamma_{\text{shock}}(\phi)$ if $\delta$ is small.

**Lemma 6.7.** There exist $\tilde{C} > 0$ and $\delta_0 \in (0, 1)$ depending only on the data such that, if $\sigma, \varepsilon > 0$ and $M_1, M_2 \geq 1$ in (5.15) satisfy (5.16), and $\delta \in (0, \delta_0)$, then any solution $\psi \in C(\Omega^{+}(\phi)) \cap \Omega^{1}(\Omega^{+}(\phi) \setminus \Gamma_{\text{sonic}}) \cap C^{2}(\Omega^{+}(\phi))$ of (6.13) and (5.30)–(5.33) is in $C^{1, \alpha}(\mathbb{B}_{\phi}(P_1) \cap \Omega^{+}(\phi)) \cap C^{2, \alpha/2}(\mathbb{B}_{\phi}(P_1) \cap \Omega^{+}(\phi))$, for sufficiently small $\phi > 0$ depending only on the data and $\delta$, and satisfies
\[
(6.45) \quad \| \psi \|^{(1-\alpha, \{P_1\})}_{2, \alpha/2, \Omega^{+}(\phi)} \leq C(\delta, \psi)\sigma,
\]
where $C$ depends only on the data, $\delta$, and $\| \psi \|^{(1-\alpha, \{\Omega^{+}(\phi)\})}_{C^{1, \alpha/2}(\Omega^{+}(\phi))}$. Moreover, for $\delta$ as above,
\[
(6.46) \quad |\psi(x)| \leq \tilde{C}(\delta) \text{dist}(x, P_1)^{1+\alpha} \quad \text{for any } x \in \Omega^{+}(\phi),
\]
where $\tilde{C}$ depends only on the data and $\delta$, and is independent of $\psi$.

**Proof.** In Steps 1–3 of this proof below, the positive constants $C$ and $L_i, 1 \leq i \leq 4$, depend only on the data.

**Step 1.** We work in the $(x, y)$-coordinates. Then the point $P_1$ has the coordinates $(0, y_{P_1})$ with $y_{P_1} = \pi/2 + \arctan(|\xi_1|/\eta_1) - \theta_w > 0$. From (5.25), (5.26), we have
\[
\Omega^{+}(\phi) \cap \mathbb{B}_{\phi}(P_1) = \{x > 0, y < \tilde{f}_{\phi}(x) \} \cap \mathbb{B}_{\phi}(P_1),
\]
where $\tilde{f}_{\phi}(0) = y_{P_1}$, $\tilde{f}_{\phi}'(0) > 0$, and $\tilde{f}_{\phi} > y_{P_1}$ on $\mathbb{R}_+$ by (5.7) and (5.26).
Step 2. We change the variables in such a way that $P_1$ becomes the origin and the second-order part of (6.13) at $P_1$ becomes the Laplacian. Denote

$$\mu = \sqrt{\hat{a}_{11}(P_1)/\hat{a}_{22}(P_1)}.$$  \hspace{1cm} (6.47)

Then, using (6.22) and $\hat{a}_{12}(P_1) = 0$, we have

$$\sqrt{c_2 \delta}/2 \leq \mu \leq \sqrt{2c_2 \delta}.$$  \hspace{1cm} (6.48)

Now we introduce the variables

$$(X, Y) := (x/\mu, y P_1 - y).$$

Then, for $\varphi = \varepsilon$, we have

$$\Omega^+(\varphi) \cap B_\varepsilon = \{ X > 0, \ Y > F(X) \} \cap B_\varepsilon,$$  \hspace{1cm} (6.49)

where $F(X) = \int_0^X - \hat{f}_\varphi(\mu X).$ By (5.26), we have $0 < \hat{f}_\varphi'(X) \leq C$ for all $X \in [0, 2\varepsilon]$ if $\hat{C}$ is sufficiently large in (5.46) so that $2\varepsilon \leq \kappa$. With this, we use $\hat{f}_\varphi(0) = y P_1$ and (6.48) to obtain

$$F(0) = 0, \quad -L_1 \sqrt{\delta} \leq F'(X) < 0 \quad \text{for} \quad X \in [0, \varepsilon].$$  \hspace{1cm} (6.50)

We now write $\psi$ in the $(X, Y)$-coordinates. Introduce the function

$$\nu(X, Y) := \psi(x, y) = \psi(\mu X, y P_1 - Y).$$

Since $\psi$ satisfies (6.6) and the boundary conditions (5.32) and (6.19), then $\nu$ satisfies

$$\nu(X, Y) := \frac{1}{\mu^2} \hat{a}_{11} \nu_{XX} - \frac{2}{\mu} \hat{a}_{12} \nu_{XY} + \hat{a}_{22} \nu_{YY} + \frac{1}{\mu} \hat{a}_1 \nu_X - \hat{a}_2 \nu_Y = 0$$  \hspace{1cm} \text{in} \quad \{ X > 0, \ Y > F(X) \} \cap B_\varepsilon,  \hspace{1cm} (6.51)

$$Bu := \frac{1}{\mu} \hat{b}_1 \nu_X - \hat{b}_2 \nu_Y + \hat{b}_3 \nu = 0$$  \hspace{1cm} \text{on} \quad \{ X > 0, \ Y = F(X) \} \cap B_\varepsilon,  \hspace{1cm} (6.52)

$$v = 0$$  \hspace{1cm} \text{on} \quad \{ X = 0, \ Y > 0 \} \cap B_\varepsilon,  \hspace{1cm} (6.53)

where

$$\hat{a}_{1j}(X, Y) = \hat{a}_{1j}(\mu X, y P_1 - Y), \quad \hat{a}_1(X, Y) = \hat{a}_1(\mu X, y P_1 - Y),$$

$$\hat{b}_1(X, Y) = \hat{b}_1(\mu X, y P_1 - Y).$$

In particular, from (6.20), (6.22), and (6.47), we have

$$\hat{a}_{ij}, \hat{a}_i \in \mathcal{C}^{a/2}(\{X > 0, \ Y > F(X)\} \cap B_\varepsilon).$$  \hspace{1cm} (6.54)

$$\hat{a}_{22}(0, 0) = \frac{1}{\mu^2} \hat{a}_{11}(0, 0), \quad \hat{a}_{12}(0, 0) = \hat{a}_2(0, 0) = 0.$$  \hspace{1cm} (6.55)
(6.56) \[ |\tilde{a}_{1i}(X, Y) - \tilde{a}_{1i}(0, 0)| \leq C |(X, Y)|^\alpha \quad \text{for } i = 1, 2, \]
(6.57) \[ |\tilde{a}_{12}(X, Y)| + |\tilde{a}_{21}(X, Y)| + |\tilde{a}_{2}(X, Y)| \leq C |X|^{1/2}, \quad |\tilde{a}_1(X, Y)| \leq C. \]

From (6.8), there exists \( L_2 > 0 \) such that

(6.58) \[-L_2^{-1} \leq \tilde{b}_1(X, Y) \leq -L_2 \quad \text{for any } (X, Y) \in \{ X > 0, Y = F(X) \} \cap B_0. \]

Moreover, (6.7) implies

(6.59) \[ (\tilde{b}_1, \tilde{b}_2) \cdot \nu_F > 0 \quad \text{on } \{ X > 0, Y = F(X) \} \cap B_0, \]

where \( \nu_F = (\nu_F(X, Y) \) is the interior unit normal at \( (X, Y) \in \{ X > 0, Y = F(X) \} \cap B_0 \). Thus condition (6.52) is oblique.

**Step 3.** We use the polar coordinates \((r, \theta)\) on the \((X, Y)\)-plane, i.e.,

\[ (X, Y) = (r \cos \theta, r \sin \theta). \]

From (6.50), we have \( F, F' < 0 \) on \((0, \varrho)\), which implies that \((X^2 + F(X)^2)' > 0\) on \((0, \varrho)\). Then it follows from (6.50) that, if \( \delta > 0 \) is a small constant depending only on the data and \( \varrho \) is a small constant depending only on the data and \( \delta \), there exists a function \( \theta_F \in C^1(\mathbb{R}_+) \) and a constant \( L_3 > 0 \) such that

(6.60) \[ \{ X > 0, Y > F(X) \} \cap B_0 = \{ 0 < r < \varrho, \theta_F(r) < \theta < \pi/2 \} \]

with

(6.61) \[-L_3 \sqrt{\delta} \leq \theta_F(r) \leq 0.\]

Choosing sufficiently small \( \delta_0 > 0 \), we show that, for any \( \delta \in (0, \delta_0) \), a function

(6.62) \[ w(r, \theta) = r^{1+\alpha} \cos G(\theta), \quad \text{with } G(\theta) = \frac{3 + \alpha}{2} \left( \theta - \frac{\pi}{4} \right), \]

is a positive supersolution of (6.51)-(6.53) in \( \{ X > 0, Y > F(X) \} \cap B_0 \).

By (6.49) and (6.60), (6.61), we find that, when \( 0 < \delta \leq \delta_0 \leq \left( \frac{3+\alpha}{\delta(3+\alpha/2)} \right)^{2}, \]

\[ \frac{\pi}{2} + \frac{1-\alpha}{16} \pi \leq G(\theta) \leq \frac{\pi}{2} - \frac{1-\alpha}{8} \pi \quad \text{for all } (r, \theta) \in \Omega^+(\phi) \cap B_0. \]

In particular,

(6.63) \[ \cos(G(\theta)) \geq \sin \left( \frac{1-\alpha}{16} \pi \right) > 0 \quad \text{for all } (r, \theta) \in \Omega^+(\phi) \cap B_0 \setminus \{ X = Y = 0 \}, \]

which implies

\[ w > 0 \quad \text{in } \{ X > 0, Y > F(X) \} \cap B_0. \]
By (6.60), (6.61), we find that, for all \( r \in (0, \varrho) \) and \( \delta \in (0, \delta_0) \) with small \( \delta_0 > 0 \),
\[
\cos(\theta_F(r)) \geq 1 - C\delta_0 > 0, \quad \left| \sin(\theta_F(r)) \right| \leq C\sqrt{\delta_0}.
\]

Now, possibly further reducing \( \delta_0 \), we show that \( w \) is a supersolution of (6.52).

Using (6.48), (6.52), (6.58), the above estimates of \( (\theta_F, G(\theta_F)) \) derived above, and
the fact that \( \theta = \theta_F \) on \( \{ X > 0, Y = F(X) \} \cap B_\varrho \), we have

\[
Bw \leq \frac{b_1}{\mu} r^\alpha \left( (\alpha + 1) \cos(\theta_F) \cos(G(\theta_F)) + \frac{3 + \alpha}{2} \sin(\theta_F) \sin(G(\theta_F)) \right)
+ C r^{a-1} |\bar{b}_2| + C r^{a+1} |\bar{b}_3|
\leq -r^\alpha \left( 1 - C\delta_0 \right) \left( \frac{\sin(\frac{1-\alpha}{16}\pi)}{CL_2\sqrt{\delta_0}} - CL_2 - C \right) < 0,
\]

if \( \delta_0 \) is sufficiently small. We now fix \( \delta_0 \) that satisfies all the smallness assumptions
made above.

Finally, we show that \( w \) is a supersolution of (6.51) in \( (X, Y) \in \{ X > 0, Y > F(X) \} \cap B_\varrho \) if \( \varrho \) is small. Denote by \( A_0 \) the operator obtained by fixing the co-
efficients of \( A \) in (6.51) at \( (X, Y) = (0, 0) \). Then \( A_0 = \tilde{a}_{22}(0, 0)A \) by (6.55). By
(6.22), we obtain \( \tilde{a}_{22}(0, 0) = -\tilde{a}_{22}(0, 0) \geq 1/(4\tilde{c}_2) > 0 \). Now, by an explicit
calculation and using (6.48), (6.55)-(6.57), (6.60), and (6.63), we find that, for
\( \delta \in (0, \delta_0) \) and \( (X, Y) \in \{ X > 0, Y > F(X) \} \cap B_\varrho \),
\[
Aw(r, \theta) = a_2(0, 0)\Delta w(r, \theta) + (A - A_0)w(r, \theta)
\leq \tilde{a}_{22}(0, 0) r^{a-1} \left( (\alpha + 1)^2 - \frac{3 + \alpha}{2} \right) \cos(\theta)
+ C r^{a-1} \left( \frac{1}{\mu} |\tilde{a}_{11}(X, Y) - \tilde{a}_{11}(0, 0)| + |\tilde{a}_{22}(X, Y) - \tilde{a}_{22}(0, 0)| \right)
+ C \frac{r^{a-1}}{\mu} |\tilde{a}_{12}(X, Y)| + C \frac{r^{a-1}}{\mu} |\tilde{a}_{21}(X, Y)| + C r^{a} |\tilde{b}_2(X, Y)|
\leq r^{a-1} \left( \frac{(1-\alpha)(5+3\alpha)}{8\tilde{c}_2} \sin(\frac{\pi}{16}) + C \frac{\pi^2/2}{\sqrt{\delta}} \right) < 0
\]
for sufficiently small \( \varrho > 0 \) depending only on the data and \( \delta \).

Thus, all the estimates above hold for small \( \delta_0 > 0 \) and \( \varrho > 0 \) depending only
on the data.

Now, since
\[
\min_{\{ X \geq 0, Y \geq F(X) \} \cap \partial B_\varrho} w(X, Y) = L_4 > 0,
\]
We rescale \( z = (x, y) \) near \( z_0 \):

\[
Z = (X, Y) := \frac{1}{d}(x - x_0, y - y_0).
\]

Since \( B_d(z_0) \cap (\partial \Omega^+ (\phi) \setminus \Gamma_{\text{shock}}) = \emptyset \), then, for \( \rho \in (0, 1) \), the domain obtained by rescaling \( \Omega^+ (\phi) \cap B_{\rho d}(z_0) \) is

\[
\hat{\Omega}_{\rho}^{z_0} := B_1 \cap \left\{ \frac{Y}{\hat{d}} < \hat{F}(X) := \hat{f}_\phi(x_0 + \hat{d}_X) - \hat{f}_\phi(x_0) \right\},
\]

where \( \hat{f}_\phi \) is the function in (5.25). Note that \( y_0 = \hat{f}_\phi(x_0) \) since \( (x_0, y_0) \in \Gamma_{\text{shock}} \).

Since \( L \geq 1 \), we have

\[
\| \hat{F} \|_{C^{2,\alpha}([-1,1])} \leq \| \hat{f}_\phi \|_{2,\alpha, \mathbb{R}_+}^{(-1-\alpha,0)}
\]

and \( \| \hat{f}_\phi \|_{2,\alpha, \mathbb{R}_+}^{(-1-\alpha,0)} \) is estimated in terms of the data by (5.26).

Define

\[
v(Z) = \frac{1}{d^{1+\alpha}} \psi(z_0 + \hat{d} Z) \quad \text{for } Z \in \hat{\Omega}_1^{z_0}.
\]

Then

\[
\| v \|_{L^{\infty}(\hat{\Omega}_1^{z_0})} \leq C
\]

by (6.46) with \( C \) depending only on the data.

Since \( \psi \) satisfies (6.19) in \( \Omega^+ (\phi) \cap \mathbb{R}_+^d \) and the oblique derivative condition (6.6) on \( \Gamma_{\text{shock}} \cap \mathbb{R}_+^d \), then \( v \) satisfies an equation and an oblique derivative condition of the similar form in \( \hat{\Omega}_1^{z_0} \) and on \( \partial \hat{\Omega}_1^{z_0} \cap \{ Y = \hat{F}(X) \} \), respectively, whose coefficients satisfy properties (6.8) and (6.21) with the same constants as for the original equations, where we have used \( \hat{d} \leq 1 \) and the \( C^{1,2} \)-estimates of the coefficients of the equation depending only on the data, \( \delta \), and \( \hat{\psi} \). Then, from the standard local estimates for linear oblique derivative problems, we have

\[
\| v \|_{C^{2,\alpha/2}(\hat{\Omega}_1^{z_0})} \leq C,
\]

with \( C \) depending only on the data, \( \delta \), and \( \hat{\psi} \).

We obtain similar estimates for cases (i) and (ii), by using the interior estimates for elliptic equations for case (i) and the local estimates for the Dirichlet problem for linear elliptic equations for case (ii).

Writing the above estimates in terms of \( \psi \) and using the fact that the whole domain \( \Omega^+ (\phi) \cap B_d(P_1) \) is covered by the subdomains in (i)–(iii), we obtain (6.45) by an argument similar to the proof of [20, Th. 4.8] (see also the proof of Lemma A.3 below).
LEMMA 6.8. There exist $\tilde{C} > 0$ and $\delta_0 \in (0, 1)$ depending only on the data such that, if $\alpha, \varepsilon > 0$ and $M_2, M_2 \geq 1$ in (5.15) satisfy (5.16), and $\delta \in (0, \delta_0)$, there exists a unique solution $\psi \in C^{2, \alpha, \delta}(\Omega^+)$ of (5.13) and (5.30)–(5.33). The solution $\psi$ satisfies (6.9)–(6.10).

Proof. In this proof, for simplicity, we write $\Omega^+$ for $\Omega^+ (\phi)$ and denote by $\Gamma_1, \Gamma_2, \Gamma_3,$ and $\Gamma_{\text{wedge}},$ and $\Gamma_{\text{sonic}}$ respectively the relative interiors of the curves $\Gamma_{\text{shock}}(\phi), \Sigma_0(\phi), \Gamma_{\text{wedge}},$ and $\Gamma_{\text{sonic}}$ respectively.

We first prove the existence of a solution for a general problem $\mathcal{P}$ of the form

$$
\sum_{i,j=1}^{2} a_{ij}(\xi, \eta) \mu_i \mu_j \leq \lambda_1 \mu^2 \quad \text{for all } (\xi, \eta) \in \Omega^+, \mu \in \mathbb{R}^2,
$$

and

$$
\sum_{i=1}^{2} b_{ij}(\xi, \eta) \nu_i (\xi, \eta) \geq \lambda_2.
$$

$$
\frac{b_{ij}(k, k)}{b_{ij}(k, k)} (P_k) - \frac{b_{ij}(k-1, k-1)}{b_{ij}(k-1, k-1)} (P_k) \geq \lambda_2 \quad \text{for } k = 2, 3,
$$

$$
\|a_{ij}\|_{C^0(\Omega^+)} + \|b_{ij}(k, k)\|_{C^{1, \alpha}(\partial \Omega^+)} \leq L \text{ for some } L > 0.
$$

First we derive an a priori estimate of a solution of problem $\mathcal{P}$. For that, we define the following norm for $\psi \in C^{k, \beta}(\Omega^+), k = 0, 1, 2, \ldots, \text{ and } \beta \in (0, 1)$:

$$
\|\psi\|_{k, \beta} := \sum_{i=2}^{3} \|\psi\|_{k, \beta, B_2(p_i) \cap \Omega^+} + \sum_{i=1, 4} \|\psi\|_{k, \beta, B_2(p_i) \cap \Omega^+} + \|\psi\|_{C^{k, \beta}(\Omega^+ \setminus \bigcup_{i=2}^{4} B_2(p_i))},
$$

where $\rho$ is chosen small so that the balls $B_2(p_i)$ for $i = 1, \ldots, 4$ are disjoint. Define $C^{*, k, \beta} = \{\psi \in C^{*, k, \beta}: \|\psi\|_{*, k, \beta} < \infty\}$. Then $C^{*, k, \beta}$ with norm $\|\cdot\|_{*, k, \beta}$ is a Banach space. Similarly, define

$$
\|g_k\|_{*, \beta} := \sum_{i=2}^{3} \|g_k\|_{k, \beta, B_2(p_i) \cap \Gamma_k} + \sum_{i=1, 4} \|g_k\|_{k, \beta, B_2(p_i) \cap \Gamma_k} + \|g_k\|_{C^{1, \beta}(\bigcup_{i=2}^{4} B_2(p_i))},
$$
Step 1. Since a solution \( \psi \in C^{(1-\alpha_{1,(P_{2,3})})} \) of (6.1), (5.30)-(5.32), and (6.70) with \( \alpha \in [0,1] \) is the solution of the linear problem for (6.13) with \( \hat{\psi} := \psi \) and boundary conditions (5.30)-(5.32) and (6.70). Thus, estimates (6.9) and (6.10) with constant \( C \) depending only on the data follow directly from Lemma 6.4.

Step 2. Now, from Lemma 5.2(ii), (6.1) is linear in \( \Omega^+(\phi) \cap \{c_2 - r > 4e\} \), i.e., (6.1) is (6.13) in \( \Omega^+(\phi) \cap \{c_2 - r > 4e\} \), with coefficients \( b_{ij} = a_{ij}(\xi, \eta) \) defined by (5.35). Then, by Lemma 5.2(ii),

\[
a_{ij} \in C^{(1-\alpha_{1,(P_{2,3})})}(\Omega^+(\phi) \cap \{c_2 - r > 4e\})
\]

with the norm estimated in terms of the data. Also, \( \Gamma_{\text{shock}}(\phi) \) and the coefficients \( b_{ij} \) of (6.3) satisfy (5.24) and (6.4), (6.5). Then, repeating the proof of Lemma 6.5 with the use of the \( L^\infty \) estimates of \( \psi \) obtained in Step 1 of the present proof, we conclude that \( \psi \in C^{(1-\alpha_{1,(P_{2,3})})}(\Omega^+(\phi) \cap \{c_2 - r > 6e\}) \) with

\[
(6.71) \quad \| \psi \|_{C^{(1-\alpha_{1,(P_{2,3})})}(\Omega^+(\phi) \cap \{c_2 - r > 6e\})} \leq C\sigma
\]

for \( C \) depending only on the data.

Step 3. Now we prove (6.11) for all \( s \in (0, c_2/2) \). If \( s \geq 6e \), then (6.11) follows from (6.71). Thus, it suffices to consider the case \( s \in (0, 6e) \) and show that

\[
(6.72) \quad \| \psi \|_{C^{(s/(2e + s/4))(\Omega^+(\phi) \cap \{s/2 < c_2 - r < s/2 \} \cap \{|\phi| < s/8\})}} \leq C(s) \sigma
\]

with \( C \) depending only on the data and \( s \). Indeed, (6.71) and (6.72) imply (6.11).

In order to prove (6.72), it suffices to prove the existence of \( C(s) \) depending only on the data and \( s \) such that

\[
(6.73) \quad \| \psi \|_{C^{(s/(2e + s/4))(\Omega^+(\phi) \cap \{s/2 < c_2 - r < s/2 \} \cap \{|\phi| < s/8\})}} \leq C(s) \| \psi \|_{L^\infty(\Omega^+(\phi) \cap \{s/2 < c_2 - r < s/2 \} \cap \{|\phi| < s/8\})}
\]

for all \( z := (\xi, \eta) \in \Omega^+(\phi) \cap \{s/2 < c_2 - r < s/2 \} \cap \{|\phi| < s/8\} \) with \( \text{dist}(z, \partial \Omega^+(\phi)) > s/8 \) and such that

\[
(6.74) \quad \| \psi \|_{C^{(s/(2e + s/4))(\Omega^+(\phi) \cap \{s/2 < c_2 - r < s/2 \} \cap \{|\phi| < s/8\})}} \leq C(s) \| \psi \|_{L^\infty(\Omega^+(\phi) \cap \{s/2 < c_2 - r < s/2 \} \cap \{|\phi| < s/8\})}
\]

for all \( z \in (\Gamma_{\text{shock}}(\phi) \cup \Gamma_{\text{wedge}}(\phi)) \cap \{s/2 < c_2 - r < s/2 \} \cap \{|\phi| < s/8\} \). Note that all the domains in (6.73) and (6.74) lie within \( \Omega^+(\phi) \cap \{s/4 < c_2 - r < s/4 \} \). We can assume that \( e < c_2/24 \). Since (6.1) is uniformly elliptic in \( \Omega^+(\phi) \cap \{s/4 < c_2 - r < s/4 \} \), linear and oblique with \( C^{(\alpha_{1,(P_{2,3})})}-\text{coefficients estimated in terms of the data, then (6.73) follows from Theorem A.1 and (6.74) follows from Theorem A.4 (in Appendix A). Since}

\[
\| \psi \|_{L^\infty(\Omega^+(\phi) \cap \{s/4 < c_2 - r < s/4 \})} \leq 1 \] by (6.9), the constants in the local estimates depend only on the ellipticity, the constants in Lemma 5.2(iii), and, for the case of (6.74), also
\( \tilde{A}^{(2)}_{ij} \) and \( \tilde{A}^{(1)}_i \) satisfy the property in Lemma 5.3(iii). The property in Lemma 5.3(iv) is now improved to

\[
\| \tilde{A}^{(2)}_{12}(x, y) \|_{C^1(B^{+1/2}_1 \setminus \tilde{B}^{1/2}_1)} \leq C |x|, \quad \|D(\tilde{A}^{(2)}_{12}, \tilde{A}^{(1)}_2)(x, y)\| \leq C |x|^{1/2}.
\]

Combining the estimates in Theorems A.1 and A.3, with the argument that has led to (6.34), we have

\[
\| \tilde{v} \|_{C^{2, \alpha}(B^{+1/2}_1 \setminus \tilde{B}^{1/2}_1)} \leq C,
\]

where \( C \) depends only on the data and \( \delta > 0 \) by (6.76), since \( \tilde{A}^{(2)}_{ij} \) and \( \tilde{A}^{(1)}_i \) satisfy (A.2) with the constants depending only on the data and \( \delta \). In particular, \( C \) in (6.82) is independent of \( \rho \).

We now use the domain \( D^{++} \) introduced in Step 2 of the proof of Lemma 6.6. We prove that, for any \( g \in C^\alpha(D^{++}) \) with \( \|g\|_{C^\alpha(D^{++})} \leq 1 \), there exists a unique solution \( w \in C^{2, \alpha}(D^{++}) \) of the problem:

\[
\begin{align*}
\tilde{A}^{(2)}_{11} w_{xx} + \tilde{A}^{(2)}_{22} w_{yy} + \tilde{A}^{(1)}_1 w_x &= g \quad \text{in } D^{++}, \\
 w &= 0 \quad \text{on } \partial D^{++} \cap \{x = 0, y > 0\}, \\
 w_y &= w_y = 0 \quad \text{on } \partial D^{++} \cap \{x > 0, y = 0\}, \\
 w &= 0 \quad \text{on } \partial D^{++} \cap \{x > 0, y > 0\},
\end{align*}
\]

with \( (A^{(2)}_{ij}, A^{(1)}_i) = (A^{(2)}_{ij}, A^{(1)}_i)(D, w, x, y) \). Moreover, we show

\[
\| w \|_{C^{2, \alpha}(\tilde{D}^{++})} \leq C,
\]

where \( C \) depends only on the data and is independent of \( \rho \). For that, similar to Step 2 of the proof of Lemma 6.6, we consider the even reflection \( D^+ \) of the set \( D^{++} \), and the even reflection of \( (v, g, \tilde{A}^{(2)}_{11}, \tilde{A}^{(2)}_{22}, \tilde{A}^{(1)}_1) \) from \( B^{++}_2 \) to \( \tilde{B}^{++}_2 \), without change of notation, where the even reflection of \( (A^{(2)}_{ij}, A^{(2)}_{ij}, A^{(1)}_i, A^{(1)}_i) \), which depends on \( (p, x, y) \), is defined by

\[
\tilde{A}^{(2)}_{ij}(p, x, -y) = \tilde{A}^{(2)}_{ij}(p, x, y), \quad \tilde{A}^{(1)}_i(p, x, -y) = \tilde{A}^{(1)}_i(p, x, y)
\]

for \( (x, y) \in B^{++}_2 \).

Also, denote by \( \tilde{v} \) the restriction of (the extended) \( v \) to \( \partial D^+ \). It follows from (6.78), (6.79) and (6.82) that \( \tilde{v} \in C^{2, \alpha}(\partial D^+) \) with

\[
\| \tilde{v} \|_{C^{2, \alpha}(\partial D^+)} \leq C,
\]

depending only on the data and \( \delta \). Furthermore, the extended \( g \) satisfies \( g \in C^\alpha(\tilde{D}^+) \) with

\[
\| g \|_{C^\alpha(\tilde{D}^+)} = \| g \|_{C^\alpha(D^{++})} \leq 1. \quad \text{The extended } \tilde{A}^{(2)}_{11}, \tilde{A}^{(2)}_{22}, \text{ and }
\]
$A^{(0)}_1$ satisfy (A.2) and (A.3) in $D^+$ with the same constants as the estimates satisfied by $A_{ij}$ and $A_i$ in $\Omega^+(\theta)$. We consider the Dirichlet problem
\[ A_{11}^{(0)} w_{xx} + A_{22}^{(0)} w_{yy} + A_1^{(0)} w_x = g \quad \text{in} \quad D^+, \]
\[ w = \tilde{v} \quad \text{on} \quad \partial D^+. \]
with $(A_{ij}^{(0)}, A_i^{(0)}) := (A_{ij}, A_i)(Dw, x, y)$. By the Maximum Principle,
\[ \|w\|_{L^{\infty}(D^+)} \leq \|\tilde{v}\|_{L^{\infty}(D^+)}. \]
Thus, using (6.88), we obtain an estimate of $\|w\|_{L^{\infty}(D^+)}$. Now, using Theorems A.1 and A.3 and the estimates of $\|g\|_{C^0(\overline{D^+})}$ and $\|\tilde{v}\|_{C^{2,\alpha}(\partial D^+)}$ discussed above, we obtain the a-priori estimate for the $C^{2,\alpha}$-solution $w$ of (6.89) and (6.90):
\[ \|w\|_{C^{2,\alpha}(D^+)} \leq C, \]
where $C$ depends only on the data and $\delta$. Moreover, for every $\tilde{w} \in C^{1,\alpha}(\overline{D^+})$, the existence of a unique solution $w \in C^{2,\alpha}(D^+)$ of the linear Dirichlet problem, obtained by substituting $\tilde{w}$ into the coefficients of (6.89), follows from [20, Th. 6.8]. Now, by a standard application of the Leray-Schauder Theorem, there exists a unique solution $w \in C^{2,\alpha}(D^+)$ of the Dirichlet problem (6.89) which satisfies (6.91).

From the structure of (6.89), especially the fact that $\tilde{w}^{(0)}$, $\tilde{w}^{(1)}$, and $\tilde{w}^{(2)}$ are independent of $p_2$ by Lemma 5.3 (iii), and from the symmetry of the domain and the coefficients and right-hand sides obtained by the even extension, it follows that $\tilde{w}$, defined by $\tilde{w}(x, y) = w(x, -y)$, is also a solution of (6.89), (6.90). By uniqueness for problem (6.89), (6.90), we find $w(x, y) = w(x, -y)$ in $D^+$. Thus, $w$ restricted to $D^{++}$ is a solution of (6.83)–(6.86), where (6.84) follows from (6.78) and (6.90). Moreover, (6.91) implies (6.87).

The uniqueness of a solution $w \in C^{2,\alpha}(D^{++})$ of (6.83)–(6.86) follows from the Comparison Principle (Lemma 6.3).

Now we prove the existence of a solution $w \in C^{2,\alpha}(D^{++})$ of the problem:
\[ A_{11}^{(0)} w_{xx} + 2A_{12}^{(0)} w_{xy} + A_{22}^{(0)} w_{yy} + A_1^{(0)} w_x + A_2^{(0)} w_y = 0 \quad \text{in} \quad D^{++}, \]
\[ w = 0 \quad \text{on} \quad \partial D^{++} \cap \{x = 0, y > 0\}, \]
\[ w_v = w_y = 0 \quad \text{on} \quad \partial D^{++} \cap \{y = 0, x > 0\}, \]
\[ w = v \quad \text{on} \quad \partial D^{++} \cap \{x > 0, y = 0\}, \]
where $(A_{ij}^{(0)}, A_i^{(0)}) := (A_{ij}, A_i)(Dw, x, y)$. Moreover, we prove that $w$ satisfies
\[ \|w\|_{C^{2,\alpha}(D^{++})} \leq C \]
for $C > 0$ depending only on the data and $\delta$. 

\[ (6.93) \]
Since $\psi$ is a solution of the linear equation (6.13) for $\hat{\psi} = \psi$ and satisfies the boundary conditions (5.30)-(5.33), it follows from Lemma 6.7 that $\psi$ satisfies (6.46) with constant $C$ depending only on the data and $\delta$.

Now we follow the argument of Lemma 6.7 (Step 4): We consider cases (i)-(iii) and define the function $v(X, Y)$ by (6.64). Then $v$ is a solution of the nonlinear (6.2). We apply the estimates in Appendix A. From Lemma 5.3 and the properties of the Laplacian in polar coordinates, the coefficients of (6.2) satisfy (A.2)-(A.3) with $\lambda$ depending only on the data and $\delta$. It is easy to see that $v$ defined by (6.64) satisfies an equation of the similar structure and properties (A.2)-(A.3) with the same $\lambda$, where we use that $0 \leq \bar{d} \leq 1$. Also, $v$ satisfies the same boundary conditions as in the proof of Lemma 6.7 (Step 4). Furthermore, since $\psi$ satisfies (6.46), we obtain the $L^\infty$ estimates of $v$ in terms of the data and $\delta$, e.g., $v$ satisfies (6.65) in case (iii). Now we obtain the $C^{2, \alpha}$-estimates of $v$ by using Theorem A.1 for case (i), Theorem A.3 for case (ii), and Theorem A.4 for case (iii). Writing these estimates in terms of $\psi$, we obtain (6.96), similar to the proof of Lemma 6.7 (Step 4).

**Step 6.** Finally, we prove the comparison principle, assertion (iv). The function $u = \psi_1 - \psi_2$ is a solution of a linear problem of form (6.13), (5.30), (5.32), and (5.33) with right-hand sides $N_\delta(\psi_1) - N_\delta(\psi_2)$ and $B_k(\psi_1) - B_k(\psi_2)$ for $k = 1, 2, 3$, respectively, and $u \geq 0$ on $\Gamma_{\text{sonic}}$. Now the comparison principle follows from Lemma 6.3.

Using Lemma 6.8 and the definition of map $\hat{J}$ in (6.12), and using Lemma 6.9 and the Leray-Schauder Theorem, we conclude the proof of Proposition 6.1.

**Proposition 6.2.** Let $\sigma, \varepsilon, M_1$, and $M_2$ be as in Proposition 6.1. Then there exists a solution $\psi \in C(\Omega^+(\phi)) \cap C^1(\Omega^+(\phi) \setminus \Gamma_{\text{sonic}}) \cap C^2(\Omega^+(\phi))$ of problem (5.29)-(5.33) such that $\psi$ satisfies (6.9)-(6.11).

**Proof.** Let $\delta \in (0, \delta_0)$. Let $\psi_j$ be a solution of (6.1) and (5.30)-(5.33) obtained in Proposition 6.1. Using (6.11), we can find a sequence $\delta_j$ for $j = 1, \ldots$ and $\psi \in C^1(\Omega^+(\phi) \setminus \Gamma_{\text{sonic}}) \cap C^2(\Omega^+(\phi))$ such that, as $j \to \infty$, we have

(i) $\delta_j \to 0$;

(ii) $\psi_{s_j} \to \psi$ in $C^1(\Omega^+(\phi))$ for every $s \in (0, c_2/2)$, where $\Omega^+_s(\phi) = \Omega^+(\phi) \cap \{c_2 - r > s\}$;

(iii) $\psi_{s_j} \to \psi$ in $C^2(\Omega^+_s(\phi))$ for every compact $K \subset \Omega^+_s(\phi)$.

Then, since each $\psi_{s_j}$ satisfies (6.1), (5.30), and (5.32)-(5.33), it follows that $\psi$ satisfies (5.29), (5.30), and (5.32)-(5.33). Also, since each $\psi_{s_j}$ satisfies (6.9)-(6.11),
LEMMA 7.2. A solution \( \psi \in C(\Omega^+(\phi)) \cap C^1(\Omega^+(\phi) \setminus \Gamma_{\text{sonic}}) \cap C^2(\Omega^+(\phi)) \) of (5.29)-(5.33) is unique.

Proof. If \( \psi_1 \) and \( \psi_2 \) are two solutions, then we repeat the proof of Lemma 7.1 to show that \( \psi_1 - \psi_2 \) cannot achieve a negative minimum in \( \Omega^+(\phi) \) and in the relative interiors of \( \Gamma_{\text{shock}}(\phi) \) and \( \Gamma_{\text{wedge}} \). Now (5.29) is linear, uniformly elliptic near \( \Sigma_0 \) (by Lemma 5.2), and the function \( \psi_1 - \psi_2 \) is \( C^1 \) up to the boundary in a neighborhood of \( \Sigma_0 \). Then the boundary condition (5.33) combined with Hopf's Lemma yields that \( \psi_1 - \psi_2 \) cannot achieve a minimum in the relative interior of \( \Sigma_0 \). By the argument of Step (iii) in the proof of Lemma 6.3, \( \psi_1 - \psi_2 \) cannot achieve a negative minimum at the points \( P_2 \) and \( P_3 \). Thus, \( \psi_1 \geq \psi_2 \) in \( \Omega^+(\phi) \) and, by symmetry, the opposite is also true. \( \square \)

LEMMA 7.3. There exists \( \tilde{C} > 0 \) depending only on the data such that, if \( \sigma, \epsilon, M_1, \) and \( M_2 \) satisfy (5.16), the solution \( \psi \in C(\Omega^+(\phi)) \cap C^1(\Omega^+(\phi) \setminus \Gamma_{\text{sonic}}) \cap C^2(\Omega^+(\phi)) \) of (5.29)-(5.33) satisfies

\[
0 \leq \psi(x, y) \leq \frac{3}{5(y+1)} x^2 \quad \text{in} \quad \Omega^+(\phi) := \Omega^+_{\text{shock}}(\phi).
\]

Proof. We first notice that \( \psi \geq 0 \) in \( \Omega^+(\phi) \) by Proposition 6.2. Now we make an estimate (7.1). Set

\[
w(x, y) := \frac{3}{5(y+1)} x^2.
\]

We first show that \( w \) is a supersolution of (5.29). Since (5.29) rewritten in the \( (x, y) \)-coordinates in \( \Omega^+(\phi) \) has form (5.42), we write it as

\[
\mathcal{N}_1(\psi) + \mathcal{N}_2(\psi) = 0,
\]

where

\[
\mathcal{N}_1(\psi) = \left( 2x - (y+1) \frac{x}{5(y+1)} \right) \frac{x}{5(y+1)} \psi_{xx} + \frac{1}{5(y+1)} \psi_{xy} - \psi_x,
\]

\[
\mathcal{N}_2(\psi) = O_1^\phi \psi_{xx} + O_2^\phi \psi_{xy} + O_3^\phi \psi_{yy} - O_4^\phi \psi_x + O_5^\phi \psi_y.
\]

Now we substitute \( w(x, y) \). By (5.37),

\[
\mathcal{N}_1(\psi) = \xi^1 \left( \frac{6}{5(y+1)} \right) = \frac{6}{5(y+1)};
\]

thus

\[
\mathcal{N}_1(w) = -\frac{6}{25(y+1)} x.
\]

Using (5.44), we have

\[
|\mathcal{N}_2(w)| \leq \frac{6}{5(y+1)} O_1^\phi (Dw, x, y) + \frac{6x}{5(y+1)} O_2^\phi (Dw, x, y) \leq C x^{3/2} \leq C \epsilon^{1/2} x,
\]
where the last inequality holds since \( x \in (0, 2\varepsilon) \) in \( \Omega'(\phi) \). Thus, if \( \varepsilon \) is small, we find

\[
N(w) < 0 \quad \text{in} \quad \Omega'(\phi).
\]

The required smallness of \( \varepsilon \) is achieved if (5.16) is satisfied with large \( \hat{C} \).

Also, \( w \) is a supersolution of (5.30): Indeed, since (5.30) rewritten in the \((x, y)\)-coordinates has form (6.6), estimates (6.8) hold, and \( x > 0 \), we find

\[
M(w) = \beta_1(x, y) \frac{6}{5(y + 1)} x + \beta_3(x, y) \frac{3}{5(y + 1)} x^2 < 0 \quad \text{on} \quad \Gamma_{\text{shock}}(\phi) \cap \partial \Omega'.
\]

Moreover, on \( \Gamma_{\text{wedge}} \), \( w_w = w_y = 0 = \psi_w \). Furthermore, \( w = 0 = \psi \) on \( \Gamma_{\text{sonic}} \) and, by (6.9), \( \psi \leq w \) on \( \{ x = 2\varepsilon \} \) if

\[
C \sigma \leq \varepsilon^2,
\]

where \( C \) is a large constant depending only on the data, i.e., if (5.16) is satisfied with large \( \hat{C} \). Thus, \( \psi \leq w \) in \( \Omega'(\phi) \) by Lemma 7.1.

We now estimate the norm \( \| \psi \|_{2,0, \Omega'(\phi)}^{(par)} \) in the subdomain \( \hat{\Omega}'(\phi) := \Omega^+(\phi) \cap \{ c_2 - r < 2\varepsilon \} \) of \( \Omega'(\phi) := \Omega^+(\phi) \cap \{ c_2 - r < 2\varepsilon \} \).

**Lemma 7.4.** There exist \( \hat{C}, C > 0 \) depending only on the data such that, if \( \sigma, \varepsilon, M_1, \) and \( M_2 \) satisfy (5.16), the solution \( \psi \in C(\Omega^+(\phi)) \cap C^1(\Omega^+(\phi) \setminus \Gamma_{\text{sonic}}) \cap C^2(\Omega^+(\phi)) \) of (5.29)–(5.33) satisfies

\[
\| \psi \|_{2,0, \hat{\Omega}'(\phi)}^{(par)} \leq C.
\]

**Proof.** We assume \( \hat{C} \) in (5.16) is sufficiently large so that \( \sigma, \varepsilon, M_1, \) and \( M_2 \) satisfy the conditions of Lemma 7.3.

Step 1. We work in the \((x, y)\)-coordinates and, in particular, we use (5.25)–(5.26). We can assume \( \varepsilon < \kappa/20 \), which can be achieved by increasing \( \hat{C} \) in (5.16).

For \( z := (x, y) \in \hat{\Omega}'(\phi) \) and \( \rho \in (0, 1) \), define

\[
\hat{R}_{z, \rho} := \{(s, t) : |x - s| < \frac{\rho}{4} x, |y - t| < \frac{\rho}{4} \sqrt{x} \}, \quad R_{z, \rho} := \hat{R}_{z, \rho} \cap \Omega^+(\phi).
\]

Since \( \Omega'(\phi) = \Omega^+(\phi) \cap \{ c_2 - r < 2\varepsilon \} \), then, for any \( z \in \hat{\Omega}'(\phi) \) and \( \rho \in (0, 1) \),

\[
R_{z, \rho} \subset \Omega^+(\phi) \cap \{(s, t) : \frac{3}{4} x < s < \frac{5}{4} x \} \subset \Omega'(\phi).
\]

For any \( z \in \hat{\Omega}'(\phi) \), we have at least one of the following three cases:

(i) \( R_{z, 1/10} = \hat{R}_{z, 1/10} \);

(ii) \( z \in R_{z, 1/2} \) for \( z_w = (x, 0) \in \Gamma_{\text{wedge}} \);

(iii) \( z \in R_{z, 1/2} \) for \( z_y = (x, \hat{f}_y(x)) \in \Gamma_{\text{shock}}(\phi) \).
Thus, it suffices to make the local estimates of $D\psi$ and $D^2\psi$ in the following rectangles with $z_0 := (x_0, y_0)$:

(i) $R_{z_0, 1/20}$ for $z_0 \in \hat{\Omega}'(\phi)$ and $R_{z_0, 1/10} = R_{z_0, 1/10}$;

(ii) $R_{z_0, 1/2}$ for $z_0 \in \Gamma_{\text{wedge}} \cap \{x < \varepsilon\}$;

(iii) $R_{z_0, 1/2}$ for $z_0 \in \Gamma_{\text{shock}}(\phi) \cap \{x < \varepsilon\}$.

**Step 2.** We first consider case (i) in Step 1. Then

$$R_{z_0, 1/10} = \left\{ \left( x_0 + \frac{x_0}{4} S, y_0 + \frac{\sqrt{x_0}}{4} T \right) : (S, T) \in Q_{1/10} \right\},$$

where $Q_{\rho} := (-\rho, \rho)^2$ for $\rho > 0$.

Rescale $\psi$ in $R_{z_0, 1/10}$ by defining

$$\psi^{(z_0)}(S, T) := \frac{1}{x_0} \psi \left( \frac{x_0}{x_0} S, y_0 + \frac{\sqrt{x_0}}{4} T \right) \quad \text{for } (S, T) \in Q_{1/10}.$$

Then, by (7.1) and (7.4),

$$\|\psi^{(z_0)}\|_{C(Q_{1/10})} \leq 1/(\gamma + 1).$$

Moreover, since $\psi$ satisfies (5.42)-(5.43) in $R_{z_0, 1/10}$, then $\psi^{(z_0)}$ satisfies

$$\left( 1 + \frac{1}{4} S \right) \left( 2 - (\gamma + 1) \xi_1 \left( \frac{4\psi^{(z_0)}}{S} \right) + x_0 O_{1, \phi}^{(z_0)} \right) \psi^{(z_0)} + x_0 O_{2, \phi}^{(z_0)} = 0$$

in $Q_{1/10}$, where

$$\frac{1}{c_2} \frac{1 + S/4}{2c_2} \frac{\gamma + 1}{\gamma} \left( \frac{4\psi^{(z_0)}}{\phi_T^{(z_0)}} \right) \frac{8x_0}{(c_2 - x_0(1+S/4))^2} \phi_T^{(z_0)}.$$
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\[
\begin{align*}
\partial_{\phi,z_0}^2 & (p, S, T) = \frac{1}{c_2(c_2 - x_0(1 + S/4))^2} \{ (1 + S/4)(2c_2 - x_0(1 + S/4)) \\
& - (\gamma - 1) \left( x_0 \phi_{z_0}^2 + (c_2 - x_0(1 + S/4))(1 + S/4) \xi_1 \left( \frac{4p_1}{1 + S/4} \right) + 8x_0 \phi_{\phi_{z_0}}^2 \right) \}, \\
\partial_{\phi,z_0} & (p, S, T) = \frac{1}{c_2 - x_0(1 + S/4)} \left\{ 1 + \frac{S/4 - \gamma - 1}{c_2} \left( x_0 \phi_{z_0}^2 + 8x_0 \phi_{\phi_{z_0}}^2 \right) \right\}, \\
\partial_{\phi,z_0}^2 & (p, S, T) = \frac{8}{c_2(c_2 - x_0(1 + S/4))^2} \left( 4x_0 \phi_{\phi_{z_0}}^2 + 2c_2 - 2x_0(1 + S/4) \phi_{z_0}^2 \right),
\end{align*}
\]

where \( \phi_{z_0} \) is the rescaled \( \phi \) as in (7.5). By (7.4) and \( \phi \in \mathcal{U} \), we have

\[ ||\phi_{z_0}||_{C^2, a(Q_{1/10})} \leq CM_1, \]

and thus

\[ ||\phi_{z_0}||_{C^1(\mathcal{O}_{1/10} \times \mathbb{R}^2)} \leq C(1 + M_1^2), \quad k = 1, \ldots, 5. \]

Now, since every term \( G_k(\phi_{z_0}) \) in (7.7) is multiplied by \( x_0^{\beta_k} \) with \( \beta_k \geq 1 \) and \( x_0 \in (0, \varepsilon) \), condition (5.16) (possibly after increasing \( \tilde{C} \)) depending only on the data implies that (7.7) satisfies conditions (A.2, A.3) in \( Q_{1/10} \) with \( \lambda > 0 \) depending only on \( c_2, \) i.e., on the data by (4.31). Then, using Theorem A.1 and (7.6), we find

\[ ||\psi_{z_0}||_{C^2, a(Q_{1/20})} \leq C. \]

Step 3. We then consider case (ii) in Step 1. Let \( z_0 \in \Gamma_{\text{wedge}} \cap \{ x < \varepsilon \} \). Using (5.25) and assuming that \( \sigma \) and \( \varepsilon \) are sufficiently small depending only on the data, we have \( R_{z_0,1} \cap \partial \Omega^+(\phi) \subset \Gamma_{\text{wedge}} \) and thus, for any \( \rho \in (0, 1], \)

\[ R_{z_0,\rho} = \left\{ \left( x_0 + \frac{x_0}{4} S, y_0 + \frac{\sqrt{x_0}}{4} T \right) : (S, T) \in Q_{\rho} \cap \{ T > 0 \} \right\}. \]

The choice of parameters for that can be made as follows: First choose \( \sigma \) small so that \( |\xi - \xi_1| \leq |\xi|/10 \), where \( \xi \) is defined by (3.3), which is possible since \( \xi_1 \to \xi \) as \( \theta_{\text{in}} \to \pi/2 \), and then choose \( \varepsilon < (|\xi|/10)^2 \).

Define \( \psi_{z_0}(S, T) \) by (7.5) for \( (S, T) \in Q_1 \cap \{ T > 0 \} \). Then, by (7.1) and (7.4),

\[ ||\psi_{z_0}||_{C(\mathcal{O}(\cap T > 0))} \leq 1/(\gamma + 1). \]
Moreover, similar to Step 2, \( \psi^{(z_0)} \) satisfies (7.7) in \( Q_1 \cap \{ T > 0 \} \), and the terms \( \partial^k \psi^{(z_0)} \) satisfy estimate (7.9) in \( Q_1 \cap \{ T > 0 \} \). Then, as in Step 2, we conclude that (7.7) satisfies conditions (A.2) \((A.3)\) in \( Q_1 \cap \{ T > 0 \} \) if (5.16) holds with sufficiently large \( \hat{C} \). Moreover, since \( \psi \) satisfies (5.32), it follows that
\[
\partial_T \psi^{(z_0)} = 0 \quad \text{on} \quad \{ T = 0 \} \cap Q_1.
\]

Then, from Theorem A.4,
\[
\| \psi^{(z_0)} \|_{C^{2, \alpha}(Q_{1/2} \cap \{ T \geq 0 \})} \leq C.
\]

Step 4. We now consider case (iii) in Step 1. Let \( z_0 \in \Gamma_{\text{shock}}(\psi) \cap \{ x < \epsilon \} \).

Using (5.25) and the fact that \( y_0 = \hat{f}_\phi(x_0) \) for \( z_0 \in \Gamma_{\text{shock}}(\psi) \cap \{ x < \epsilon \} \), and assuming that \( \sigma \) and \( \epsilon \) are small as in Step 3, we have \( R_{z_0, 1} \cap \partial \Omega^+ (\phi) \subset \Gamma_{\text{shock}}(\psi) \) and thus, for any \( \rho \in (0, 1) \),
\[
R_{z_0, \rho} = \left\{ \left( x_0 + \frac{x_0}{4} S, y_0 + \frac{\sqrt{x_0}}{4} T \right) : (S, T) \in Q_\rho \cap \{ T < \epsilon^{1/4} F^{(z_0)}(S) \} \right\}
\]
with
\[
F^{(z_0)}(S) = 4\frac{\hat{f}_\phi (x_0 + \frac{x_0}{4} S)}{\epsilon^{1/4} \sqrt{x_0}} - \hat{f}_\phi (x_0).
\]

Then we use (5.27) and \( x_0 \in (0, 2\epsilon) \) to obtain
\[
F^{(z_0)}(0) = 0,
\]
\[
\| F^{(z_0)} \|_{C^1((-1/2, 1/2)^2)} \leq \frac{\| \hat{f}_\phi \|_{L^\infty((0, 2\epsilon) \times x_0)}}{\epsilon^{1/4} \sqrt{x_0}} \leq C(1 + M_1 \epsilon^{1/4}).
\]
\[
\| F^{(z_0)} \|_{C^2((-1/2, 1/2)^2)} \leq \frac{\| \hat{f}_\phi \|_{L^\infty((0, 2\epsilon) \times x_0)}^2}{4 \epsilon^{1/4} \sqrt{x_0}} \leq C(1 + M_1) \epsilon^{5/4},
\]
and thus, from (5.16),
\[
\| F^{(z_0)} \|_{C^{2, \alpha}((-1/2, 1/2)^2)} \leq C / \hat{C} \leq 1
\]
if \( \hat{C} \) is large. Define \( \psi^{(z_0)}(S, T) \) by (7.5) for \( (S, T) \in Q_1 \cap \{ T < \epsilon^{1/4} F^{(z_0)}(S) \} \).

Then, by (7.1) and (7.4),
\[
\| \psi^{(z_0)} \|_{C^0(\Gamma_{\text{shock}}(\psi))} \leq \epsilon/ (y + 1).
\]

Similar to Steps 2,3, \( \psi^{(z_0)} \) satisfies (7.7) in \( Q_1 \cap \{ T < \epsilon^{1/4} F^{(z_0)}(S) \} \) and the terms \( \partial^k \psi^{(z_0)} \) satisfy estimate (7.9) in \( Q_1 \cap \{ T < \epsilon^{1/4} F^{(z_0)}(S) \} \). Then, as in Steps 2,3, we conclude that (7.7) satisfies conditions (A.2) \((A.3)\) in \( Q_1 \cap \{ T < \epsilon^{1/4} F^{(z_0)}(S) \} \) if (5.16) holds with sufficiently large \( \hat{C} \). Moreover, \( \psi \) satisfies (5.30) on \( \Gamma_{\text{shock}}(\psi) \),
which can be written in form (6.6) on \(\Gamma_{\text{shock}}(\phi) \cap \mathcal{D}'\). This implies that \(\psi(\varepsilon_0)\)
 satisfies
\[
\partial_T \psi(\varepsilon_0) = \varepsilon^{1/4} \left( B_2 \partial_T \psi(\varepsilon_0) + B_3 \psi(\varepsilon_0) \right)
\]
on \(\{ T = \varepsilon^{1/4} F(\varepsilon_0)(S) \} \cap Q_{1/2}, \)
where
\[
B_2(S, T) = -\frac{\sqrt{x_0}}{\varepsilon^{1/4} \beta_1} \left( \frac{x_0}{4} S, y_0 + \frac{\sqrt{x_0}}{4} T \right)
\]
and
\[
B_3(S, T) = -\frac{x_0}{4\varepsilon^{1/4} \beta_1} \left( \frac{x_0}{4} S, y_0 + \frac{\sqrt{x_0}}{4} T \right)
\]
From (6.8),
\[
( B_2, B_3 )_{1, \sigma, \Omega \cap \{ T \leq \varepsilon^{1/4} F(\varepsilon_0)(S) \} } \leq C \varepsilon^{1/4} M_1 \leq \frac{C}{\tilde{C}} \leq 1.
\]
Now, if \(\varepsilon\) is sufficiently small, it follows from Theorem A.2 that
\[
\| \psi(\varepsilon_0) \|_{C^{2, \sigma}(\Omega_{1/2} \cap \{ T \leq \varepsilon^{1/4} F(\varepsilon_0)(S) \})} \leq C.
\]
The required smallness of \(\varepsilon\) is achieved by choosing large \(\tilde{C}\) in (5.16).

Step 5. Combining (7.10), (7.12), and (7.15) with an argument similar to the proof of [20, Th. 4.8] (see also the proof of Lemma A.3 below), we obtain (7.2). \(\square\)

Now we define the extension of solution \(\psi\) from the domain \(\Omega^+(\phi)\) to the domain \(\Omega\).

**Lemma 7.5.** There exist \(\tilde{C}, C_1 > 0\) depending only on the data such that, if 
\(\sigma, \varepsilon, M_1, M_2\) satisfy (5.16), there exists \(C_2(\varepsilon)\) depending only on the data and \(\varepsilon\) and, for any \(\psi \in \mathcal{K}\), there exists an extension operator
\[
\mathcal{P}_\phi : C^{1, \sigma}(\Omega^+(\phi)) \cap C^{2, \sigma}(\Omega^+(\phi) \setminus \Gamma_{\text{sonic}} \cup \Sigma_0) \rightarrow C^{1, \sigma}(\mathcal{D}) \cap C^{2, \sigma}(\mathcal{D})
\]
satisfying the following two properties:

(i) If \(\psi \in C^{1, \sigma}(\Omega^+(\phi)) \cap C^{2, \sigma}(\Omega^+(\phi) \setminus \Gamma_{\text{sonic}} \cup \Sigma_0)\) is a solution of problem
(5.29)–(5.33), then
\[
\| \mathcal{P}_\phi \psi \|_{2, \sigma, \Omega} \leq C_1\sigma.
\]
(7.16)
\[
\| \mathcal{P}_\phi \psi \|_{\omega, \Omega} \leq C_2(\varepsilon)\sigma;
\]
(7.17)
(ii) Let \(\beta \in (0, \alpha)\). If a sequence \(\phi_k \in \mathcal{K}\) converges to \(\phi\) in \(C^{1, \beta}(\mathcal{D})\), then \(\phi \in \mathcal{K}\).
Furthermore, if \(\psi_k \in C^{1, \sigma}(\Omega^+(\phi_k)) \cap C^{2, \sigma}(\Omega^+(\phi_k) \setminus \Gamma_{\text{sonic}} \cup \Sigma_0)\) and \(\psi \in C^{1, \sigma}(\Omega^+(\phi)) \cap C^{2, \sigma}(\Omega^+(\phi) \setminus \Gamma_{\text{sonic}} \cup \Sigma_0)\) are the solutions of problems
(5.29)–(5.33) for \(\phi_k\) and \(\phi\) respectively, then
\[
\mathcal{P}_{\phi_k} \psi_k \rightarrow \mathcal{P}_\phi \psi \text{ in } C^{1, \beta}(\mathcal{D}).
\]
Furthermore, using the second estimate in (5.27), noting that \( M_{2\sigma} \leq 1 \) by (5.16), and using the definition of \( \mathcal{P}_{\phi} \) and the fact that the change of coordinates \((x, y) \to (\xi, \eta)\) is smooth and invertible in \( S \cap \{ v_{2} < x < \kappa \} \), we find that, in the \((\xi, \eta)\)-coordinates,

\[
\| \mathcal{P}_{\phi}^{1/2} \psi \|_{C^{2,\alpha}(\Omega \cap \{ v_{2} \leq r \leq \kappa \})} \leq C \| \psi \|_{C^{2,\alpha}(\Omega \cap \{ v_{2} \leq r \leq \kappa \})}.
\]

**Step 3.** Now we define an extension operator in the \((\xi, \eta)\)-coordinates. Let

\[
\mathcal{E}_{2} : C^{1}([-1, 1] \times (-v_{2}, \eta_{1})) \cap C^{2}([0, 1] \times (-v_{2}, \eta_{1}))
\]

\[
\to C^{1}([-1, 1] \times (-v_{2}, \eta_{1})) \cap C^{2}([-1, 1] \times (-v_{2}, \eta_{1}))
\]

be defined by

\[
\mathcal{E}_{2} \psi(X, Y) := \sum_{i=1}^{3} a_{i} \psi \left( -\frac{X}{i}, Y \right) \quad \text{for} \quad (X, Y) \in (-1, 0) \times (-v_{2}, \eta_{1}),
\]

where \( a_{1}, a_{2}, \) and \( a_{3} \) are the same as in (7.22).

Let \( \tilde{\Omega}_{2} := \Omega^{+}(\phi) \cap \{ 0 \leq \eta \leq \eta_{1} \} \). Define the mapping \( \Psi : \tilde{\Omega}_{2} \to (0, 1) \times (-v_{2}, \eta_{1}) \) by

\[
\Psi(\xi, \eta) := \left( \frac{\xi - f_{\phi}(\eta)}{\eta \cot \theta_{w} - f_{\phi}(\eta)}, \eta \right),
\]

where \( f_{\phi}(\cdot) \) is the function from (5.21), (5.22). Then the inverse of \( \Psi \) is

\[
\Psi^{-1}(X, Y) = \left( f_{\phi}(Y) + X Y \cot \theta_{w} - f_{\phi}(Y), Y \right),
\]

and thus, from (5.24),

\[
\| \psi \|_{C^{1,\alpha,\Omega_{2}^{\phi}}} \leq C \| \psi \|_{C_{2,\alpha,\Omega_{2}^{\phi}}(-v_{2}, \eta_{1})} \leq C \| \psi \|_{C^{2,\alpha}(\Omega \cap \{ v_{2} < \eta < \eta_{1} \})} \leq C.
\]

Moreover, by (5.24), for sufficiently small \( \epsilon \) and \( \sigma \) (which are achieved by choosing large \( C \) in (5.16)), we have \( \Omega \cap \{ -v_{2} < \eta < \eta_{1} \} \subset \psi^{-1}(\{ -1, 1 \} \times \{ -v_{2}, \eta_{1} \}) \). Define

\[
\mathcal{P}_{\phi}^{2} := \mathcal{E}_{2}(\psi \circ \psi^{-1}) \circ \psi \quad \text{on} \quad \Omega \cap \{ -v_{2} < \eta < \eta_{1} \}.
\]

Then \( \mathcal{P}_{\phi}^{2} \psi \in C^{1,\alpha,\Omega_{2}^{\phi}} \cap C^{2,\alpha,\Omega_{2}^{\phi}} \) since \( \Omega \setminus \Omega^{+}(\phi) \subset \Omega \cap \{ -v_{2} < \eta < \eta_{1} \} \). Furthermore, using (7.27) and the definition of \( \mathcal{P}_{\phi}^{2} \), we find that, for any \( s \in (-v_{2}, \eta_{1}) \),

\[
\| \psi \|_{C^{1,\alpha,\Omega_{2}^{\phi}}} \leq C(\eta_{1} - s) \| \psi \|_{C_{2,\alpha,\Omega^{+}(\phi) \cap \{ \eta \leq s \}}}.
\]

Choosing \( C \) large in (5.16), we have \( \epsilon < \kappa/100 \). Then (5.25) implies that there exists a unique point \( P' = \Gamma_{\text{shock}}(\phi) \cap \{ v_{2} - r = \kappa/8 \} \). Let \( P' = (\xi', \eta') \) in the \((\xi, \eta)\)-coordinates. Then \( \eta' > 0 \). Using (7.18) and (7.20), we find

\[
\Omega^{+}(\phi) \cap \{ \eta = -v_{2} \} \cup \{ \eta = \eta_{1} \}.
\]
Also, $\kappa/C \leq \eta_1 - \eta' \leq C\kappa$ by (5.22), (5.24), and (4.3). These facts and (7.28) with $s = \eta'$ imply

\begin{equation}
\|\mathcal{P}_{\phi}^2 \psi\|_{2,\alpha,\Omega\cap(c_2-r>\kappa/8)} \leq C \|\psi\|_{2,\alpha,\Omega^+(\phi)\cap(c_2-r>\kappa/8)}.
\end{equation}

**Step 4.** Finally, we choose a cutoff function $\xi \in C^\infty(\mathbb{R})$ satisfying

$\xi \equiv 1$ on $(-\infty, \kappa/4)$, $\xi \equiv 0$ on $(3\kappa/4, \infty)$, $\xi' \leq 0$ on $\mathbb{R}$,

and define

$\mathcal{P}_{\phi} \psi := \xi(c_2-r)\mathcal{P}_{\phi}^1 \psi + (1 - \xi(c_2-r))\mathcal{P}_{\phi}^2 \psi$ in $\mathbb{R}$.

Since $\mathcal{P}_{\phi}^k \psi = \psi$ on $\Omega^+(\phi)$ for $k = 1, 2$, so is $\mathcal{P}_{\phi} \psi$. Also, from the properties of $\mathcal{P}_{\phi}^k$ above, $\mathcal{P}_{\phi} \psi \in C^{1,\alpha}(\mathbb{R}) \cap C^{2,\alpha}(\mathbb{R})$ if

$\psi \in C^{1,\alpha}(\Omega^+(\phi)) \cap C^{2,\alpha}(\Omega^+(\phi) \setminus \Gamma_{\text{sonic}} \cup \Sigma_0)$.

If such $\psi$ is a solution of (5.29)–(5.33), then we prove (7.16) (7.17): $\mathcal{P}_{\phi} \psi \equiv \mathcal{P}_{\phi}^1 \psi$ on $\mathbb{R}$ by the definition of $\xi$ and by $\varepsilon < \kappa/100$. Thus, since (7.16) has been proved in Step 2 for $\mathcal{P}_{\phi}^1 \psi$, we obtain (7.16) for $\mathcal{P}_{\phi} \psi$. Also, $\psi$ satisfies (6.11) by Proposition 6.2. Using (6.11) with $s = \varepsilon/2$, (7.26), and (7.29), we obtain (7.17). Assertion (i) is then proved.

**Step 5.** Finally we prove assertion (ii). Let $\phi_k \in \mathfrak{K}$ converge to $\phi$ in $C^{1,\beta}(\mathbb{R})$. Then obviously $\phi \in \mathfrak{K}$. By (5.20) (5.22), it follows that

\begin{equation}
\mathcal{P}_{\phi_k} \psi_k \rightarrow \mathcal{P}_{\phi} \psi \text{ in } C^{1,\beta}([-\nu_2, \eta_1]),
\end{equation}

where $\mathcal{P}_{\phi_k}, \mathcal{P}_{\phi} \in C^{1,\alpha}([-\nu_2, \eta_1])$ are the functions from (5.21) corresponding to $\phi_k, \phi$, respectively. Let $\psi_k, \psi \in C^{1,\alpha}(\Omega^+(\phi_k)) \cap C^{2,\alpha}(\Omega^+(\phi_k) \setminus \Gamma_{\text{sonic}} \cup \Sigma_0)$ be the solutions of problems (5.29)–(5.33) for $\phi_k, \phi$. Let $\{\psi_{k_m}\}$ be any subsequence of $\{\psi_k\}$. By (7.16) (7.17), it follows that there exist a further subsequence $\{\phi_{k_{m_n}}\}$ and a function $\tilde{\psi} \in C^{1,\alpha}(\mathbb{R}) \cap C^{2,\alpha}(\mathbb{R})$ such that

$\mathcal{P}_{\phi_{k_{m_n}}} \psi_{k_{m_n}} \rightarrow \tilde{\psi}$ in $C^{2,\alpha/2}$ on compact subsets of $\mathbb{R}$ and in $C^{1,\alpha/2}(\mathbb{R})$.

Then, using (7.30) and the convergence $\phi_k \rightarrow \phi$ in $C^{1,\beta}(\mathbb{R})$, we prove (by the argument as in [10, p. 479]) that $\tilde{\psi}$ is a solution of problem (5.29)–(5.33) for $\phi$.

By uniqueness in Lemma 7.2, $\tilde{\psi} = \psi$ in $\Omega^+(\phi)$. Now, using (7.30) and the explicit definitions of extensions $\mathcal{P}_{\phi}^1$ and $\mathcal{P}_{\phi}^2$, it follows by the argument as in [10,
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with
\[
\hat{O}_k(x, y) = O^\psi_k(D\psi(x, y), x, y) \quad \text{for } k = 1, \ldots, 5.
\]

for \(O^\psi_k\) defined by (5.43) with \(\phi = \psi\). From (5.37), we have
\[
\xi_1(A) = A.
\]

Thus we can rewrite (8.11) in the form
\[
a_{11}v_{xx} + a_{12}v_{xy} + a_{22}v_{yy} + bv + cv = -A((y + 1)A - 1) + E(x, y),
\]

with
\[
b(x, y) = 1 - (y + 1)\left(\frac{\xi_1(A - \frac{y}{x})}{\xi_1(A - \frac{y}{x} - A)} + \frac{\xi_1(A - \frac{y}{x})}{\xi_1(A - \frac{y}{x} - A)}\right),
\]
\[
c(x, y) = (y + 1)A\left(\frac{\xi_1(A - \frac{y}{x})}{\xi_1(A - \frac{y}{x} - A)} - \int_0^1 \frac{\xi_1(A - s\frac{y}{x})}{\xi_1(A - \frac{y}{x} - A)} ds\right),
\]

where \(v\) and \(v_x\) are evaluated at the point \((x, y)\).

Since \(\psi \in \mathcal{K}\) and \(v\) is defined by (8.4), we have
\[
a_{11}, b, c \in C(\Omega_{4\varepsilon}^+ \setminus \{x = 0\}).
\]

Combining (8.12) with (5.16), (5.37), (5.45), and (8.14), we obtain that, for sufficiently large \(\hat{C}\) depending only on the data,
\[
a_{11} \geq \frac{1}{6}x, \quad a_{22} \geq \frac{1}{2c_2}, \quad |a_{12}| \leq \frac{1}{3\sqrt{c_2}}x^{1/2} \quad \text{on } \Omega_{2\varepsilon}^+.
\]

Thus, \(4a_{11}a_{22} - (a_{12})^2 \geq \frac{2}{9c_2}x\) on \(\Omega_{2\varepsilon}^+\), which implies that (8.15) is elliptic on \(\Omega_{2\varepsilon}^+\) and uniformly elliptic on every compact subset of \(\Omega_{3\varepsilon}^+ \setminus \{x = 0\}\).

Furthermore, using (5.39) and (8.17) and noting \(A > 0\) and \(x > 0\), we have
\[
\xi_1(A) = A.
\]

Now we estimate \(E(x, y)\). Using (8.14), (5.43), (4.50), and \(\psi \in \mathcal{K}\), we find that, on \(\Omega_{2\varepsilon}^+\),
\[
|\partial_x \hat{O}_1| \leq C(x + |\psi| + |D\psi| + x|\psi_{xx}| + |\psi_x\psi_{xx}| + |\psi_{xx}| + |D\psi|^2) \leq CM_1x,
\]
\[
|\partial_x \hat{O}_2| \leq C(|D\psi| + |D\psi|^2 + |\psi_x\psi_{xx}| + (1 + |\psi_x|)|\psi_{xx}|) \leq CM_1x^{1/2}(1 + M_1x),
\]
\[
|\partial_x \hat{O}_3| \leq C(1 + |\psi| + |\psi_x\phi'_{x}| + (1 + |D\psi|)|D^2\psi| + |D\psi|^2)
\]
\[
\leq CM_1(1 + M_1x).
\]
where we have used the fact that \( |x_t^E(s)| \leq C \) on \( \Gamma \). Combining these estimates with (8.13), (8.14), (5.44), and \( \psi \in \mathcal{H} \), we obtain from (8.13) that
\[
|x_n^E(x,y)| \leq CM_1^2(x + M_1|y|) \leq C\tilde{C} \quad \text{on} \quad \Omega_{2e}^+.
\]
From this and \((\gamma + 1)A > 1\), we conclude that the right-hand side of (8.15) is strictly negative in \( \Omega_{2e}^+ \) if \( \tilde{C} \) is sufficiently large, depending only on the data.

We fix \( \tilde{C} \) satisfying all the requirements above (thus depending only on the data). Then we have
\[
(8.19) \quad a_{11}v_{xx} + a_{12}v_{xy} + a_{22}v_{yy} + b v_x + c v < 0 \quad \text{on} \quad \Omega_{2e}^+;
\]
the equation is elliptic in \( \Omega_{2e}^+ \) and uniformly elliptic on compact subsets of \( \Omega_{2e}^+ \setminus \{ x = 0 \} \), and (8.18) holds. Moreover, \( v \) satisfies (8.5) and the boundary conditions (8.6)–(8.8) and (8.10). Then it follows that
\[
v \geq 0 \quad \text{on} \quad \Omega_{2e}^+.
\]
Indeed, let \( z_0 := (x_0, y_0) \in \Omega_{2e}^+ \) be a minimum point of \( v \) over \( \Omega_{2e}^+ \) and \( v(z_0) < 0 \).

Then, by (8.6)–(8.7) and (8.10), either \( z_0 \) is an interior point of \( \Omega_{2e}^+ \) or \( z_0 \in \Gamma_{\text{wedge}} \cap \{ 0 < x < 2e \} \). If \( z_0 \) is an interior point of \( \Omega_{2e}^+ \), then (8.19) is violated since (8.19) is elliptic, \( v(z_0) < 0 \), and \( c(z_0) \leq 0 \) by (8.18). Thus, the only possibility is \( z_0 \in \Gamma_{\text{wedge}} \cap \{ 0 < x < 2e \} \), i.e., \( z_0 = (x_0, 0) \) with \( x_0 > 0 \). Then, by (8.2), there exists \( \rho > 0 \) such that \( B_{\rho/2}(z_0) \cap \{ y > 0 \} \). (8.19) is uniformly elliptic in \( B_{\rho/2}(z_0) \cap \{ y \geq 0 \} \), with the coefficients \( a_{ij}, b, c \in C(B_{\rho/2}(z_0) \cap \{ y \geq 0 \}) \). Since \( v(z_0) < 0 \) and \( v \) satisfies (8.5), then, reducing \( \rho > 0 \) if necessary, we have \( v < 0 \) in \( B_{\rho}(z_0) \cap \{ y > 0 \} \). Thus, \( c \leq 0 \) on \( B_{\rho}(z_0) \cap \{ y > 0 \} \) by (8.18). Moreover, \( v(x, y) \) is not a constant in \( B_{\rho/2}(x_0) \cap \{ y \geq 0 \} \) since its negative minimum is achieved at \( (x_0, 0) \) and cannot be achieved in any interior point, as we showed above. Thus, \( \partial_y v(z_0) > 0 \) by Hopf's Lemma, which contradicts (8.8). Therefore, \( v \geq 0 \) on \( \Omega_{2e}^+ \) so that (8.3) holds on \( \Omega_{2e}^+ \). Then, using (8.9), we obtain (8.3) on \( \Omega_{2e}^+ \).

Now bounding \( \psi_x \) from below, we first prove the following lemma in the \((\xi, \eta)\)-coordinates.

**Lemma 8.1.** If \( \tilde{C} \) in (5.16) is sufficiently large, depending only on the data, then
\[
(8.20) \quad \psi_\eta \leq 0 \quad \text{in} \quad \Omega^+.
\]

**Proof.** We divide the proof into six steps.

**Step 1.** Set \( w = \psi_\eta \). From \( \psi \in \mathcal{H} \) and (8.1),
\[
(8.21) \quad w 
\in \mathcal{C}^0, \mathcal{C}^1(\overline{\Omega^+}) \cap \mathcal{C}^2(\overline{\Omega^+} \setminus (\Gamma_{\text{sonic}} \cup \Sigma_0)) \cap \mathcal{C}^2(\Omega^+) \).
\]
Since \( \psi \in \mathcal{H} \), we have \( \psi \in C^2(\Omega^+ \setminus \Gamma_{\text{sonic}} \cup \Sigma_0) \). Thus we can differentiate (8.27) in the direction tangential to \( \Gamma_{\text{wedge}} \), i.e., apply \( \partial_\xi := \cos \theta_w \partial_x + \sin \theta_w \partial_y \) to (8.27).

Differentiating and substituting the right-hand side of (8.23) for \( \psi_{\xi \xi} \), we have

(8.28)
\[
\frac{1}{2} \left( \cos(2\theta_w) + \frac{\hat{A}_{12}}{\hat{A}_{11}} \sin(2\theta_w) \right) w_x + \frac{1}{2} \sin(2\theta_w) \left( 1 + \frac{\hat{A}_{22}}{\hat{A}_{11}} \right) w_y = 0 \quad \text{on} \quad \Gamma_{\text{wedge}}.
\]

This condition is oblique if \( \sigma \) is small: Indeed, since the unit normal on \( \Gamma_{\text{wedge}} \) is \((- \sin \theta_w, \cos \theta_w)\), we use (3.1) and (8.22) to find

\[
\left( \cos(2\theta_w) + \frac{\hat{A}_{12}}{\hat{A}_{11}} \sin(2\theta_w), \frac{1}{2} \sin(2\theta_w) \right) \begin{pmatrix} (- \sin \theta_w, \cos \theta_w) \end{pmatrix} \geq 1 - C\sigma \geq \frac{1}{2}.
\]

**Step 5.** In this step, we derive the condition for \( w \) on \( \Gamma_{\text{shock}} \). Since \( \psi \) is a solution of (5.29)–(5.33) for \( \phi = \psi \), the Rankine-Hugoniot conditions hold on \( \Gamma_{\text{shock}} \): Indeed, the continuous matching of \( \psi \) with \( \psi_1 - \psi_2 \) across \( \Gamma_{\text{shock}} \) holds by (5.21)–(5.23) since \( \phi = \psi \). Then (4.28) holds and the gradient jump condition (4.29) can be written in form (4.42). On the other hand, \( \psi \) on \( \Gamma_{\text{shock}} \) satisfies (5.30) with \( \phi = \psi \), which is (4.42). Thus, \( \psi \) satisfies (4.29).

Since \( \psi \in \mathcal{H} \) which implies \( \psi \in C^2(\Omega^+ \setminus \Gamma_{\text{sonic}} \cup \Sigma_0) \), we can differentiate (4.29) in the direction tangential to \( \Gamma_{\text{shock}} \). The unit normal \( \nu_\xi \) on \( \Gamma_{\text{shock}} \) is given by (4.30). Then the vector

(8.29)
\[
\tau_\xi := (\tau_1, \tau_2) := \begin{pmatrix} \nu_2 + \psi_{\eta} \nu_1 - \nu_2 \\ u_1 - u_2 \end{pmatrix}
\]

is tangential to \( \Gamma_{\text{shock}} \). Note that \( \tau_\xi \neq 0 \) if \( \tilde{C} \) in (5.16) is sufficiently large, since

(8.30)
\[
|D_{\eta} \psi| \leq C(\sigma + \epsilon) \quad \text{in} \quad \Omega^+, \quad |u_2| + |v_2| \leq C\sigma,
\]

and \( u_1 > 0 \) from \( \psi \in \mathcal{H} \) and Section 3.2. Thus, we can apply the differential operator \( \partial_\xi = \tau_1 \partial_\eta + \tau_2 \partial_\eta \) to (4.29).

In the calculation below, we use the notation in Section 4.2. We showed that condition (4.29) can be written in form (4.33), where \( F(p, z, u_2, \xi, \eta) \) is defined by (4.34)–(4.36) and satisfies (4.37). Also, we denote

(8.31)
\[
\tilde{\tau}(p, u_2, v_2) = (\tilde{\tau}^1, \tilde{\tau}^2)(p, u_2, v_2) := \begin{pmatrix} \nu_2 + p_2 \\ u_1 - u_2 \end{pmatrix}
\]

and

(8.32)
\[
\hat{\psi}_\xi := \frac{u_2 + p_2}{u_1 - u_2}, \quad 1 - \frac{p_1}{u_1 - u_2}.
\]
where \( p = (p_1, p_2) \in \mathbb{R}^2 \) and \( z \in \mathbb{R} \). Then \( \hat{z} \in C^\infty(B_{\delta_2}(0) \times B_{u_1/50}(0)) \). Now, applying the differential operator \( \partial_{\xi} \), we obtain that \( \psi \) satisfies

\[
(8.32) \quad \Phi(D^2 \psi, D\psi, \psi, u_2, v_2, \xi, \eta) = 0 \quad \text{on } \Gamma_{\text{shock}},
\]

where

\[
(8.33) \quad \Phi(R, p, z, u_2, v_2, \xi, \eta) = \sum_{i,j=1}^{2} \hat{\xi}^i F_{pj} R_{ij} + \sum_{i=1}^{2} \hat{\xi}^i (F_{z} p_i + F_{\xi_1}) \text{ for } R = (R_{ij})_{i,j=1}^{2},
\]

and, in both (8.33) and the calculation below, \( D_{(\xi_1, \xi_2)} F \) denotes as \( D_{(\xi, \eta)} F \), \( (F_{pj}, F_{z}, F_{\xi_1}) \) as \( (F_{pj}, F_{z}, F_{\xi_1}) \), \((\hat{\xi}, \hat{\eta})\) as \((\hat{\xi}, \hat{\eta})\) and \( \hat{\rho} \) as \( \hat{\rho}(p, z, \xi, \eta) \), with \( \hat{\rho}(\cdot) \) and \( \hat{\eta}(\cdot) \) defined by (4.35) and (4.36), respectively. By explicit calculation, we apply (4.34)–(4.36) and (8.31) to obtain that, for every \((p, z, u_2, v_2, \xi, \eta)\),

\[
(8.34) \quad \sum_{i=1}^{2} \hat{\xi}^i (F_{z} p_i + F_{\xi_1}) = (\rho_1 - \hat{\rho}) \hat{\xi} \cdot \hat{\eta} = 0.
\]

We note that (4.28) holds on \( \Gamma_{\text{shock}} \). Using (8.32) and (8.34) and expressing \( \xi \) from (4.28), we see that \( \psi \) satisfies

\[
(8.35) \quad \Phi(D^2 \psi, D\psi, \psi, u_2, v_2, \eta) = 0 \quad \text{on } \Gamma_{\text{shock}},
\]

where

\[
(8.36) \quad \Phi(R, p, z, u_2, v_2, \eta) = \sum_{i,j=1}^{2} \hat{\xi}^i \Psi_{pj} (p, z, u_2, v_2, \eta) R_{ij},
\]

\( \Psi \) is defined by (4.39) and satisfies \( \Psi \in C^\infty(\mathbb{S}) \) with \( \|\Psi\|_{C^k(\mathbb{S})} \) depending only on the data and \( k \in \mathbb{N}, \) and \( \mathbb{S} = B_{\delta_2}(0) \times (-\delta, \delta) \times B_{u_1/50}(0) \times (-6\delta/5, 6\delta/5) \).

Now, from (4.34)–(4.36), (4.39), and (8.31), we find

\[
\hat{\xi}((0, 0), 0, 0) = (0, \hat{\eta}),
\]

\[
D_p \Psi((0, 0), 0, 0, 0, \eta) = \left( \rho_2 (\varepsilon_2^2 - \gamma^2) \right) \left( \frac{\rho_2 - \rho_1}{u_1} - \rho_2^2 \hat{\eta} \right) \eta.
\]

Thus, by (8.36), we obtain that, on \( \mathbb{R}^{2 \times 2} \times \delta \),

\[
(8.37) \quad \Phi(R, p, z, u_2, v_2, \eta) = \rho_2 (\varepsilon_2^2 - \gamma^2) R_{21} + \left( \frac{\rho_2 - \rho_1}{u_1} - \rho_2^2 \right) \eta R_{22}
\]

\[
+ \sum_{i,j=1}^{2} \hat{\xi}^i (p, z, u_2, v_2, \eta) R_{ij}.
\]
of $w$ cannot be achieved in the interior of $\Omega^+$, unless $w$ is constant on $\Omega^+$, by the Strong Maximum Principle. Since $w$ satisfies the oblique derivative conditions (8.28) and (8.39) on the straight segment $\Gamma_{\text{wedge}}$ and on the curve $\Gamma_{\text{shock}}$ that is $C^{2,\alpha}$ in its relative interior, and since (8.24) is uniformly elliptic in a neighborhood of any point from the relative interiors of $\Gamma_{\text{wedge}}$ and $\Gamma_{\text{shock}}$, it follows from Hopf's Lemma that the maximum of $w$ cannot be achieved in the relative interiors of $\Gamma_{\text{wedge}}$ and $\Gamma_{\text{shock}}$, unless $w$ is constant on $\Omega^+$. Now conditions (8.25); (8.26) imply that $w \leq 0$ on $\Omega^+$. This completes the proof. □

Using Lemma 8.1 and working in the $(x, y)$-coordinates, we have

PROPOSITION 8.2. If $\hat{C}$ in (5.16) is sufficiently large, depending only on the data, then

$$\psi_x \geq -\frac{4}{3(y + 1)} x \quad \text{in} \quad \Omega^+ \cap \{x \leq 4\varepsilon\}.$$  

Proof. By definition of the $(x, y)$-coordinates in (4.47), we have

$$\psi_y = -\sin \theta \psi_x + \frac{\cos \theta}{r} \psi_y,$$

where $(r, \theta)$ are the polar coordinates in the $(\xi, \eta)$-plane.

From (7.20), it follows that, for sufficiently small $\sigma$ and $\varepsilon$, depending only on the data,

$$\eta \geq \eta^* \quad \text{for all} \quad (\xi, \eta) \in \mathcal{B} \cap \{c_2 - r < 4\varepsilon\},$$

where $(l(\eta^*), \eta^*)$ is the unique intersection point of the segment $\{(l(\eta), \eta): \eta \in (0, \eta_1)\}$ with the circle $\partial B_{c_2 - 4\varepsilon}(0)$. Let $\tilde{\eta}^*$ be the corresponding point for the case of normal reflection, i.e., $\tilde{\eta}^* = \sqrt{(\tilde{c}_2 - 4\varepsilon)^2 - \tilde{\xi}^2}$. By (3.5), $\eta^* \geq \sqrt{\tilde{c}_2^2 - \tilde{\xi}^2}/2 > 0$ if $\varepsilon$ is sufficiently small. Also, from (4.3), (4.4) and (3.24), and using the convergence $(\theta_0, c_2, \tilde{\xi}) \to (\pi/2, \tilde{c}_2, \tilde{\xi})$ as $\theta_0 \to \pi/2$, we obtain $\eta^* \geq \tilde{\eta}^*/2$ and $\tilde{c}_2 \leq 2\tilde{c}_2$ if $\alpha$ and $\varepsilon$ are sufficiently small. Thus, we conclude that, if $\hat{C}$ in (5.16) is sufficiently large depending only on the data, then, for every $(\xi, \eta) \in \mathcal{B} \cap \{c_2 - r < 4\varepsilon\}$, the polar angle $\theta$ satisfies

$$\sin \theta = \frac{\eta}{\sqrt{\tilde{\xi}^2 + \eta^2}} > 0, \quad |\cot \theta| = \left|\frac{\tilde{\xi}}{\eta}\right| \leq \frac{8\tilde{c}_2}{\sqrt{\tilde{\xi}^2 + \eta^2}} \leq C.$$

From (8.41), (8.42) and Lemma 8.1, we find that, on $\Omega^+ \cap \{c_2 - r < 4\varepsilon\}$,

$$\psi_x = -\frac{1}{\sin \theta} \psi_y + \frac{\cot \theta}{r} \psi_y \geq \frac{\cot \theta}{r} \psi_y \geq -C |\psi_y|.$$
Note that $\psi \in \mathcal{X}$ implies $|\psi_x(x,y)| \leq M_1 x^{3/2}$ for all $(x,y) \in \Omega^+ \cap \{c_2-r < 2\varepsilon\}$.

Then, using (8.43) and (5.16) and choosing large $\hat{C}$, we have

$$\psi_x \geq -\frac{4}{3(y+1)}x \quad \text{in } \Omega^+ \cap \{x \leq 2\varepsilon\}.$$ 

Also, $\psi \in \mathcal{X}$ implies

$$|\psi_x| \leq M_2 \sigma \leq \frac{4}{3(y+1)}(2\varepsilon) \quad \text{on } \Omega^+ \cap \{2\varepsilon \leq x \leq 4\varepsilon\},$$

where the second inequality holds by (5.16) if $\hat{C}$ is sufficiently large depending only on the data. Thus, (8.40) holds on $\Omega^+_{4\varepsilon}$.

\[\square\]

9. Proof of the Main Theorem

Let $\hat{C}$ be sufficiently large to satisfy the conditions in Propositions 7.1 and 8.1, 8.2. Then, by Proposition 7.1, there exist $\sigma_0, \varepsilon > 0$ and $M_1, M_2 \geq 1$ such that, for any $\sigma \in (0, \sigma_0]$, there exists a solution $\psi \in \mathcal{X}(\sigma, \varepsilon, M_1, M_2)$ of problem (5.29)–(5.33) with $\phi = \psi$. Fix $\sigma \in (0, \sigma_0]$ and the corresponding "fixed point" solution $\psi$, which, by Propositions 8.1, 8.2, satisfies

$$|\psi_x| \leq \frac{4}{3(y+1)}x \quad \text{in } \Omega^+ \cap \{x \leq 4\varepsilon\}.$$

Then, by Lemma 5.4, $\psi$ satisfies (4.19) in $\Omega^+ (\psi)$. Moreover, $\psi$ satisfies properties (i)–(v) in Step 10 of Section 5.6 by following the argument in Step 10 of Section 5.6. Then, extending the function $\varphi = \psi + \psi_2$ from $\Omega := \Omega^+(\psi)$ to the whole domain $\Lambda$ by using (1.20) to define $\varphi$ in $\Lambda \setminus \Omega$, we obtain

$$\varphi \in W^{1,\infty}_{\text{loc}}(\Lambda) \cap \left( \bigcup_{i=0}^{2} C^1(\Lambda_i \cup \Sigma) \right) \cap C^{1,1}(\Lambda_i),$$

where the domains $\Lambda_i$, $i = 0, 1, 2$, are defined in Step 10 of Section 5.6. From the argument in Step 10 of Section 5.6, it follows that $\varphi$ is a weak solution of Problem 2, provided that the reflected shock $S_1 = P_0 P_1 P_2 \cap \Lambda$ is a $C^2$-curve.

Thus, it remains to show that $S_1 = P_0 P_1 P_2 \cap \Lambda$ is a $C^2$-curve. By definition of $\varphi$ and since $\psi \in \mathcal{X}(\sigma, \varepsilon, M_1, M_2)$, the reflected shock $S_1 = P_0 P_1 P_2 \cap \Lambda$ is given by $S_1 = \{\xi = f_{S_1}(\eta) : \eta P_2 < \eta P_1\}$, where $\eta P_2 = -v_2, \eta P_1 = \left( \frac{x}{v_1}, \frac{\theta_1 \sin \theta_1}{v_1} \right), \eta P_0 = (0,0)$, and

$$f_{S_1}(\eta) = \begin{cases} f(\eta) & \text{if } \eta \in (\eta P_0, \eta P_1), \\ \eta P_2 & \text{if } \eta \in (\eta P_1, \eta P_2) \end{cases}$$

where $f(\eta)$ is defined by (4.3), $\eta P_2 > 0$ is defined by (4.6), and $\eta P_1 > \eta P_0$ if $\sigma$ is sufficiently small, which follows from the explicit expression of $\eta P_1$ given above and the fact that $(\delta, c_2, \xi) \to (\underline{c}, c_2, 0)$ as $\theta_0 \to \pi/2$. The function $f_{S_1}$ is defined by (5.21) for $\phi = \psi$. 

12 places respectively.
Thus we need to show that \( f_{S_1} \in C^2(\{\eta_{P_0}, \eta_{P_1}\}) \). By (4.3) and (5.24), it suffices to show that \( f_{S_1} \) is twice differentiable at the points \( \eta_{P_1} \) and \( \eta_{P_2} \).

First, we consider \( f_{S_1} \) near \( \eta_{P_2} \). We change the coordinates to the \((x, y)\)-coordinates in (4.47). Then, for sufficiently small \( \varepsilon_1 > 0 \), the curve \( \{ x = f_{S_1}(x), 0 < r < o_2 + \varepsilon_1 \} \) has the form \( \{ y = \hat{f}_{S_1}(x) + \varepsilon_1 \} \), where

\[
\hat{f}_{S_1}(x) = \begin{cases} \hat{f}_\psi(x) & \text{if } x \in (0, \varepsilon_1), \\ \hat{f}_0(x) & \text{if } x \in (-\varepsilon_1, 0), \end{cases}
\]

with \( \hat{f}_0 \) and \( \hat{f}_\psi \) defined by (5.9) and (5.25) for \( \phi = \psi \). In order to show that \( f_{S_1} \) is twice differentiable at \( \eta_{P_1} \), it suffices to show that \( \hat{f}_{S_1} \) is twice differentiable at \( x = 0 \).

From (5.26), (5.27), and (5.9), it follows that \( \hat{f}_{S_1} \in C^1((-\varepsilon_1, \varepsilon_1)) \). Moreover, from (5.3), (5.6), (5.22), and (5.27), we write \( \varphi_1, \varphi_2, \) and \( \psi \) in the \((x, y)\)-coordinates to obtain that

\[
\hat{f}_{S_1}^{\prime}(x) = \begin{cases} \frac{\partial_x (\varphi_1 - \varphi_2 - \psi)}{\partial_y (\varphi_1 - \varphi_2)} (x, \hat{f}_{S_1}(x)) & \text{if } x \in (0, \varepsilon_1), \\ \frac{\partial_x (\varphi_1 - \varphi_2)}{\partial_y (\varphi_1 - \varphi_2)} (x, \hat{f}_{S_1}(x)) & \text{if } x \in (-\varepsilon_1, 0), \end{cases}
\]

and that \( \hat{f}_{S_1}^{\prime}(x) \) is given for \( x \in (-\varepsilon_1, \varepsilon_1) \) by the second line of the right-hand side of (9.3). Using (5.3) and \( \psi \in \mathcal{K} \) with (5.16) for sufficiently large \( \mathcal{C} \), we have

\[
|\hat{f}_{S_1}^{\prime}(x) - \hat{f}_0^{\prime}(x)| \leq C|D_{(x, y)}(x, \hat{f}_\psi(x))| \quad \text{for all } x \in (0, \varepsilon_1).
\]

Since \( \psi \) satisfies (5.30) with \( \phi = \psi \), it follows that, in the \((x, y)\)-coordinates, \( \psi \) satisfies (6.6) on \( \{ y = \hat{f}_\psi(x) \mid x \in (0, \varepsilon_1) \} \), and (6.8) holds. Then it follows that

\[
|\psi_x(x, \hat{f}_\psi(x))| \leq C(|\psi_y(x, \hat{f}_\psi(x))| + |\psi_x(x, \hat{f}_\psi(x))|) \leq Cx^{3/2},
\]

where the last inequality follows from \( \psi \in \mathcal{K} \). Combining this with (9.2), (9.4), and \( \hat{f}_{S_1}, \hat{f}_0 \in C^1((-\varepsilon_1, \varepsilon_1)) \) yields

\[
|\hat{f}_{S_1}^{\prime}(x) - \hat{f}_0^{\prime}(x)| \leq Cx^{3/2} \quad \text{for all } x \in (-\varepsilon_1, \varepsilon_1).
\]

Then it follows that \( \hat{f}_{S_1}^{\prime}(x) - \hat{f}_0^{\prime}(x) \) is differentiable at \( x = 0 \). Since

\[
\hat{f}_0 \in C^\infty((-\varepsilon_1, \varepsilon_1)),
\]

we conclude that \( \hat{f}_{S_1} \) is twice differentiable at \( x = 0 \). Thus, \( f_{S_1} \) is twice differentiable at \( \eta_{P_1} \).

In order to prove the \( C^2 \)-smoothness of \( f_{S_1} \) up to \( \eta_{P_2} \), we extend the solution \( \phi \) and the free boundary function \( f_{S_1} \) into \( \{ \eta < -\varepsilon_2 \} \) by the even reflection about the line \( \Sigma_0 \subset \{ \eta = -\varepsilon_2 \} \) so that \( P_2 \) becomes an interior point of the shock curve. Note that we continue to work in the shifted coordinates defined
in Section 4.1; that is, for $(\xi, \eta)$ such that $\eta < -v_2$ and $(\xi, -2v_2 - \eta) \in \Omega^+(\psi)$, we define $(\varphi, \varphi_1)(\xi, \eta) = (\varphi, \varphi_1)(\xi, -2v_2 - \eta)$ and $f_{\varphi_1}(\eta) = -2v_2 - \eta$ for $\varphi_1$ given by (4.15). Denote $\Omega^+_{\varphi_1}(P_2) := B_{\delta}(P_2) \cap \{ \xi > f_{\varphi_1}(\eta) \}$ for sufficiently small $\delta_1 > 0$.

From $\varphi \in C^{1,\alpha}(\Omega^+(\psi)) \cap C^{2,\alpha}(\Omega^+(\psi))$ and (4.13), we have

$$\varphi \in C^{1,\alpha}(\Omega^+_{\varphi_1}(P_2)) \cap C^{2,\alpha}(\Omega^+_{\varphi_1}(P_2)).$$

Also, the extended function $\varphi_1$ is in fact given by (4.15). Furthermore, from (5.20) and (5.22), we can see that the same is true for the extended functions and hence

$$\{ \xi > f_{\varphi_1}(\eta) \} \cap B_{\delta_1}(P_2) = \{ \varphi < \varphi_1 \} \cap B_{\delta_1}(P_2),$$

$$f_{\varphi_1} \in C^{1,\alpha}(\Omega^+_{\varphi_1}(P_2)).$$

Furthermore, from (1.8), (1.9) and (4.13), it follows that the extended $\varphi$ satisfies (1.8) with (1.9) in $\Omega^+_{\varphi_1}(P_2)$, where we have used the form of equation, i.e., the fact that there is no explicit dependence on $(\xi, \eta)$ in the coefficients and that the dependence of $D\varphi$ is only through $|D\varphi|$. Finally, the boundary conditions (4.9) and (4.10) are satisfied on $\Gamma_{\varphi_1}(P_2) := \{ \xi = f_{\varphi_1}(\eta) \} \cap B_{\delta_1}(P_2)$. (1.8) is uniformly elliptic in $\Omega^+_{\varphi_1}(P_2)$ for $\varphi$, which follows from $\varphi = \varphi_2 + \psi$ and Lemmas 5.2 and 5.4. Condition (4.10) is uniformly oblique on $\Gamma_{\varphi_1}(P_2)$ for $\varphi$, which follows from Section 4.2.

Next, we rewrite (1.8) in $\Omega^+_{\varphi_1}(P_2)$ and the boundary conditions (4.9) and (4.10) on $\Gamma_{\varphi_1}(P_2)$ in terms of $u := \varphi_1 - \varphi$. Substituting $u + \varphi_1$ for $\varphi$ into (1.8) and (4.10), we obtain that $u$ satisfies

$$F(D^2u, Du, u, \xi, \eta) = 0 \text{ in } \Omega^+_{\varphi_1}(P_2), \quad u = G(Du, u, \xi, \eta) = 0 \text{ on } \Gamma_{\varphi_1}(P_2),$$

where the equation is quasilinear and uniformly elliptic, the second boundary condition is oblique, and the functions $F$ and $G$ are smooth. Also, from (5.20) which holds for the even extensions as well, we find that $\partial_{\delta} u > 0$ on $\Gamma_{\varphi_1}(P_2)$. Then, applying the hodograph transform of [28, §3], i.e., changing $(\xi, \eta) \rightarrow (X, Y) = (u(\xi, \eta), \eta)$, and denoting the inverse transform by $(X, Y) \rightarrow (\xi, \eta) = (v(X, Y), Y)$, we obtain

$$v \in C^{1,\alpha}(B^+_{\delta}(\{(0, -v_2)\})) \cap C^{2,\alpha}(B^+_{\delta}(\{(0, -v_2)\})),$$

where $B^+_{\delta}(\{(0, -v_2)\}) := B_{\delta}(\{(0, -v_2)\}) \cap \{ X > 0 \}$ for small $\delta > 0$, $v(X, Y)$ satisfies a uniformly elliptic quasilinear equation

$$\bar{F}(D^2v, Dv, v, X, Y) = 0 \quad \text{in } B^+_{\delta}(\{(0, -v_2)\})$$

and the oblique derivative condition

$$\bar{G}(Dv, v, Y) = 0 \quad \text{on } \partial B^+_{\delta}(\{(0, -v_2)\}) \cap \{ X = 0 \},$$

where $\bar{F}, \bar{G}$ are related to $F, G$. This completes the proof.
and the functions $\bar{F}$ and $\bar{G}$ are smooth. Then, from the local estimates near the boundary in the proof of [32, Th. 2], $v \in C^{2,\alpha}(B_{\delta/2}^+(0, -\theta_2))$. Since $f_{S_1}(\eta) = v(0, \eta)$, it follows that $f_{S_1}$ is $C^{2,\alpha}$ near $\gamma^2$. It remains to prove the convergence of the normal reflection solution as $\theta_\infty \to \pi/2$. Let $\theta_\infty \to \pi/2$ as $i \to \infty$. Denote by $\phi^i$ and $f^i$ the corresponding solution and the free-boundary function respectively, i.e., $P_0, P_1, P_2 \cap \Lambda$ for each $i$ is given by $\{\xi = f^i(\eta) : \eta \in (\gamma_1, \gamma_2)\}$. Denote by $\psi^\infty$ and $f^\infty(\eta) = \xi^\infty$ the solution and the reflected shock for the normal reflection respectively. For each $i$, we find that $\psi^i - \psi^\infty = \psi^i$ in the subsonic domain $\Omega^+_i$, where $\psi^i$ is the corresponding “fixed point solution” from Proposition 7.1, and $\psi^i \in K(\pi/2 - \theta_\infty, \epsilon^i, M_1^i, M_2^i)$ with (5.16). Moreover, $f^i$ satisfies (5.24). We also use the convergence of state (2) to the corresponding state of the normal reflection obtained in Section 3.2. Then we conclude that, for a subsequence, $f^i \to f^\infty$ in $C^{1,\alpha}_{\text{loc}}$ and $\phi^i \to \phi^\infty$ in $C^1$ on compact subsets of $\{\xi > \xi^\infty\}$ and $\{\xi < \xi^\infty\}$. Also, we obtain $\|D\phi^i, \phi^i\|_{L^\infty(K)} \leq C(K)$ for every compact set $K \subset \bar{\Omega} : = \{\xi \leq \xi^\infty, \eta \geq 0\}$. Then $\phi^i \to \phi^\infty$ in $W_{\text{loc}}^{1,1}(\bar{\Omega}^\infty)$ by the Dominated Convergence Theorem. Since such a converging subsequence can be extracted from every sequence $\theta_\infty \to \pi/2$, it follows that $\phi^\infty$ as $\theta_\infty \to \pi/2$.

**Appendix A. Estimates of solutions to elliptic equations**

In this appendix, we make some careful estimates of solutions of boundary value problems for elliptic equations in $\mathbb{R}^2$, which are applied in Sections 6 and 7. Throughout the appendix, we denote by $(x, y)$ or $(X, Y)$ the coordinates in $\mathbb{R}^2$, by $\mathbb{R}^2_+: = \{y > 0\}$, and, for $z = (x, 0)$ and $r > 0$, denote $B_r^+(z) : = B_r(z) \cap \mathbb{R}^2_+$ and $\Sigma_r(z) : = B_r(z) \cap \{y = 0\}$. We also denote $B_r : = B_r(0)$, $B_r^+ : = B_r^+(0)$, and $\Sigma_r : = \Sigma_r(0)$.

We consider an elliptic equation of the form

\[(A.1) \quad A_{11}u_{xx} + 2A_{12}u_{xy} + A_{22}u_{yy} + A_1u_x + A_2u_y = f,\]

where $A_{ij} = A_{ij}(D, x, y)$, $A_i = A_i(D, x, y)$, and $f = f(x, y)$. We study the following three types of boundary conditions: (i) the Dirichlet condition, (ii) the oblique derivative condition, (iii) the “almost tangential derivative” condition.

One of the new ingredients in our estimates below is that we do not assume that the equation satisfies the “natural structure conditions”, which are used in the earlier related results; see, e.g., [20, Ch. 15] for the interior estimates for the Dirichlet problem and [37] for the oblique derivative problem. For (A.1), the natural structure conditions include the requirement that $|p| |DpA_{ij}| \leq C$ for all $p \in \mathbb{R}^2$.

Note that equations (5.42) and (5.49) do not satisfy this condition because of the term $\lambda^2 \xi^1 (\xi_2^1)^2$ in the coefficient of $\psi_{xx}$. Thus we have to derive the estimates...
for the equations without the "natural structure conditions". We consider only the two-dimensional case here.

The main point at which the "natural structure conditions" are needed is the gradient estimates. The interior gradient estimates and global gradient estimates for the Dirichlet problem, without requiring the natural structure conditions, were obtained in the earlier results in the two-dimensional case; see Trudinger [47] and references therein. However, it is not clear how this approach can be extended to the oblique and "almost tangential" derivative problems. We also note a related result by Lieberman [34] for fully nonlinear equations and the boundary conditions without obliqueness assumption in the two-dimensional case, in which the Hölder estimates for the gradient of a solution depend on both the bounds of the solution and its gradient.

In this appendix, we present the $C^{2,\alpha}$-estimates of the solution only in terms of its $C$-norm. For simplicity, we restrict to the case of quasilinear (A.1) and linear boundary conditions, which is the case for the applications in this paper. Below, we first present the interior estimate in the form that is used in the other parts of this paper. Then we give a proof of the $C^{2,\alpha}$-estimates for the "almost tangential" derivative problem. Since the proofs for the Dirichlet and oblique derivative problems are similar to that for the "almost tangential" derivative problem, we just sketch these proofs.

**Theorem A.1.** Let $u \in C^{2}(B_2)$ be a solution of equation (A.1) in $B_2$. Let $A_{ij}(p, x, y), A_i(p, x, y)$, and $f(x, y)$ satisfy that there exist constants $\lambda > 0$ and $\alpha \in (0, 1)$ such that

(A.2) $\lambda^2|\mu|^2 \leq \sum_{i,j=1}^{n} A_{ij}\mu_i\mu_j \leq \lambda^{-1}|\mu|^2 \quad$ for all $(x, y) \in B_2, \ p, \mu \in \mathbb{R}^2$,

(A.3) $\|A_{ij}, A_i\|_{C^\alpha(\mathbb{R}^2 \times \overline{B_2})} + \|D_p(A_{ij}, A_i)\|_{C(\mathbb{R}^2 \times \overline{B_2})} + \|f\|_{C^\alpha(\overline{B_2})} \leq \lambda^{-1}$.

Assume that $\|u\|_{C(\overline{B_2})} \leq M$. Then there exists $C > 0$ depending only on $(\lambda, M)$ such that

(A.4) $\|u\|_{C^{2,\alpha}(\overline{B_2})} \leq C(\|u\|_{C(\overline{B_2})} + \|f\|_{C^\alpha(\overline{B_2})})$.

**Proof.** We use the standard interior Hölder seminorms and norms as defined in [20, Eqs. (4.17) and (6.10)]. By [20, Th. 12.4], there exists $\beta \in (0, 1)$ depending only on $\lambda$ such that

$[u]_{1, \beta, B_2} \leq C(\lambda)(\|u\|_{0, B_2} + \|f - A_1D_1u - A_2D_2u\|_{0, B_2})$

$\leq C(\lambda, M)(1 + \|f\|_{0, B_2} + \|Du\|_{0, B_2})$. 

$[u]_{0, \beta, B_2}$ 

$[f]_{0, \beta, B_2}$ 

$[Du]_{0, \beta, B_2}$
Then, applying the interpolation inequality ([20, (6.82)]) with the argument similar to that for the proof of [20, Th. 12.4], we obtain
\[ \|u\|_{1,\beta, B_2}^* \leq C \left( \lambda, M \right)(1 + \|f\|_{0, B_2}^{(2)}). \]

Now we consider (A.1) as a linear elliptic equation
\[ \sum_{i=1}^{n} a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} a_i(x) u_{x_i} = f(x) \quad \text{in } B_{3/2} \]
with coefficients \( a_{ij}(x) = A_{ij}(D u(x), x) \) and \( a_i = A_i(D u(x), x) \) in \( C^{\beta}(B_{3/2}) \)
satisfying
\[ \|\left( a_{ij}, a_i \right)\|_{C^\beta(B_{3/2})} \leq C(\lambda, M). \]

We can assume \( \beta \leq \alpha \). Then the local estimates for linear elliptic equations yield
\[ \|u\|_{C^{2,\beta}(B_{3/4})} \leq C(\lambda, M) \left( \|u\|_{C(B_{3/2})} + \|f\|_{C^\beta(B_{3/2})} \right). \]

With this estimate, we have \( \|(a_{ij}, a_i)\|_{C^\alpha(B_{3/4})} \leq C(\lambda, M) \). Then the local estimates for linear elliptic equations in \( B_{3/4} \) yield (A.4).

Now we make the estimates for the "almost tangential derivative" problem.

**Theorem A.2.** Let \( \lambda > 0, \alpha \in (0, 1) \), and \( \epsilon \geq 0 \). Let \( \Phi \in C^{2,\alpha}(R) \) satisfy
\[ \Phi \|C_{2,\alpha}(R) \leq \lambda^{-1}, \]
and denote \( \Omega_R^+ := B_R \cap \{ y > \epsilon \Phi(x) \} \) for \( R > 0 \). Let \( u \in C^{2,\beta}(B_{3/2}^+) \cap C^1(B_{3/2}^+) \)

satisfy (A.1) in \( \Omega_R^+ \) and
\[ u = \epsilon b(x, y) u_y + c(x, y) u \quad \text{on } \Gamma_\Phi := B_2 \cap \{ y = \epsilon \Phi(x) \}. \]

Let \( A_{ij}(p, x, y), A_i(p, x, y), b(x, y), c(x, y), \) and \( f(x, y) \) satisfy that there exists constant \( \lambda > 0 \) such that
\[ \lambda |\mu|^2 \leq \sum_{i,j=1}^{n} A_{ij} \mu_i \mu_j \leq \lambda^{-1} |\mu|^2 \quad \text{for } (x, y) \in \Omega_R^+, p, \mu \in R^2, \]

\[ \|(A_{ij}, A_i)\|_{C^{\alpha}(\overline{\Omega_R^+} \times R^2)} + \|D_p(A_{ij}, A_i)\|_{C(\overline{\Omega_R^+} \times R^2)} + \|f\|_{C^{\alpha}(\overline{\Omega_R^+})} \leq \lambda^{-1}, \]

\[ \|(b, c)\|_{C^{1,\alpha}(\overline{\Omega_R^+})} \leq \lambda^{-1}. \]

Assume that \( \|u\|_{C^{2,\alpha}(\Omega_R^+)} \leq M \). Then there exist \( \epsilon_0(\lambda, M, \alpha) > 0 \) and \( C(\lambda, M, \alpha) > 0 \) such that, if \( \epsilon \in (0, \epsilon_0) \),
\[ \|u\|_{C^{2,\alpha}(\Omega_R^+)} \leq C \left( \|u\|_{C(\overline{\Omega^+_R})} + \|f\|_{C^\alpha(\overline{\Omega^+_R})} \right). \]
To prove this theorem, we first flatten the boundary part $\Gamma_\Phi$ by defining the variables $(X, Y) = \Psi(x, y)$ with $(X, Y) = (x, y - \varepsilon \Phi(x))$. Then $(x, y) = \Psi^{-1}(X, Y) = (X, Y + \varepsilon \Phi(X))$. From (A.5), we have
\[
\| \Psi - \text{Id} \|_{C^2, \alpha(\bar{\Omega}_{\varepsilon}^+)} + \| \Psi^{-1} - \text{Id} \|_{C^2, \alpha(\bar{B}_{\varepsilon}^+)} \leq \varepsilon \lambda^{-1}.
\]

Then, for sufficiently small $\varepsilon$ depending only on $\lambda$, the transformed domain $\Omega_{\varepsilon}^+ := \Psi(\Omega_{\varepsilon}^+)$ satisfies
\[
B_{2 - 2\varepsilon/\lambda}^+ \subset \Omega_{\varepsilon}^+ \subset B_{2 + 2\varepsilon/\lambda}^+, \quad \Omega_{\varepsilon}^+ \subset \mathbb{R}^2_+ := \{ Y > 0 \}, \quad \partial \Omega_{\varepsilon}^+ \cap \{ Y = 0 \} = \Psi(\Gamma_{\Phi});
\]
and the function
\[
v(X, Y) = u(x, y) := u(\Psi^{-1}(X, Y))
\]

satisfies an equation of form (A.1) in $\Omega_{\varepsilon}^+$ with (A.7), (A.8) and the corresponding elliptic constants $\lambda/2$; and the boundary condition for $v$ by an explicit calculation is
\[
v_x = \varepsilon \left( b(\Psi^{-1}(X, 0)) + \Phi'(X) \right) v_Y + c(\Psi^{-1}(X, 0)) v \quad \text{on} \quad \partial \Omega_{\varepsilon}^+ \cap \{ Y = 0 \};
\]
i.e., it is of form (A.6) with (A.9) satisfied on $\partial \Omega_{\varepsilon}^+$ with elliptic constant $\lambda/4$.

Moreover, by (A.11) and (A.12), it suffices for this theorem to show the following estimate for $v(X, Y)$:
\[
\| v \|_{2, \alpha, B_{\varepsilon/5}^+} \leq C(\lambda, M, \alpha) \left( \| f \|_{0, B_{2 - 2\varepsilon/\lambda}^+} + \| f \|_{0, B_{2 - 2\varepsilon/\lambda}^+} \right).
\]
That is, we can consider the equation in $B_{2 - 2\varepsilon/\lambda}^+$ and condition (A.13) on $\Sigma_{2 - 2\varepsilon/\lambda}$ or, by rescaling, we can simply consider our equation in $B_{2}^+$ and condition (A.13) on $\Sigma_2 := B_2 \cap \{ Y = 0 \}$. In other words, without loss of generality, we can assume $\Phi \equiv 0$ in the original problem.

For simplicity, we use the original notation $(X, Y, u(X, Y))$ to replace the notation $(X, Y, v(X, Y))$. Then we assume that $\Phi \equiv 0$. Thus, (A.1) is satisfied in the domain $B_{2}^+$, the boundary condition (A.6) is prescribed on $\Sigma_2 = B_2 \cap \{ Y = 0 \}$, and conditions (A.7)-(A.9) hold in $B_{2}^+$. Also, we use the partially interior norms [20, Eq. 4.29] in the domain $B_{2}^+ \cup 2 \Sigma_2 \Sigma 2$ with the related distance function $d_2 := \text{dist}(x, \partial B_{2}^+ \setminus \Sigma_2)$. The universal constant $C$ in the argument below depends only on $\lambda$ and $M$, unless otherwise specified.

As in [20, §13.2], we introduce the functions $w_i = D_i u$ for $i = 1, 2$. Then we conclude from (A.1) that $w_1$ and $w_2$ are weak solutions of the following equations of divergence form:
\begin{align}
A.15) \quad & D_1 \left( \frac{A_{11}}{A_{22}} D_1 w_1 + \frac{2A_{12}}{A_{22}} D_2 w_1 \right) + D_{22} w_1 = D_1 \left( \frac{f}{A_{22}} - \frac{A_1}{A_{22}} D_1 u - \frac{A_2}{A_{22}} D_2 u \right), \\
A.16) \quad & D_{11} w_2 + D_2 \left( \frac{2A_{12}}{A_{11}} D_1 w_2 + \frac{A_{22}}{A_{11}} D_2 w_2 \right) = D_2 \left( \frac{f}{A_{11}} - \frac{A_1}{A_{11}} D_1 u - \frac{A_2}{A_{11}} D_2 u \right).
\end{align}

From (A.6), we have
\begin{align}
A.17) \quad & w_1 = g \quad \text{on } \Sigma_2,
\end{align}
where
\begin{align}
A.18) \quad & g := \epsilon b w_2 + c u \quad \text{for } B_2^+.
\end{align}

We first obtain the following Hölder estimates of \( D_1 u \).

\textbf{Lemma A.1.} There exist \( \beta \in (0, \alpha] \) and \( C > 0 \) depending only on \( \lambda \) such that,

for any \( z_0 \in B_2^+ \cup \Sigma_2, \)
\begin{align}
A.19) \quad \& \quad \|D_{x_0}^\beta [w_1]_{0, \beta, B_{2R}^+(z_0)} \|_{B_2^+} \\
\quad \quad \quad \quad \leq C \left( \|D u, f\|_{0, \beta, B_{2R}^+(z_0)} + d_{x_0}^\beta [g]_{0, \beta, B_{2R}^+(z_0)} \right).
\end{align}

\textit{Proof.} We first prove that, for \( z_1 \in \Sigma_2 \) and \( B_{2R}^+(z_1) \subset B_2^+ \),
\begin{align}
A.20) \quad \& \quad R^\beta [w_1]_{0, \beta, B_{2R}^+(z_1)} \leq C \left( \|D u, R f\|_{0, \beta, B_{2R}^+(z_1)} + R^\beta [g]_{0, \beta, B_{2R}^+(z_1)} \right).
\end{align}

We rescale \( u, w_1, \) and \( f \) in \( B_{2R}^+(z_1) \) by defining
\begin{align}
A.21) \quad \& \quad \hat{u}(Z) = \frac{1}{2 R} u(z_1 + 2 R Z), \quad \hat{f}(Z) = 2 R f(z_1 + 2 R Z) \quad \text{for } Z \in B_1^+,
\end{align}
and \( \hat{w}_i = D Z_i \hat{u} \). Then \( \hat{w}_1 \) satisfies an equation of form (A.15) in \( B_1^+ \) with \( u \) replaced by \( \hat{u} \) whose coefficients \( A_{ij} \) and \( A_i \) satisfy (A.7) with unchanged constants (this holds for (A.8) since \( R \leq 1 \)). Then, by the elliptic version of [36, Th. 6.33] stated in the parabolic setting (it can also be obtained by using [36, Lemma 4.5] instead of [20, Lemma 8.23] in the proofs of [20, Th. 8.27, 8.29] to achieve \( \alpha = \alpha_0 \) in [20, Th. 8.29]), we find constants \( \tilde{\beta}(\lambda) \in (0, 1) \) and \( C(\lambda) \) such that
\begin{align}
A.22) \quad \& \quad \|\hat{w}_1\|_{0, \beta, B_{1/2}^+(z_1)} \leq C \left( \|D \hat{u}, \hat{f}\|_{0, \beta, B_1^+} + \|\hat{w}_1\|_{0, \beta, B_1^+} \right)
\end{align}
for \( \beta \leq \min(\tilde{\beta}, \alpha) \). Rescaling back and using (A.17), we have (A.20).
Let \( \eta \in C^1_0(B_{2R}(\hat{\xi})) \) and \( \xi = \eta^2(w_1 - g) \). Note that \( \eta \in W^{1,2}_0(B_{2R}(\hat{\xi}) \cap B^+_2) \) by (A.17). We use \( \xi \) as a test function in the weak form of (A.15):

\[
\int_{B^+_2} \frac{1}{A_{22}} \sum_{i,j=1}^2 A_{ij} D_i w_1 D_j \xi \, dz = \int_{B^+_2} \frac{1}{A_{22}} \left( -\sum_{i=1}^2 A_i D_i u + f \right) D_1 \xi \, dz,
\]

and apply (A.7), (A.8) and (A.23) to obtain

\[
\int_{B^+_2} |Dw_1|^2 \eta^2 \, dz \leq C \int_{B^+_2} \left( (\delta + \epsilon)|Dw_1|^2 + \epsilon|D^2u|^2 \right) \eta^2
\]

\[
+ \left( \frac{1}{\delta} + 1 \right) ((D\eta)^2 + \eta^2)(w_1 - g)^2 + ((D\eta)^2 + \eta^2)(f^2 \eta^2) \right) \, dz,
\]

where \( C \) depends only on \( \lambda \), and the sufficiently small constant \( \delta > 0 \) will be chosen below. Since

\[
|Dw_1|^2 = (D_{11} u)^2 + (D_{12} u)^2,
\]

it remains to estimate \( |D_{22} u|^2 \). Using the ellipticity property (A.7), we can express \( D_{22} u \) from (A.1) to obtain

\[
\int_{B^+_2} |D_{22} u|^2 \eta^2 \, dz \leq C(\lambda) \int_{B^+_2} ((D_{11} u)^2 + |D_{12} u|^2 + |D u|^2 + f^2) \eta^2 \, dz.
\]

Combining this with (A.29) (A.30) yield

\[
\int_{B^+_2} |D^2 u|^2 \eta^2 \, dz \leq C \int_{B^+_2} \left( (\delta + \epsilon)|D^2 u|^2 \right)
\]

\[
+ \left( \frac{1}{\delta} + 1 \right) ((D\eta)^2 + \eta^2)(w_1 - g)^2 + ((D\eta)^2 + \eta^2)(f^2 \eta^2) \right) \, dz.
\]

Choose \( \epsilon_0 = \delta = (4C)^{-1} \). Then, when \( \epsilon \in (0, \epsilon_0) \),

\[
\int_{B^+_2} |D^2 u|^2 \eta^2 \, dz \leq C \int_{B^+_2} ((D\eta)^2 + \eta^2)(w_1 - g)^2 + ((D\eta)^2 + \eta^2)(f^2 \eta^2) \, dz.
\]

Now we make a more specific choice of \( \eta \): In addition to \( \eta \in C^1_0(B_{2R}(\hat{\xi})) \), we assume that \( \eta \equiv 1 \) on \( B_R(\hat{\xi}) \), \( 0 \leq \eta \leq 1 \) on \( \mathbb{R}^2 \), and \( |D\eta| \leq 10/R \). Also, since \( B_{2R}(\hat{\xi}) \cap \Sigma_2 \neq \emptyset \), then, for any fixed \( z^* \in B_{2R}(\hat{\xi}) \cap \Sigma_2 \), we have \( |z - z^*| \leq 2R \) for any \( z \in B_{2R}(\hat{\xi}) \). Moreover, \( (w_1 - g)(z^*) = 0 \) by (A.17). Then, since \( B_{2R}(\hat{\xi}) \subset B_{d_{01}/16(\epsilon_0)} \), we find from (A.19), (A.24), and (A.27) that, for any \( \epsilon \in B_{2R}(\hat{\xi}) \cap B^+_2 \),
global solutions of shock reflection by large-angle wedges

\[ \frac{1}{2} (w_1 - g)(z) \]

\[ \leq \frac{C}{d_{\tau_0}} \left( \| (Du, f) \|_{0,0,B_{\tau_0/2}(\tau_0) \cap B_2^+} + d_{\tau_0} \| g \|_{0,0,B_{\tau_0/2}(\tau_0) \cap B_2^+} \right) \| z - z^* \|^\beta \\
+ \| g \|_{0,0,B_{\tau_0/2}(\tau_0) \cap B_2^+} \| z - z^* \|^\beta \\
\leq C \left( \frac{1}{d_{\tau_0}} \| (Du, f) \|_{0,0,B_{\tau_0/2}(\tau_0) \cap B_2^+} + e [Du]_{0,0,B_{\tau_0/2}(\tau_0) \cap B_2^+} \\
+ \| u \|_{0,0,B_{\tau_0/2}(\tau_0) \cap B_2^+} R^\beta \right) R^\beta. \]

Using this estimate and our choice of \( \eta \), we obtain from (A.32) that

\[ \int_{B_R(z) \cap B_2^+} |D^2 u|^2 dz \]

\[ \leq C \left( \frac{1}{d_{\tau_0}} \| (Du, f) \|_{0,0,B_{\tau_0/2}(\tau_0) \cap B_2^+} + e^2 [Du]_{0,0,B_{\tau_0/2}(\tau_0) \cap B_2^+}^2 \\
+ C \left( \| u \|_{0,0,B_{\tau_0/2}(\tau_0) \cap B_2^+}^2 + \| f \|_{0,0,B_{\tau_0/2}(\tau_0) \cap B_2^+}^2 \right) (R^{2\beta} + R^2) \right) \]

which implies (A.26) for case (i).

Now we consider case (ii): \( \tilde{z} \in B_2^+ \) and \( R > 0 \) satisfy \( B_R(\tilde{z}) \subset B_{\tau_0/2}(\tau_0) \)

\( \) and \( B_{2R}(\tilde{z}) \cap \Sigma_2 = \emptyset \). Then \( B_{2R}(\tilde{z}) \subset B_{\tau_0/16}(\tau_0) \cap B_2^+ \). Let \( \eta \in C^1_\beta (B_{2R}(\tilde{z})) \)

\( \) and \( \xi = \eta^2 (w_1 - w_1(\tilde{z})) \). Note that \( \xi \in W_0^{1,2} (B_2^+) \) since \( B_{2R}(\tilde{z}) \subset B_2^+ \). Thus we can use \( \xi \) as a test function in (A.28). Performing the estimates similar to those that have been done to obtain (A.32), we have

\[ \int_{B_2^+} |D^2 u|^2 \eta^2 dz \leq C (\lambda) \int_{B_2^+} \left( \| (Du, f) \|^2 + \| f \|^2 \right) \eta^2 dz. \]

Choose \( \eta \in C^1_\beta (B_{2R}(\tilde{z})) \) so that \( \eta \equiv 1 \) on \( B_R(\tilde{z}) \), \( 0 \leq \eta \leq 1 \) on \( \mathbb{R}^2 \), and \( |D \eta| \leq 10/R \).

Note that, for any \( z \in B_{2R}(\tilde{z}) \),

\[ |w_1(z) - w_1(\tilde{z})| \leq C \left( \frac{1}{d_{\tau_0}} \| (Du, f) \|_{0,0,B_{\tau_0/2}(\tau_0) \cap B_2^+} + e [Du]_{0,0,B_{\tau_0/2}(\tau_0) \cap B_2^+} \\
+ \| u \|_{0,0,B_{\tau_0/2}(\tau_0) \cap B_2^+} R^\beta \right) R^\beta. \]

by (A.19) since \( B_{2R}(\tilde{z}) \subset B_{\tau_0/16}(\tau_0) \cap B_2^+ \). Now we obtain (A.26) from (A.33)
similar to that for case (i). Then Lemma A.2 is proved.

\[ \Box \]
LEMMA A.3. Let $\beta$ and $\varepsilon_0$ be as in Lemma A.2. Then, for $\varepsilon \in (0, \varepsilon_0)$, there exists $C(\lambda)$ such that

$$
\frac{3}{4} (u)^*_{1, \beta, B_2^+ \cup \Sigma_2} \leq C \left( \| u \|^*_{1, 0, B_2^+ \cup \Sigma_2} + \varepsilon [u]^*_{1, \beta, B_2^+ \cup \Sigma_2} + \| f \|^*_{0, 0, B_2^+} \right),
$$

where $[\cdot]^*$ and $\| \cdot \|^*$ denote the standard partially interior seminorms and norms (20, Eq. 4.29).

Proof. Estimate (A.34) follows directly from Lemma A.2 and an argument similar to the proof of [20, Th. 4.8]. Let $z_1, z_2 \in B_2^+$ with $d_{z_1} \leq d_{z_2}$ (thus $d_{z_1} z_2 = d_{z_2}$) and let $|z_1 - z_2| \leq d_{z_1}/64$. Then $z_2 \in B_{d_{z_0} / 2}(x_0) \cap B_2^+$ and, by Lemma A.2 applied to $x_0 = z_1$, we find

$$
\frac{d_{z_1} + \beta |Du(z_1) - Du(z_2)|}{|z_1 - z_2|^\beta} \leq C \left( \| u \|^*_{1, 0, B_{d_{z_1}}(z_1) \cap B_2^+} + \varepsilon d_{z_1} + \beta |Du|_{0, 0, B_{d_{z_1} / 2}(z_1) \cap B_2^+} + \| f \|^*_{0, 0, B_2^+} \right),
$$

where the last inequality holds since $2d_{z} \geq d_{z_1}$ for all $z \in B_{d_{z_1}}(z_1) \cap B_2^+$. If $z_1, z_2 \in B_2^+$ with $d_{z_1} \leq d_{z_2}$ and $|z_1 - z_2| \geq d_{z_1}/64$, then

$$
\frac{d_{z_1} + \beta |Du(z_1) - Du(z_2)|}{|z_1 - z_2|^\beta} \leq 64 (d_{z_1} |Du(z_1)| + d_{z_2} |Du(z_2)|) \leq 64 \| u \|^*_{1, 0, B_2^+ \cup \Sigma_2}.
$$

Now we can complete the proof of Theorem A.2. For sufficiently small $\varepsilon_0 > 0$ depending only on $\lambda$, when $\varepsilon \in (0, \varepsilon_0)$, we use Lemma A.3 to obtain

$$
[u]^*_{1, \beta, B_2^+ \cup \Sigma_2} \leq C(\lambda) \left( \| u \|^*_{1, 0, B_2^+ \cup \Sigma_2} + \| f \|^*_{0, 0, B_2^+} \right),
$$

We use the interpolation inequality [20, Eq. (6.89)] to estimate

$$
\| u \|^*_{1, 0, B_2^+ \cup \Sigma_2} \leq C(\beta, \delta) \| u \|_{0, B_2^+} + \delta [u]^*_{1, \beta, B_2^+ \cup \Sigma_2}
$$

for $\delta > 0$. Since $\beta = \beta(\lambda)$, we choose sufficiently small $\delta(\lambda) > 0$ to find

$$
[u]^*_{1, \beta, B_2^+ \cup \Sigma_2} \leq C(\lambda) \left( \| u \|_{0, 0, B_2^+} + \| f \|_{0, 0, B_2^+} \right)
$$

from (A.35). In particular, we obtain a global estimate in a smaller half-ball:

$$
[u]^*_{1, \beta, B_{d_{z_0} / 5}^+} \leq C(\lambda) \left( \| u \|_{0, 0, B_2^+} + \| f \|_{0, 0, B_2^+} \right).
$$
We can assume $\beta \leq \alpha$. Now we consider (A.15) as a linear elliptic equation
\begin{equation}
(A.38) \quad \sum_{i,j=1}^{2} D_i(a_{ij}(x,y) D_j w_1) = D_1 F \quad \text{in } B_{y/5}^+.
\end{equation}
where $a_{ij}(x,y) = (A_{ij}/A_{22})(Du(x,y),x,y)$ for $i+j < 4, a_{22} = 1$, and $F(x,y) = (A_1 D_1 u + A_2 D_2 u + f)/A_{22}$ with $(A_{ij}, A_i) = (A_{ij}, A_i)(Du(x,y),x,y)$. Then (A.36), combined with (A.8), implies
\begin{equation}
(A.39) \quad \|a_{ij}\|_{0,\beta,B_{y/5}} \leq C(\lambda, M).
\end{equation}
From now on, $d_z$ denotes the distance related to the partially interior norms in $B_{y/5}^+ \cup \Sigma_{y/5}$, i.e., for $z \in B_{y/5}^+$, $d_z := \text{dist}(z, \partial B_{y/5}^+ \setminus \Sigma_{y/5})$. Now, similar to the proof of Lemma A.1, we rescale (A.38) and the Dirichlet condition (A.17) from the balls $B_R(\epsilon_1) \subset B_{y/5}^+$ and $B_R(z_1) \subset B_{y/5}^+$ with $R \leq 1$ to $B = B_1^+$ or $B = B_1$, respectively, by defining
\begin{align*}
(\tilde{w}_1, \tilde{g}, \tilde{a}_{ij})(Z) &= (w_1, g, a_{ij})(z_1 + RZ), \quad \tilde{F}(Z) = RF(z_1 + RZ) \quad \text{for } Z \in B.
\end{align*}
Then $\sum_{i,j=1}^{2} D_i(\tilde{a}_{ij}(x,y) D_j \tilde{w}_1) = D_1 \tilde{F}$ in $B$, the ellipticity of this rescaled equation is the same as that for (A.38), and $\|\tilde{a}_{ij}\|_{0,\beta,B} \leq C$ for $C = C(\lambda, M)$ in (A.39), where we have used $R \leq 1$. This allows us to apply the local $C^{1,\beta}$ interior and boundary estimates for the Dirichlet problem [20, Th. 8.32, Cor. 8.36] to the rescaled problems in the balls $B_{d_z \epsilon_0}^+(\epsilon_1)$ and $B_{d_z \epsilon_0} \subset B_{y/5}^+$ as in Lemma A.1. Then, scaling back and multiplying by $d_{z_0}^2$, applying the covering argument as in Lemma A.1, and recalling the definition of $F$, we obtain that, for any $z_0 \in B_{y/5}^+ \cup \Sigma_{y/5}$,
\begin{equation}
(A.40) \quad d_{z_0}^2 |[w_1]|_{1,\beta,B_{d_z \epsilon_0} / 16(z_0) \cap B_{y/5}^+} + d_{z_0}^2 |w_1|_{1,0,B_{d_z \epsilon_0} / 16(z_0) \cap B_{y/5}^+} \leq C \left( \|D_0 u\|_{0,0,B_{d_z \epsilon_0} / 16(z_0) \cap B_{y/5}^+} + d_{z_0}^{1+\beta} |[u]|_{1,\beta,B_{d_z \epsilon_0} / 16(z_0) \cap B_{y/5}^+} \right)
\end{equation}
where we have used $d_{z_0} < 2$. Recall that $Du_1 = (D_1 u, D_1 u)$. Expressing $D_{22} u$ from (A.1) by using (A.7), (A.8) and (A.36) to estimate the $H^\alpha$ norms of $D_{22} u$, in terms of the norms of $D_{11} u, D_{12} u$, and $Du$, and by using (A.18) and (A.9) to estimate the terms involving $g$ in (A.40), we obtain from (A.40) that, for every
\[ d_{\gamma_2}^2 + \beta (D^2 u)_{0,0, i_{d,0}/16(z_0)} \cap B_{r/5}^+ + d_{\gamma_2}^2 (D^2 u)_{0,0, i_{d,0}/16(z_0)} \cap B_{r/5}^+ \]
\[ \leq C \left( d_{\gamma_2} \| D u \|_{C(B_{d,0}/2(z_0) \cap B_{r/5}^+)} + d_{\gamma_2}^{1+\beta} \| u \|_{1,0,h \cap B_{d,0}/2(z_0) \cap B_{r/5}^+} \right) \]
\[ + d_{\gamma_2} \| u \|_{1,0,h \cap B_{d,0}/2(z_0) \cap B_{r/5}^+} + \| f \|_{0,0, h \cap B_{d,0}/2(z_0) \cap B_{r/5}^+} \]
\[ + \| e_{d_{\gamma_2}}^{2+\beta} (D^2 u)_{0,0, i_{d,0}/2(z_0) \cap B_{r/5}^+} + d_{\gamma_2}^2 (D^2 u)_{0,0, i_{d,0}/2(z_0) \cap B_{r/5}^+} \). \]

From this estimate, the argument of Lemma A.3 implies

(A.41)

\[ \| u \|_{2,1; B_{r/3}^+ \cap \Sigma_{r/3}} \leq C \left( \| u \|_{1,0, B_{r/3}^+ \cap \Sigma_{r/3}} + \| f \|_{0,0, B_{r/3}^+} \right). \]

Thus, reducing \( \epsilon_0 \) if necessary and using (A.37), we conclude

(A.42)

\[ \| u \|_{2,1; B_{r/3}^+ \cap \Sigma_{r/3}} \leq C(\lambda, M)(\| u \|_{0, B_{r}^+} + \| f \|_{0,0, B_{r}^+}). \]

Estimate (A.42) implies a global estimate in a smaller ball and, in particular,

\[ \| u \|_{1,0, B_{r/3}^+} \leq C(\lambda, M)(\| u \|_{0, B_{r}^+} + \| f \|_{0,0, B_{r}^+}). \]

Now we can repeat the argument, which leads from (A.37) to (A.42) with \( \beta \) replaced by \( \alpha \), in \( B_{r/3}^+ \) (and, in particular, further reducing \( \epsilon_0 \) depending only on \( \lambda, M, \alpha \)) to obtain

(A.43)

\[ \| u \|_{2,1; B_{r/3}^+ \cap \Sigma_{r/3}} \leq C(\lambda, M, \alpha)(\| u \|_{0, B_{2}^+} + \| f \|_{0,0, B_{2}^+}). \]

which implies (A.14) and hence (A.10) for the original problem. Theorem A.2 is proved.

Now we show that the estimates also hold for the Dirichlet problem.

THEOREM A.3. Let \( \lambda > 0 \) and \( \alpha \in (0, 1) \). Let \( \Phi \in C^{2,\alpha}(\mathbb{R}) \) satisfy (A.5) and
\[ \Omega^+_2 := \Omega \cap \{ y > \Phi(x) \} \] for \( R > 0 \). Let \( u \in C^2(\Omega^+_2) \cap C(\Omega^+_2) \) satisfy (A.1) in
\[ \Omega^+_2 \] and

(A.44)

\[ u = g \quad \text{on} \quad \Gamma_\Phi := \partial \Omega \cap \{ y = \Phi(x) \}, \]

where \( A_{ij} = A_{ij}(u, x, y) \) and \( A_{ij} = A_{ij}(u, x, y), i, j = 1, 2 \), and \( f = f(x, y) \) satisfy (A.7) and (A.8), and \( g = g(x, y) \) satisfies

(A.45)

\[ \| g \|_{C^{2,\alpha}(\Omega^+_2)} \leq \lambda^{-1}, \]

with \( \lambda, \alpha \) as defined above. Assume that \( \| u \|_{C(\Omega^+_2)} \leq M \). Then

Proof. By replacing \( u \) with \( u - g \), we can assume without loss of generality that \( g = 0 \). Also, by flattening the boundary as in the proof of Theorem A.2, we
can assume $\Phi \equiv 0$. That is, we have reduced to the case when (A.1) holds in $B_1^+$ and $u = 0$ on $\Sigma_2$. Thus, $u_x = 0$ on $\Sigma_2$. Then estimate (A.45) follows from Theorem A.2.

We now derive the estimates for the oblique derivative problem.

**Theorem A.4.** Let $\lambda > 0$ and $\alpha \in (0, 1)$. Let $\Phi \in C^{2,\alpha}(\mathbb{R})$ satisfy (A.5) and

$$\Omega_2^+ := B_R \cap \{ y > \Phi(x) \} \text{ for } R > 0. \text{ Let } u \in C^2(\Omega_2^+) \cap C^1(\overline{\Omega_2^+}) \text{ satisfy}
$$

(A.46) $A_{11}u_{xx} + 2A_{12}u_{xy} + A_{22}u_{yy} + A_{11}u_x + A_{22}u_y = 0 \text{ in } \Omega_2^+.$

(A.47) $b_1 u_x + b_2 u_y + cu = 0 \text{ on } \Gamma_\Phi := B_2 \cap \{ y = \Phi(x) \},$

where $A_{ij} = A_{ij}(Du, x, y)$ and $A_i = A_i(Du, x, y)$, $i, j = 1, 2$, satisfy (A.7), (A.8), and $b_i = b_i(x, y)$, $i = 1, 2$, and $c = c(x, y)$ satisfy the following obliqueness condition and $C^{1,\alpha}$-bounds:

(A.48) $b_2(x, y) \geq \lambda$ for $(x, y) \in \Gamma_\Phi.$

(A.49) $\| (b_1, b_2, c) \|_{C^{1,\alpha}(\overline{\Omega_2^+})} \leq \lambda^{-1}.$

Assume that $\| u \|_{C^2(\overline{\Omega_2^+})} \leq M$. Then there exists $C = C(\lambda, M, \alpha) > 0$ such that

(A.50) $\| u \|_{C^{2,\alpha}(\overline{\Omega_2^+})} \leq C \| u \|_{C^2(\overline{\Omega_2^+})}.$

**Proof.** Step 1. First, we flatten the boundary $\Gamma_\Phi$ by the change of coordinates $(X, Y) = \Psi(x, y) = (x, y - \Phi(x))$. Then $(x, y) = \Psi^{-1}(X, Y) = (X, Y + \Phi(x)).$

From (A.5), $\| \Psi \|_{C^{2,\alpha}(\overline{\Omega_2^+})} + \| \Psi^{-1} \|_{C^{2,\alpha}(\overline{\Omega_2^+})} \leq C(\lambda)$, where $\Omega_2^+ := \Psi(\Omega_2^+)$ satisfies $\Omega_2^+ \subset \mathbb{R}_+^2 := \{ Y > 0 \}$ and $\Gamma_0 := \partial \Omega_2^+ \cap \{ Y = 0 \} = \Psi(\Gamma_\Phi).$ By a standard calculation, $u(X, Y) = u(x, y) := u(\Psi^{-1}(X, Y))$ satisfies the equation of form (A.46) in $\Omega_2^+$ and the oblique derivative condition of form (A.47) on $\Gamma_0$, where (A.7), (A.8), and (A.48), (A.49) are satisfied with modified constant $\lambda > 0$ depending only on $\lambda$. Also, $\| \Psi \|_{C(\overline{\Omega_2^+})} \leq M$. Thus, (A.50) follows from

(A.51) $\| u \|_{C^{2,\alpha}(\overline{\Omega_2^+})} \leq C(\lambda, M, \alpha) \| u \|_{C(\overline{\Omega_2^+})}.$

Next we note that, in order to prove (A.51), it suffices to prove that there exist $K$ and $C$ depending only on $(\lambda, M, \alpha)$ such that, if $u$ satisfies (A.5), (A.47) in $B_1^+$ and $\Sigma_1 := B_1 \cap \{ y = 0 \}$ respectively, (A.7), (A.8), and (A.48), (A.49) hold in $B_1^+$, and $| u | \leq M$ in $B_1^+$, then

(A.52) $\| u \|_{C^{2,\alpha}(\overline{B_1^+})} \leq C \| u \|_{C(\overline{B_1^+})}.$

Indeed, if (A.52) is proved, then, using also the interior estimates (A.4) in Theorem A.1 and applying the scaling argument similar to the proof of Lemma A.1, we
obtain that, for any $z_0 \in \mathbb{R}^2_+ \cup \Sigma_2$,
\[
d_{z_0}^{2+\alpha} ||v||_{C^{2,\alpha}(B_{\delta z_0}(z_0) \cap \mathbb{R}^2_+)} \leq C ||v||_{C(\delta z_0) \cap \mathbb{R}^2_+)}.
\]

From this, we use the argument of the proof of Lemma A.3 to obtain (A.51).

Thus it remains to show (A.52). First we make a linear change of variables to normalize the problem so that

\begin{align*}
(A.53) \quad b_1(0) &= 0, \quad b_2(0) = 1
\end{align*}

for the modified problem. Let

\[
(X, Y) = \tilde{\Psi}(x, y) := \frac{1}{b_2(0)} (b_2(0)x - b_1(0)y, y),
\]

Then

\[
(x, y) = \tilde{\Psi}^{-1}(X, Y) = (X + b_1(0)Y, b_2(0)Y), \quad |D\tilde{\Psi}| + |D\tilde{\Psi}^{-1}| \leq C(\lambda),
\]

where the estimate follows from (A.48) and (A.49). Then the function $w(X, Y) := v(x, y) = v(X + b_1(0)Y, b_2(0)Y)$ is a solution of the equation of form (A.46) in the domain $\tilde{\Psi}(B^+_1)$ and the boundary condition of form (A.47) on the boundary part $\tilde{\Psi}(\Sigma_1)$ such that (A.7), (A.8) and (A.48), (A.49) are satisfied with constant $\lambda > 0$ depending only on $\lambda$ and $\lambda_3$ holds, which can be verified by a straightforward calculation. Also, $\|w\|_{C^2(\tilde{\Psi}(B^+_1))} \leq M$.

Note that $\tilde{\Psi}(B^+_1) \subset \mathbb{R}^2_+ := \{ Y > 0 \}$ and $\tilde{\Psi}(\Sigma_1) = \partial \tilde{\Psi}(B^+_1) \cap \{ Y = 0 \}$.

Moreover, since $|D\tilde{\Psi}| + |D\tilde{\Psi}^{-1}| \leq C(\lambda)$, there exists $K_1 = K_1(\lambda) > 0$ such that, for any $r > 0$, $B_{r_1/K_1} \subset \tilde{\Psi}(B_r) \subset B_{K_1/r}$. Thus it suffices to prove

\[
\|w\|_{C^{2,\alpha}(B_{r_1/K_1})} \leq C \|w\|_{C(B^+_1)}
\]

for some $r \in (0, 1/K_1)$. This estimate implies (A.52) with $K = 2K_1/r$.

Step 2. As a result of the reduction performed in Step 1, it suffices to prove the following: There exist $\varepsilon \in (0, 1)$ and $c$ depending only on $\lambda, \alpha, M$ such that

if $u$ satisfies (A.46) and (A.47) in $B^+_2$ and on $\Sigma_2$, respectively, if (A.7), (A.8) and (A.48), (A.49) hold in $B^+_2$, and if (A.53) holds and $\|u\|_{0, B^+_2} \leq M$, then

\[
\|u\|_{2, \alpha, B^+_2} \leq C \|u\|_{0, B^+_2}.
\]

We now prove this claim. For $\varepsilon > 0$ to be chosen later, we rescale from $B^+_2$ into $B^+_2$ by defining

\begin{align*}
(A.54) \quad v(x, y) &= \frac{1}{\varepsilon} (u(\varepsilon x, \varepsilon y) - u(0, 0)) \quad \text{for} \quad (x, y) \in B^+_2.
\end{align*}
Then $v$ satisfies

$$
(A.55) \quad \mathring{A}_{11} v_{xx} + 2 \mathring{A}_{12} v_{xy} + \mathring{A}_{22} v_{yy} + \mathring{A}_1 v_x + \mathring{A}_2 v_y = 0 \text{ in } B_2^+,
$$

$$
(A.56) \quad v_y = \tilde{b}_1 v_x + \tilde{b}_2 v_y + \tilde{c} u + cu(0,0) \text{ on } \Sigma_2,
$$

where

$$
\tilde{A}_{ij}(p, x, y) = A_{ij}(p, ex, ey), \quad \tilde{A}_i(p, x, y) = \varepsilon A_i(p, ex, ey),
$$

$$
\tilde{b}_1(x, y) = -b_1(ex, ey), \quad \tilde{b}_2(x, y) = -b_2(ex, ey) + 1, \quad \tilde{c}(x, y) = -\varepsilon c(ex, ey).
$$

Then $\mathring{A}_{ij}$ and $\mathring{A}_i$ satisfy (A.7) (A.8) in $B_2^+$ and, using (A.49), (A.53), and $\varepsilon \leq 1$, we have

$$
(A.57) \quad \| (\mathring{b}_1, \mathring{b}_2, \tilde{c}) \|_{1, \varepsilon, B_2^+} \leq C \varepsilon \quad \text{for some } C = C(\lambda).
$$

Now we follow the proof of Theorem A.2. We use the partially interior norms [20, Equation 4.29] in the domain $B_2^+ \cup \Sigma_2$ whose distance function is $d_z = \text{dist}(z, \partial B_2^+ \setminus \Sigma_2)$. We introduce the functions $w_i = D_i v$, $i = 1, 2$, to conclude from (A.55) that $w_1$ and $w_2$ are weak solutions of equations

$$
(A.58) \quad D_1 \left( \frac{A_{11}}{A_{22}} D_1 w_1 + \frac{2A_{12}}{A_{22}} D_2 w_1 \right) + D_2 w_1 = -D_1 \left( \frac{A_1}{A_{22}} D_1 v + \frac{A_2}{A_{22}} D_2 v \right),
$$

$$
(A.59) \quad D_1 w_2 + D_2 \left( \frac{2A_{12}}{A_{11}} D_1 w_2 + \frac{A_{22}}{A_{11}} D_2 w_2 \right) = -D_2 \left( \frac{A_1}{A_{11}} D_1 v + \frac{A_2}{A_{11}} D_2 v \right)
$$

in $B_2^+$, respectively. From (A.56), we have

$$
(A.60) \quad w_2 = \tilde{g} \quad \text{on } \Sigma_2,
$$

where $\tilde{g} := \mathring{b}_1 v_x + \mathring{b}_2 v_y + \tilde{c} v + \tilde{c} u(0,0)$ in $B_2^+$.

Using (A.59) and the Dirichlet boundary condition (A.60) for $w_2$ and following the proof of Lemma A.1, we can show the existence of $\beta \in (0, \alpha]$ and $C$ depending only on $\lambda$ such that, for any $z_0 \in B_2^+ \cup \Sigma_2$,

$$
(A.61) \quad d_{20}^{\beta} [w_2]_{0, \beta, B_{d_{20}}^+ / \Sigma} \leq C \left( \| D v \|_{0, B_{d_{20}}^+ / \Sigma} + \| D \tilde{g} \|_{0, \beta, B_{d_{20}}^+ / \Sigma} \right).
$$

Next we obtain the Hölder estimates of $D v$ if $\varepsilon$ is sufficiently small. We first note that, by (A.57), $\tilde{g}$ satisfies

$$
(A.62) \quad | D \tilde{g} | \leq C \varepsilon \left( | D^2 v | + | D v | + | v | + \| v \|_{0, B_{2^+}} \right) \quad \text{in } B_2^+,
$$

$$
(A.63) \quad \| D \tilde{g} \|_{0, \beta, B_{d_{2}^+} / \Sigma} \leq C \varepsilon \left( \| v \|_{1, \beta, B_{d_{2}} / \Sigma} + \| v \|_{0, B_{2^+}} \right).
$$
for $C = C(\lambda)$. The term $\epsilon \|u\|_{0, B_{2e}^+}$ in (A.62) to (A.63) comes from the term $\tilde{g} u(0, 0)$ in the definition of $\tilde{g}$. We follow the proof of Lemma A.2, but we now use the integral form of (A.59) with test functions $\zeta = \eta^2 (w_2 - \tilde{g})$ and $\xi = \eta^2 (w_2 - w_2(\tilde{g}))$ to get an integral estimate of $|Dw_2|$ and thus of $|D_jw_1|$ for $i + j > 2$, and then use (A.55) to estimate the remaining derivative $D_{12}$. In these estimates, we use (A.61) to (A.63). We obtain that, for sufficiently small $\epsilon$ depending only on $\lambda$,

\[
\begin{align*}
(\text{A.64}) \quad & d^\beta_{z_0}[Dv]_{0, \beta, B_{2e}^+/2} \leq C \left( \|v\|_{0, B_{2e}^+} + \epsilon d^\beta_{z_0}[Dv]_{0, \beta, B_{2e}^+/2} \right) \\
& + \epsilon d^\beta_{z_0}[Dv]_{0, \beta, B_{2e}^+/2} \\
& \text{for any } z_0 \in B_{2e}^+ \cup \Sigma_2, \text{ with } C = C(\lambda). \quad \text{Using (A.64), we follow the proof of Lemma A.3 to obtain} \\
(\text{A.65}) \quad & [v]^*_{1, \beta, B_{2e}^+ \cup \Sigma_2} \leq C(\lambda) \left( \|v\|^*_{1, 0, B_{2e}^+ \cup \Sigma_2} + \epsilon \|u\|_{0, B_{2e}^+} \right).
\end{align*}
\]

Now we choose sufficiently small $\epsilon > 0$ depending only on $\lambda$ to have

\[
[v]^*_{1, \beta, B_{2e}^+ \cup \Sigma_2} \leq C(\lambda) \left( \|v\|^*_{1, 0, B_{2e}^+ \cup \Sigma_2} + \epsilon \|u\|_{0, B_{2e}^+} \right).
\]

Then we use the interpolation inequality, similar to the proof of (A.36), to have

\[
(\text{A.66}) \quad \|u\|^*_{1, \beta, B_{2e}^+ \cup \Sigma_2} \leq C(\lambda) \left( \|u\|^*_{0, B_{2e}^+} + \epsilon \|u\|_{0, B_{2e}^+} \right).
\]

By (A.54) with $\epsilon = \epsilon(\lambda)$ as chosen above, (A.66) implies

\[
(\text{A.67}) \quad \|u\|^*_{1, \beta, B_{2e}^+ \cup B_{2e}^+} \leq C(\lambda) \|u\|_{0, B_{2e}^+}.
\]

Then problem (A.46) and (A.47) can be regarded as a linear oblique derivative problem in $B_{7e/4}$, whose coefficients $a_{ij}(x, y) := A_{ij}(Du(x, y), x, y)$ and $a_i(x, y)$ have the estimate in $C^0, B_{7e/4}$ by a constant depending only on $\lambda, M$ from (A.67) and (A.8). Moreover, we can assume $\beta \leq \alpha$ so that (A.49) implies the estimates of $(b_i, c)$ in $C^{1, \beta}(B_{7e/4})$ with $\epsilon = \epsilon(\lambda)$. Then the standard estimates for linear oblique derivative problems [20, Lemma 6.29] imply

\[
(\text{A.68}) \quad \|u\|_{2, \beta, B_{7e/2}^+} \leq C(\lambda, M) \|u\|_{0, B_{7e/4}^+}.
\]

In particular, the $C^0, B_{5e/2}$-norms of the coefficients $(a_{ij}, a_i)$ of the linear (A.46) are bounded by a constant depending only on $\lambda, M$, which implies

\[
(\text{A.69}) \quad \|u\|_{2, \beta, B_{5e/2}^+} \leq C(\lambda, M) \|u\|_{0, B_{5e/2}^+}.
\]
by again applying [20, Lemma 6.29]. This implies the assertion of Step 2, thus
Theorem A.4.

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References

MR 86i:35036 Zbl 0614.76074
Flat free boundaries are Lipschitz, Comm. Pure Appl. Math. 42 (1989), 55–78. MR 90b:35246
Zbl 0676.35086
MR 2002h:76077 Zbl 1015.76038
MR 2001m:76055 Zbl 1017.76040
[9] T. Chang and G. Q. Chen, Some fundamental concepts about system of two spatial dimen-
[10] G.-Q. Chen and M. Feldman, Multidimensional transonic shock and free boundary prob-
Zbl 1015.35075
[12] _______, Steady transonic shocks and free boundary problems for the Euler equations in infinite
[13] _______, Existence and stability of multidimensional transonic flows through an infinite nozzle
Zbl 1114.91065
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