Family Floer program

and non-archimedean SYZ mirror construction

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SYZ's picture: Review - 1/2

- A (special) Lagrangian fibration π (possibly with singularities)

$$\begin{array}{c} X \\ \Big\downarrow \pi \\ B_0 \subset B \end{array}$$

  - B_0 is the smooth locus of π:
  - $L_q := \pi^{-1}(q)$ is the Lag. fiber over q

- By Arnold-Liouville's Theorem,
  - For $q \in B_0$, L_q must be a torus $T^n = (S^1)^n$.
  - B_0 has an integral affine structure (locally looks like $\mathbb{R}^n$)

- $X_0 \equiv \pi^{-1}(B_0) \to B_0$ gives a Lagrangian torus fibration.

SYZ's picture: Review - 2/2

- Following SYZ's idea of T-duality, the mirror could be obtained by taking a dual fibration.

$$\begin{array}{ccc} X_0 & \qquad\qquad & X_0^\vee \equiv \bigcup_{q\in B_0} H^1(L_q, U(1)) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi^\vee} \\ B_0 & & B_0 \end{array}$$

- Classically, the dual fiber is expected to be

  $H^1(L_q, U(1))$

  $= \{$ all flat $U(1)$-connections on L_q up to gauge equivalence. $\}$

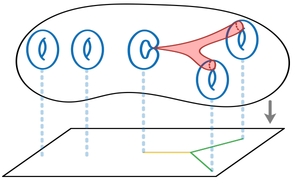
  $\cong U(1)^n \cong (S^1)^n$ is also a torus. (because L_q is a torus)

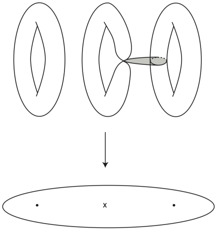
- Now, $X_0^\vee$ can be regarded as a dual torus fibration ('T-duality')

The T-duality need to be modified by so-called '**quantum corrections**' **(q.c.)** which are given by counting

holomorphic disks in $\pi_2(X, L_q)$ for $q \in B_0$ (**Lagrangian Floer theory**)

Here we still require $q \in B_0$ above, L_q is still smooth, but such a holomorphic disk can get in touch with those singular fibers over $B \setminus B_0$.





So these q.c. may include info. outside the fibration,
e.g. singular fibers
e.g. (toric) divisors

c.f. FOOO, cpt toric mfd (Maslov-two disks)

$\Longrightarrow$ q.c. is necessary

Namely, we are gonna to study **Lagrangian Floer theory** for the family $(L_q)_{q\in B_0}$ of torus fibers *simultaneously*. This gives the name **Family Floer**.

- The Family Floer theory is invented by Fukaya in around 2000; later, Tu and Abouzaid made great progress.
- Roughly, FF predicts that the dual torus fiber of $L_q,\ q\in B_0$ is **not**

$$H^1(L_q, U(1)) \cong U(1)^n$$

but (possibly a subset of) a 'non-archimedean torus'

$$H^1(L_q, U_\Lambda) \cong U_\Lambda^n$$

where U_Λ (later) is the multiplicative group of the **Novikov field**

$$\Lambda := \left\{ \sum_{i=0}^{\infty} a_i T^{E_i} \mid a_i \in \mathbb{C}, E_i \nearrow +\infty \right\}$$

This is a non-archimedean field, just like $\mathbb{C}((T))$.

Quantum correction and Family Floer - 3/3

Previous FF works more or less rely on tautological unobstructedness

Assumption: There is no holomorphic disk in $\pi_2(M, L_q)$ for all q.

Motivation # 1

Can we somehow drop or weaken this assumption? Because

(i) These disks are the 'quantum corrections' we need.

(ii) At least, Maslov-two disks $\implies$ Landau-Ginzburg potential (FOOO)

Moreover, the expected mirror $X^\vee \equiv \sqcup_q H^1(L_q; U_\Lambda)$ is just a **set** at first.
It is a very delicate issue to put an 'analytic space' structure on $X^\vee$!

Motivation # 2

We aim to develop a **rigid analytic space*** structure from the ground up.

will see: A_∞-homotopy in Lag. Floer (A) $\implies$ Isom. of rigid analytic (B)

Main theorem

Suppose we have a Lagrangian torus fibration $\pi : U \to B_0$ on an open subset U of a closed symplectic manifold (M, ω). (e.g. compact toric)

Main Theorem

Assume Maslov indices of pseudo-holomorphic disks are non-negative. Then we can associate to (M, π) a triple $(M^\vee, W^\vee, \pi^\vee)$ consisting of

1. a Λ-rigid analytic space $M^\vee$; mirror space
2. a global function $W^\vee$; Landau-Ginzburg potential
3. a projection $\pi^\vee : M^\vee \to B_0$ 'SYZ dual fibration'

unique up to isomorphism of rigid analytic spaces.

(Our mirror construction is **independent of choices!**)

- Kontsevich-Soibelman proposed to use non-archimedean geometry to study mirror symmetry. We justify this proposal in some sense.

Rigid analytic geometry: Review

- $\mathrm{val} : \sum_{i\geq 0} a_i T^{E_i} \in \Lambda$ $(a_0 \neq 0) \mapsto E_0 \in \mathbb{R}$; norm $|\cdot| = \exp(-\mathrm{val}())$;
  $\Longrightarrow$ **adic topology** on Λ; such a field is a non-archimedean field
- mul. gp. $U_\Lambda = \{\mathrm{val}(z) = 0\} = \{|z| = 1\}$; analogue of $U(1) \equiv S^1$
- Nov. ring $\Lambda_0 := \{\mathrm{val} \geq 0\}$; $\Lambda_+ := \{\mathrm{val} > 0\}$ used to hold q.c. data
- $U_\Lambda = \mathbb{C}^* \oplus \Lambda_+$; $\Lambda_0 = \mathbb{C} \oplus \Lambda_+$. $[u] \neq 0 \in \pi_2(M, L)$; $E(u) > 0$

Algebraic/analytic geom. over $\mathbb{C}$	**Rigid analytic geom. over Λ**
Polynomial alg. $R_n = \mathbb{C}[z_1, \ldots, z_n]$	Tate's algebra $T_n := \Lambda\langle z_1, \ldots, z_n\rangle$ $= \{f = \sum a_\nu \mathbf{z}^\nu \mid \mathrm{val}(a_\nu) \to 0\}$ $= \{f \mid f$ converges on unit ball $B^n_\Lambda\}$
$\mathrm{Spec}(R_n) \cong \mathbb{C}^n$, affine space	$\mathrm{Sp}(T_n) \cong B^n_\Lambda = \{(z_i) \in \Lambda^n \mid \lvert z_i\rvert \leq 1\}$
Affine scheme $\mathrm{Spec}(R_n/\mathfrak{a}) = V(\mathfrak{a})$	Affinoid space $\mathrm{Sp}(T_n/\mathfrak{a}) = V(\mathfrak{a})$
Variety/scheme	Rigid analytic space/variety (Defn)
$\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$, $z_i \mapsto \log \lvert z_i\rvert$ - a torus fibration - fiber is topologically $T^n = U(1)^n$	**trop** : $(\Lambda^*)^n \to \mathbb{R}^n$, $z_i \mapsto \mathrm{val}(z_i)$ - 'a non-archimedean torus fibration' - fiber at 0 is our nonar. torus U^n_Λ

Local chart of $M^\vee$ - 1/6

Claim: $\pi^\vee : M^\vee \to B_0$ is locally **trop** : $(\Lambda^*)^n \to \mathbb{R}^n$, $z_i \mapsto \mathrm{val}(z_i)$.

Consider the so-called polytopal domain: (an affinoid space)

$$\textbf{trop}^{-1}(\Delta) \equiv \mathrm{Sp}\ \Lambda\langle\Delta\rangle \qquad \Delta \subset \mathbb{R}^n \text{ a } \textit{rational polyhedron}$$

Here $\Lambda\langle\Delta\rangle$ is the so-called polyhedral affinoid algebra:

$$\Lambda\langle\Delta\rangle = \{ f = \sum_{\nu\in\mathbb{Z}^d} a_\nu \mathbf{z}^\nu \mid {}^{\mathrm{val}(a_\nu)+\langle\nu,u\rangle\to\infty}_{\text{for all } u\in\Delta} \} \equiv \{ f \mid f \text{ converges on } \textbf{trop}^{-1}(\Delta) \}$$

In our situation, the base B_0 locally looks like $\mathbb{R}^n$. It makes sense to define

$$\Lambda\langle\Delta, q\rangle \qquad \text{for } q \in B_0 \text{ and a small } \textit{rational polyhedron } \Delta \subset B_0$$

$GL(n,\mathbb{Z})$ preserves 'rational' condition. Think: $\Delta \subset \mathbb{R}^n$; q is like the origin.

Example: If $\Delta = \{q\}$ then $\mathrm{Sp}\ \Lambda\langle q, q\rangle \equiv \textbf{trop}^{-1}(0) \equiv U_\Lambda^n \equiv H^1(L_q; U_\Lambda)$.

Local chart of $M^\vee$ - 2/6

Claim:

Our mirror space $M^\vee$ is locally given by a closed analytic subvariety

$$V(\mathfrak{a}) := \mathrm{Sp}\left(\Lambda\langle\Delta, q\rangle/\mathfrak{a}\right) \quad \text{in the polytopal domain } \mathbf{trop}^{-1}(\Delta)$$

cut out by the ideal $\mathfrak{a}$ (defn later) of 'weak Maurer-Cartan equations''weak Maurer-Cartan equations'

- Following FOOO, we can associate to $L = L_q$ a filtered A_∞ algebra

$$(H^*_{dR}(L), \mathfrak{m}) \qquad \mathfrak{m}_k = \sum_\beta T^{E(\beta)} \mathfrak{m}_{k,\beta};$$

  where $\beta \in \pi_2(M, L)$, $E(\beta) = \omega \cap \beta$ is the energy
- $\mathfrak{m}_{k,\beta} : H^*(L) \otimes \cdots \otimes H^*(L) \to H^*(L)$ is a map of degree $2 - k - \mu(\beta)$ (counting holo disks in class β) $\mathfrak{m} = (\mathfrak{m}_k) = (\mathfrak{m}_{k,\beta})$ satisfies A_∞ eq
- Gromov's compactness $\implies$ $\mathfrak{m}$ converges for adic topology on Λ.
- Use homological perturbation to obtain $\mathfrak{m}$ (canonical model).

Local chart of $M^\vee$ - 3/6

Definition of Maurer-Cartan equation (MC eq)

$$\sum_\beta \sum_k T^{E(\beta)} \mathfrak{m}_{k,\beta}(b,\dots,b) = 0; \qquad \text{for } b \in H^1(L;\Lambda_+)$$

There is an important property of $\mathfrak{m}$: for any $b \in H^1(L)$, we have

Divisor axioms

$$\sum_{\ell=0}^k \mathfrak{m}_{k+1,\beta}(x_1,\dots,x_{\ell-1},b,x_\ell,\dots,x_k) = \partial\beta \cap b \cdot \mathfrak{m}_{k,\beta}(x_1,\dots,x_k)$$

$$\implies \mathfrak{m}_{k,\beta}(b,\dots,b) = \frac{(\partial\beta\cap b)^k}{k!}\mathfrak{m}_{0,\beta} \quad \text{using combinatorics}$$

By divisor axioms, MC eq can be transferred to (which we prefer)

$$\sum_\beta T^{E(\beta)} e^{\partial\beta\cap b}\mathfrak{m}_{0,\beta} = 0; \qquad b \in H^1(L;\Lambda_0),\ \partial\beta \in \pi_1(L) \cong \mathbb{Z}^n$$

Remark: This idea was used in FOOO's work on compact toric manifolds.

Idea: Forget about the original MC eq $\sum T^{E(\beta)} e^{\partial\beta\cap b}\mathfrak{m}_{0,\beta}$, and focus on:

MC formal power series *(will not lose any information)*

$P = \sum_\beta T^{E(\beta)} Y^{\partial\beta}\mathfrak{m}_{0,\beta}$. (a collection of series, $\mathfrak{m}_{0,\beta} \in H^*(L) \cong \mathbb{R}^N$)

- Fix a basis $(\theta_i) \subset H^1$, $Y^{\partial\beta} \longleftrightarrow Y_1^{\partial_1\beta}\cdots Y_n^{\partial_n\beta}$ with $\partial_i\beta = \partial\beta\cap\theta_i$.
- If we set $b = \sum_i x_i\theta_i$ $(x_i \in \Lambda_0)$ then $e^{\partial\beta\cap b} = (e^{x_1})^{\partial_1\beta}\cdots(e^{x_n})^{\partial_n\beta}$

$\mathbf{y} = (y_i = e^{x_i})_{i=1}^n$ is a point in U_Λ^n; Any point $\mathbf{y}$ in U_Λ^n is in this form.

Point 1: The restriction function $P|_{U_\Lambda^n}$ *'recovers'* the MC equation.

$$P(\mathbf{y}) = \sum T^{E(\beta)} y_1^{\partial_1\beta}\cdots y_n^{\partial_n\beta}\mathfrak{m}_{0,\beta}$$
$$= \sum T^{E(\beta)} e^{\partial\beta\cap b}\mathfrak{m}_{0,\beta}$$
$$= \sum T^{E(\beta)}\mathfrak{m}_{k,\beta}(b,\ldots,b) \quad \text{(DA)}$$

Define

$U_\Lambda^n \subset Domain(P) \subset (\Lambda^*)^n$,

the domain of convergence.

Local chart of $M^\vee$ - 5/6

Point 2: P converges on a bigger domain $\mathbf{trop}^{-1}(\Delta) \supset U_\Lambda^n \equiv \mathbf{trop}^{-1}(0)$.

Reverse isoperimetric inequality: $E(\beta) \geq cL(\partial\beta)$ (Groman-Solomon).
Take $\Delta \ni 0$ where $0 \leftrightarrow q \in B_0$ s.t. $\mathrm{diam}(\Delta) \leq c \implies$ Point 2

will see: P contains info. of nearby Lag. fibers over Δ. (Fukaya's trick)

Moreover, there is an important **'rigidity'** for the formal power series:

Point 3: Conversely, the function $P|_{U_\Lambda^n}$ determines the series P itself!

Let $f = \sum a_\nu z^\nu$ be a formal power series in $\Lambda[[z_1^\pm, \ldots z_n^\pm]]$. Then

Lemma X 'vanish center fiber $\implies$ vanish everywhere' **(not hard)**

If f vanishes on $U_\Lambda^n \cong \mathbf{trop}^{-1}(0)$, then $f \equiv 0$ is identically zero.

(later) $\implies$ **Wall crossing formula** $\implies$ **Transition maps well-defined**

Notice that $\mathfrak{m}_{0,\beta} \in H^{2-\mu(\beta)}(L)$; we also assume $\mu(\beta) \geq 0$. So, consider:

$$P = \left(\sum_{\mu(\beta)=2} T^{E(\beta)} Y^{\partial\beta} \mathfrak{m}_{0,\beta}\right) + \left(\sum_{\mu(\beta)=0} T^{E(\beta)} Y^{\partial\beta} \mathfrak{m}_{0,\beta}\right)$$

MC eq = W + weak MC eq

$$P =: W \cdot \mathbf{1} + \sum_{p<q} Q_{pq} \cdot \theta_{pq} \qquad \theta_{pq} := \theta_p \wedge \theta_q \in H^2(L) \text{ basis}$$

rec. Δ small, $Domain(P) \supset \mathbf{trop}^{-1}(\Delta) \implies W, Q_{pq} \in \Lambda\langle \Delta, q\rangle \cong \Lambda\langle\Delta\rangle$.

Definition: $\mathfrak{a}$ = the ideal gen. by all Q_{pq} = 'the ideal of weak MC eqs'.

(i) A **local chart** of the mirror space $M^\vee$ is defined to be $V(\mathfrak{a}) := \mathrm{Sp}\left(\Lambda\langle\Delta, q\rangle / \mathfrak{a}\right) \subset \mathbf{trop}^{-1}(\Delta)$ 'zero locus of weak MC eqs'

(ii) Moreover, this W can be viewed as a function on $V(\mathfrak{a})$; it will be a local piece of the global LG potential $W^\vee$.

Transition map - 1/5

Now that we have lots of 'local charts'. Our next step is to *glue* them!

Let $V(\mathfrak{a})$ and $V(\tilde{\mathfrak{a}})$ be the local charts as before.

A transition map $\Phi^*: V(\tilde{\mathfrak{a}}) \to V(\mathfrak{a})$

$$\Updownarrow$$

like scheme theory

An affinoid algebra homomorphism $\Phi: \Lambda\langle \Delta, q\rangle/\mathfrak{a} \to \Lambda\langle \tilde{\Delta}, \tilde{q}\rangle/\tilde{\mathfrak{a}}$

will do: First find a homo. $\Lambda\langle \Delta, q\rangle \to \Lambda\langle \tilde{\Delta}, \tilde{q}\rangle$; then pass to the quotient

Two main aspects for the construction

(I) Fukaya's trick **(II) A_∞ homotopy equivalence**

Transition map - 2/5

(I) Fukaya's trick

Choose $F \in Diff_0(M)$ s.t. $F(L) = \tilde{L}$. There is a natural identification:

$$\mathcal{M}(J, L; \beta) \cong \mathcal{M}(F_* J, \tilde{L}; \tilde{\beta})$$

$$\boxed{u \text{ is } J\text{-holomorphic}} \mapsto \boxed{F \circ u \text{ is } F_*J\text{-holomorphic}}$$

where $\beta \in \pi_2(M, L)$, $\tilde{\beta} \equiv F_*\beta \in \pi_2(M, \tilde{L})$, and $F_*J := dF \circ J \circ dF^{-1}$.

Fukaya's trick

The two A_∞ algebras $\mathfrak{m}^{J,L}$ and $\mathfrak{m}^{F_*J,\tilde{L}}$ are closely related to each other.

- 'Counting numbers' are basically the same; only the energy is varied.
- Explicitly,
$$\begin{cases} \mathfrak{m}^{F_*J,\tilde{L}}_{k,\tilde{\beta}}(x_1, \dots, x_k) = F^{-1*}\mathfrak{m}^{J,L}_{k,\beta}(F^*x_1, \dots, F^*x_k) \\ E(\tilde{\beta}) = E(\beta) + \langle \partial\beta, \tilde{q} - q\rangle E(\tilde{\beta}) = E(\beta) + \langle \partial\beta, \tilde{q} - q\rangle \end{cases}$$
- Intuitively, may call $\mathfrak{m}^{F_*J,\tilde{L}}$ the F-pushforward A_∞ algebra of $\mathfrak{m}^{J,L}$

Transition map - 3/5

- Think of $q \leftrightarrow 0$, $\tilde{q} \leftrightarrow c \in \mathbb{R}^n$. Recall $P = \sum T^{E(\beta)} Y^{\partial\beta} \mathfrak{m}_{0,\beta}^{J,L}$.

1. $P|_{U_\Lambda^n} \implies$ the MC eq of $\mathfrak{m}^{J,L}$ (said before; $U_\Lambda^n \equiv \textbf{trop}^{-1}(0)$)
2. $P|_{\textbf{trop}^{-1}(c)} \implies$ the MC eq of $\mathfrak{m}^{F_* J, \tilde{L}}$ (further using Fukaya's trick)

Using Fukaya's tricks, we justify our previous message:

P contains info. of nearby Lag. fibers (all L_c for $c \in \Delta$)

Recall: $Domain(P) \supset \textbf{trop}^{-1}(\Delta) \supset U_\Lambda^n$ for small Δ (rev. iso. ineq.)

Why is Fukaya's trick useful? **Goal**: relate $V(\mathfrak{a})$ with $V(\tilde{\mathfrak{a}})$

First, we want to compare $\mathfrak{m}^{J,L}$ and $\mathfrak{m}^{J,\tilde{L}}$ for fixed J but different $L, \tilde{L}$.
Now, only need to compare $\mathfrak{m}^{F_* J,\tilde{L}}$ and $\mathfrak{m}^{J,\tilde{L}}$ for varied J but the same $\tilde{L}$.

Transition map - 4/5

To compare $\mathfrak{m}^{F_*J,\tilde{L}}$ and $\mathfrak{m}^{J,\tilde{L}}$, we take a path $\mathbf{J} : F_*J \leftrightsquigarrow J$; $\Longrightarrow$

(II) A_∞ **homotopy equivalence** $\mathfrak{C}^F = (\mathfrak{C}^F_k) = (\mathfrak{C}^F_{k,\beta})$ a collection of op.

$$\mathfrak{C}^F \triangleq \mathfrak{C}^{F,\mathbf{J}} : \quad \mathfrak{m}^{J,\tilde{L}} \longrightarrow \mathfrak{m}^{F_*J,\tilde{L}} \quad \approx \quad \mathfrak{m}^{J,L}$$

$$\mathfrak{C}^F \triangleq \mathfrak{C}^{F,\mathbf{J}} : \quad \mathfrak{m}^{J,\tilde{L}} \longrightarrow \mathfrak{m}^{F_*J,\tilde{L}} \quad \approx \quad \mathfrak{m}^{J,L}$$

$$\mathfrak{C}^F \triangleq \mathfrak{C}^{F,\mathbf{J}} : \quad \mathfrak{m}^{J,\tilde{L}} \longrightarrow \mathfrak{m}^{F_*J,\tilde{L}} \quad \approx \quad \mathfrak{m}^{J,L}$$

its MC eq gives $V(\tilde{\mathfrak{a}})$

its MC eq gives $V(\mathfrak{a})$

Fukaya's trick

Define transition map (First step)

$$\phi^F : \Lambda\langle\Delta, q\rangle \to \Lambda\langle\tilde{\Delta}, \tilde{q}\rangle, \; Y^\alpha \mapsto T^{\langle\alpha,\tilde{q}-q\rangle} Y^\alpha \exp\langle\alpha, \sum T^{E(\beta)} Y^{\partial\beta} \mathfrak{C}^F_{0,\beta}\rangle$$

Transition map - 5/5

Main Issue: mirror construction should **not** depend on choices

But, $\phi^F : \Lambda\langle \Delta, q\rangle \to \Lambda\langle \tilde{\Delta}, \tilde{q}\rangle$ depends on choices !! e.g. F and $\mathbf{J}$.

Fortunately, this doesn't matter for the following two **claims**:
Don't forget: what we need is *not* ϕ^F *but* a quotient homomorphism:

Claim (A) ϕ^F can pass to the quotient $\Phi = [\phi^F] : \Lambda\langle \Delta, q\rangle/\mathfrak{a} \to \Lambda\langle \tilde{\Delta}, \tilde{q}\rangle/\tilde{\mathfrak{a}}$
Moreover, we will prove $\Phi(W) = \tilde{W}$ and a global LG is very possible.

Definition: This Φ defines our **transition map** $\Phi^* : V(\tilde{\mathfrak{a}}) \to V(\mathfrak{a})$.

Claim (B): The quotient Φ only depends on the homotopy class of $\mathfrak{C}^F$.
$\Longrightarrow$ Our transition map Φ^* does **not** depend on choices!

We first explain Claim (A). It will be a result of our *Wall Crossing Formula*.

Claim (A) & Wall Crossing Formula - 1/3

For $\mathfrak{m} := \mathfrak{m}^{J,L}$ and $\tilde{\mathfrak{m}} := \mathfrak{m}^{J,\tilde{L}}$, we decompose their MC power series:

- $P = W \cdot \mathbf{1} + \sum Q_{pq}\theta_{pq}$
- $\tilde{P} = \tilde{W} \cdot \tilde{\mathbf{1}} + \sum \tilde{Q}_{pq}\tilde{\theta}_{pq}$

- $\mathbf{1}, \tilde{\mathbf{1}}$ = generators of H^0
- $\theta_{pq}, \tilde{\theta}_{pq}$ = basis of H^2
- $\mathfrak{a} = (Q_{pq}),\ \tilde{\mathfrak{a}} = (\tilde{Q}_{pq})$ are ideals of weak Maurer-Cartan eqs.

Wall Crossing Formula (the key to Claim (A))

$$\phi^F(\langle \eta, P\rangle) = \langle F_*\eta, \tilde{\mathbf{1}}\rangle \tilde{W} + \sum R^{F,\eta}_{pq} \tilde{Q}_{pq}$$

- Here $\eta \in H_*(L)$ and $R^{F,\eta}_{pq} = \sum_{\tilde{\beta}} T^{E(\tilde{\beta})} Y^{\partial\tilde{\beta}} \langle F_*\eta, \mathfrak{C}^F_{1,\tilde{\beta}}(\tilde{\theta}_{pq})\rangle$.
- η dual to $\theta_{pq} \implies \langle \eta, P\rangle = Q_{pq},\ \langle F_*\eta, \tilde{\mathbf{1}}\rangle = 0 \implies \phi^F(\mathfrak{a}) \subset \tilde{\mathfrak{a}}$
- η dual to $\mathbf{1} \implies \langle \eta, P\rangle = W,\ \langle F_*\eta, \tilde{\mathbf{1}}\rangle = 1 \implies \phi^F(W) \in \tilde{W} + \tilde{\mathfrak{a}}$

WCF $\implies$ Claim (A) i.e. the quotient $\Phi = [\phi^F]$ exists and $\Phi(W) = \tilde{W}$.

Claim (A) & Wall Crossing Formula - 2/3

Now, it remains to prove the Wall Crossing Formula:

Strategy of proof ? **Lemma X !**

(Recall) If f vanishes on $U_\Lambda^n \cong \mathbf{trop}^{-1}(0)$, then $f \equiv 0$ is identically zero.

1. If we want to show an identity $f_1 \equiv f_2$ of formal power series in $\Lambda[[z_1^\pm, \dots z_n^\pm]]$, then it suffices to show $f_1 = f_2$ holds restricting to U_Λ^n.
2. Moreover, recall the Novikov field enjoys the property that every $y \in U_\Lambda$ can be represented by $y = e^x$ for some $x \in \Lambda_0$.
3. Enough to show $f_1(e^{x_1}, \dots, e^{x_n}) = f_2(e^{x_1}, \dots, e^{x_n})$. Put $b = \sum x_i \theta_i$ $\implies e^{\partial\beta \cap b} \equiv (e^{x_1})^{\partial_1 \beta} \cdots (e^{x_n})^{\partial_n \beta} =: \mathbf{y}^{\partial\beta}$, where $\partial_i \beta = \partial\beta \cap \theta_i$
4. Apply *Divisor Axioms* backward; e.g. $\mathbf{y}^{\partial\beta} \mathfrak{C}^F_{0,\beta} \rightsquigarrow \sum_k \mathfrak{C}^F_{k,\beta}(b, \dots, b)$
5. A_∞ structures in Lagrangian Floer join the game!

Example $P = \sum T^{E(\beta)} Y^{\partial\beta} \mathfrak{m}_{0,\beta} \xleftrightarrow{\text{Lemma X}} P|_{U_\Lambda^n} \xleftrightarrow{\text{Div.Axiom}} \text{MC eq}$

(recall) $\phi^F : Y^\alpha \mapsto T^{\langle \alpha, \tilde{q}-q\rangle} Y^{\tilde{\alpha}} \exp\langle \tilde{\alpha}, \sum T^{E(\beta)} Y^{\partial\beta} \mathfrak{C}^F_{0,\beta}\rangle$

- $\alpha \in \pi_1(L) \quad \longleftrightarrow \quad \tilde{\alpha} := F_*\alpha \in \pi_1(\tilde{L}) \qquad \cong \mathbb{Z}^n$.
- **Observe**: Only Maslov-zero disks contribute to 'wall-crossing': because $\mathfrak{C}^F_{0,\beta} \in H^{1-\mu(\beta)}(\tilde{L})$; we also assume $\mu(\beta) \geq 0$.

Proof of WCF: Wall Crossing Formula $\Leftrightarrow$ A_∞ equation (DA & FT)

Let $\mathbf{y} = (e^{x_1}, \ldots, e^{x_n}) \in U^n_\Lambda$ and $b = \sum x_i \theta_i$. It suffices to compute

$\phi^F(\langle \eta, P\rangle)|_{Y=\mathbf{y}}$

$= \phi^F\left(\sum \langle \eta, \mathfrak{m}_{0,\beta}\rangle T^{E(\beta)} Y^{\partial\beta}\right)|_{\mathbf{y}} = \sum \langle \eta, \mathfrak{m}_{0,\beta}\rangle T^{E(\beta)} \phi^F(Y^{\partial\beta})|_{\mathbf{y}}$

$= \sum \langle \eta, \mathfrak{m}_{0,\beta}\rangle \; T^{E(\beta)} T^{\langle \partial\beta, \tilde{q}-q\rangle} \; T^{E(\beta)} T^{\langle \partial\beta, \tilde{q}-q\rangle} \; Y^{\partial\tilde{\beta}} \; Y^{\partial\tilde{\beta}} \exp\langle \partial\tilde{\beta}, \sum T^{E(\gamma}$

$= \sum \langle \eta, \; \mathfrak{m}_{0,\beta}\rangle \; T^{E(\tilde{\beta})} \; \exp(\partial\tilde{\beta} \cap b) \; \exp\langle \partial\tilde{\beta}, \sum T^{E(\gamma)} \; \mathfrak{C}^F_{k,\gamma}(b, \ldots, b)\rangle$

Divisor Axiom of $\mathfrak{m}$ $\implies$ Things like $\mathfrak{m}(\mathfrak{C}^F(b\cdots)\cdots\mathfrak{C}^F(\cdots))$ will appear

Claim (B) & A_∞ homotopy theory

Now that Claim (A) is proved. It suffices to show Claim (B). **Recall**:

Claim (A) ϕ^F can pass to the quotient $\Phi = [\phi^F] : \Lambda\langle\Delta, q\rangle/\mathfrak{a} \to \Lambda\langle\tilde\Delta, \tilde q\rangle/\tilde{\mathfrak{a}}$

Moreover, we will prove $\Phi(W) = \tilde W$ and a global LG is very possible.

Claim (B): The quotient Φ only depends on the homotopy classhomotopy class of $\mathfrak{C}^F$.

$\Longrightarrow$ Our transition map Φ^* also does **not** depend on choices!

Note: More precisely, we need to develop an improved homotopy theory of A_∞ alg, which I call **ud-homotopy**, where **u**=unitality and **d**=divisor axiom.

There are some heavy homological algebra, but the ideas are similar to previous ones.

Claim (B) & A_∞ homotopy theory

To prove Claim (B), we further consider two aspects as follows:

(B-1) For different choices, say F & F', we claim the A_∞ homomorphisms $\mathfrak{C}^F$ and $\mathfrak{C}^{F'}$ are ud-homotopic to each other. (This is basically OK)

(B-2) We compare alg. homo. By their defining formulas, we can write

$$\phi^{F'}(Y^\alpha) = \phi^F(Y^\alpha) \cdot \exp \left\langle \alpha, \sum T^{E(\gamma)} Y^{\partial\gamma}(\mathfrak{C}^{F'}_{0,\gamma} - \mathfrak{C}^F_{0,\gamma}) \right\rangle \qquad \forall \alpha \in \pi_1 \cong \mathbb{Z}^n$$

It suffices to show (any α-component of) the '**error term**'

$$S(Y) := \sum T^{E(\gamma)} Y^\gamma (\mathfrak{C}^{F'}_{0,\gamma} - \mathfrak{C}^F_{0,\gamma})$$

is contained in the ideal $\tilde{\mathfrak{a}}$ of weak MC eqs. $\implies \Phi = [\phi^F] = [\phi^{F'}]$

Cocycle conditions use similar ideas: $\phi^{ik}(Y^\alpha)$ v.s. $\phi^{ij} \circ \phi^{jk}(Y^\alpha)$.

Sketch of proof of (B-2):

1. The homotopy condition means: $\exists\ (\mathfrak{f}_s)$ and $(\mathfrak{h}_s)$ for $s \in [0,1]$ s.t.
   - $\mathfrak{f}_0 = \mathfrak{C}^F$ and $\mathfrak{f}_1 = \mathfrak{C}^{F'}$
   - $\frac{d}{ds}\mathfrak{f}_s = \sum(-1)^*\mathfrak{h}_s(\cdots\tilde{\mathfrak{m}}\cdots) + \sum(-1)^*\mathfrak{m}\big(\mathfrak{f}_s\cdots\mathfrak{f}_s\ \mathfrak{h}_s\ \mathfrak{f}_s\cdots\mathfrak{f}_s\big)$
   - The ud-homotopy further means $\mathfrak{f}_s, \mathfrak{h}_s$ have some good properties.
2. The second bullet above allows us to make the comparison:

$$\mathfrak{C}^{F'} - \mathfrak{C}^F = \int_0^1 \tfrac{d}{ds}\mathfrak{f}_s \cdot ds$$

3. Let $\mathbf{y} = (e^{x_1}, \ldots, e^{x_n}) \in U^n_\Lambda$. Using Lemma X again, we will get

$$S(Y) \triangleq \sum T^{E(\gamma)} Y^\gamma(\mathfrak{C}^{F'}_{0,\gamma} - \mathfrak{C}^F_{0,\gamma}) = \sum \lambda_{pq}(Y)\tilde{Q}_{pq}(Y)$$

(λ_{pq} is in terms of $\mathfrak{h}_s$) $\qquad S(Y) \in \tilde{\mathfrak{a}} \implies$ $\Phi = [\phi^F] = [\phi^{F'}]$

Summary

A_∞ algebra	A_∞ homomorphism	homotopy of A_∞ homo.
$\Downarrow$	$\Downarrow$	$\Downarrow$
Maurer-Cartan eq	Wall Crossing Formula	'error term' $\in \mathfrak{a}$
$\Downarrow$	$\Downarrow$	$\Downarrow$
Local chart $V(\mathfrak{a})$	Transition map Φ^* well-defined	Transition map Φ^* choice-independent & Cocycle condition