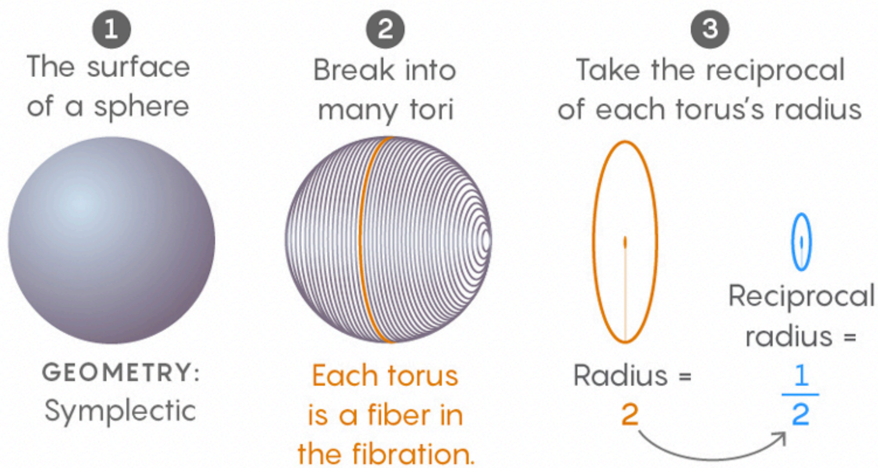
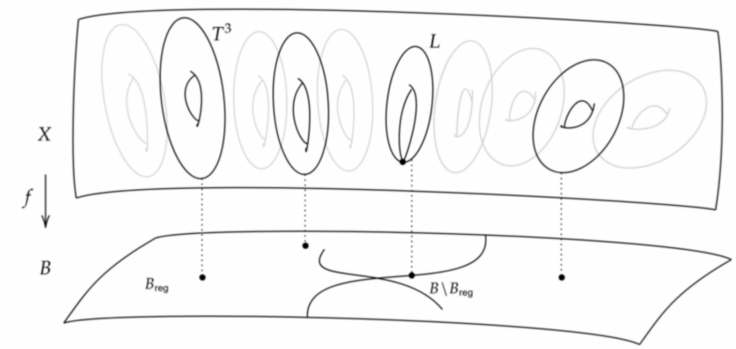
Family Floer mirror space for local SYZ singularities

Homological Mirror Symmetry Annual Meeting   Thursday, November 3, 2022

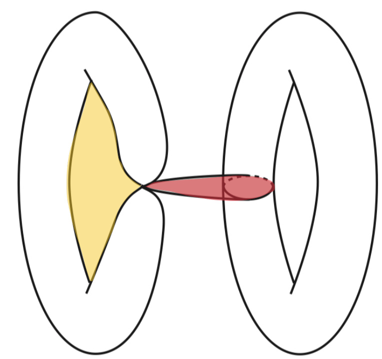
Northwestern University - Hang Yuan

Strominger-Yau-Zaslow conjecture





- In brief, SYZ says: "dualizing" a Lagrangian torus fibration reassembles a "mirror" space.
- Gross's Topological Mirror Symmetry gives great evidence of SYZ for quintic threefolds.
- Joyce negates the strong SYZ conjecture for special Lag fib (cannot match singular locus).
- Unfortunately, we don't find a good mathematically precise statement of SYZ conjecture, not to mention examples as evidence.
- <u>Key challenge</u>: what is "dualizing"? Singular Lagrangian fibers cause "quantum corrections"
- "Dualizing" ***smooth*** Lagrangian torus fibers (with the quantum correction data) has been achieved in my thesis. Today, let's further include ***singular*** Lagrangian fibers.



"quantum correction" - holomorphic disks

SYZ conjecture (a mathematically precise update)

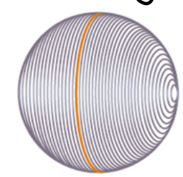
Let X be a Calabi-Yau manifold.
Then, there is a graded (zero-Maslov class) Lagrangian fibration $\pi : X \to B$.
Next, there is an analytic space $\mathscr{Y}$ over the Novikov field $\Lambda = \mathbb{C}((T^{\mathbb{R}}))$ equipped with a *tropically continuous map* $f : \mathscr{Y} \to B$ satisfying:

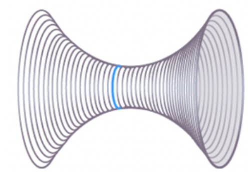
(i) π, f have the ***same*** singular locus Δ, the ***same*** smooth locus $B_0 = B \backslash \Delta$

(ii) $\pi_0 = \pi|_{B_0}$ is a Lagrangian torus fibration; $f_0 = f|_{B_0}$ is an affinoid torus fibration. They induce the ***same*** integral affine structure on B_0

(iii) f_0 is isom. to the *dual affinoid torus fibration* $\pi_0^\vee$ associated to (X, π_0)

$$\begin{tikzcd} X & X_0 \arrow[l, hook'] \arrow[d, "\pi_0"'] \arrow[r, phantom, "\substack{\text{Family Floer} \\ \text{- - - - - -} \\ \text{T-duality}}"] & X_0^\vee \arrow[r, hook, "g"] \arrow[d, "\pi_0^\vee"] & \mathscr{Y} \arrow[d, "f"] \\ B \arrow[from=u, "\pi"] & B_0 \arrow[l, hook'] & B_0 \arrow[r, hook] & B \end{tikzcd}$$





(set-level) $\pi_0^\vee : \bigcup H^1(L_q; U_\Lambda) \equiv B_0 \times U_\Lambda^n \to B_0$

where U_Λ is the unit circle in Λ (non-archimedean)

Theorem (Y 2022): The conjecture holds for any toric Calabi-Yau manifold X with Gross's special Lagrangian fibration π.

Moreover, the mirror analytic space $\mathscr{Y}$ embeds into an algebraic variety Y of the same dimension (more precisely Y^{an}).

Definition: In the above case, we say that the algebraic variety Y is **SYZ mirror** to X

Example: $Y = \{(x, y) \in \Lambda^2 \times (\Lambda^*)^{n-1} \mid x_0 x_1 = 1 + y_1 + \cdots + y_{n-1}\}$ is SYZ mirror to $X = \mathbb{C}^n \backslash \{z_1 \cdots z_n = 1\}$.

Note: its **HMS** is proved by Abouzaid-Sylvan and Gammage. Note: $\mathbb{C}^n$ can be replaced by any toric CY, and we still have some Y

- $\pi_0^\vee$ exists for graded Lag fib (no need for special Lag fib). It is derived from family Floer theory in my thesis and keep "quantum correction". Moreover, it is *unique* up to isomorphism, so the meaning of the condition (iii) is still mathematically precise.
- Even if we omit T-duality condition (iii), the affine-geometric conditions (i) (ii) are already very nontrivial evidence. For the weaker SYZ result, the proof can be even elementary: I can write the mirror $(\mathscr{Y}, f)$ explicitly, and one can directly check (i) (ii) with very standard knowledge.
- Finally, this is a ***mathematically precise*** statement of SYZ conjecture, ***with singularities***.

Review the underlying family Floer mirror construction in my thesis

Take a Lagrangian fibration with singularities

$$\pi : X \to B$$

on a symplectic manifold (X, ω). Take its smooth part:

$$\pi_0 : X_0 \to B_0$$

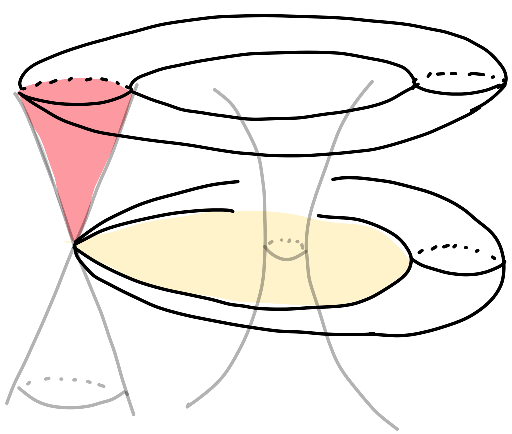
We think of holomorphic disks bounded by π_0-fibers in X (not just in X_0). The disks may meet singular π-fiber at interior points

<u>Semipositive:</u> The Maslov indices of holomorphic disks ≥ 0
(sufficient conditions: special / graded Lagrangians)

<u>Weak Unobstructedness:</u> For simplicity, let's take a sufficient condition:
Lagrangian fibers are preserved by an anti-symplectic involution φ

- e.g. complex conjugate $z_i \mapsto \bar{z}_i$ for Gross's special Lagrangian fibration
- Due to the work of Solomon, there will be some <u>pairwise</u> canceling: $\beta \leftrightarrow -\varphi_* \beta$ in $\pi_2(X, L)$. But, it never means Maslov-0 counts vanish. There is still the wall-crossing phenomenon for Maslov-0 disks.

For the singular Lagrangian fibers, we study two different types of quantum corrections



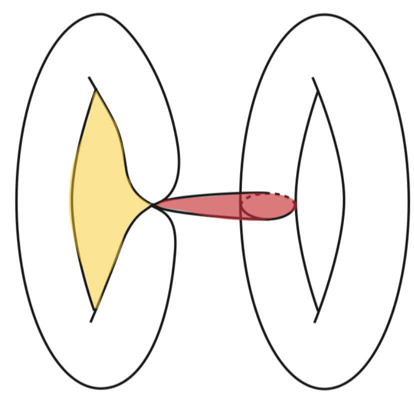
Concrete configuration

(I) Disks meet singular fibers at interior points (red)

(II) Disks meet singular fibers at boundary points (yellow)

◉ We deal with them separately, emphasizing on different aspects:

- the Floer aspect for red disk (I) (Done in my thesis)
- the NA analytic / topological aspect for yellow disks (II) (will be useful for the singular extension)



General configuration

Theorem (Y) (only use the first-type disks)

We can associate to (X, π_0) a triple $(X_0^\vee, \pi_0^\vee, W_0^\vee)$ consisting of

(a) Λ-analytic space $X_0^\vee$

(b) dual affinoid torus fibration $\pi_0^\vee : X_0^\vee \to B_0$

(c) analytic superpotential function $W_0^\vee$

unique up to isomorphism of analytic spaces.

Moreover, the integral affine structure on B_0 induced by $\pi_0^\vee$ agrees with the one induced by Lag fib π_0.

In the set-theoretic level, the $X_0^\vee$ is given by $\bigcup_{q \in B_0} H^1(L_q; U_\Lambda)$

$\Lambda = \mathbb{C}((T^{\mathbb{R}}))$ is the Novikov field. Let $\Lambda^* = \Lambda \setminus \{0\}$

- NA valuation $\mathsf{v} : \Lambda \to \mathbb{R} \cup \{\infty\}$ Or, NA norm $|z| = e^{-\mathsf{v}(z)}$.
- Novikov unitary group $U_\Lambda = \{ |z| = 1\}$, like $U(1) \cong S^1$ in $\mathbb{C}$

Affinoid torus fibration: **(dual of smooth fibers with correction)**

It is a continuous map with respect to the NA analytic topology and the manifold topology on B_0 , and it is locally modeled on the *tropicalization map:*

$$\mathfrak{trop} : (\Lambda^*)^n \to \mathbb{R}^n \qquad (y_i) \mapsto (\mathsf{v}(y_1), \ldots \mathsf{v}(y_n))$$

(a non-archimedean version of $Log : (\mathbb{C}^*)^n \to \mathbb{R}^n$)

Kontsevich-Soibelman study it and show that it induces a natural integral affine structure on B_0. I learn this name from Nicaise-Xu-Yu.

tropically continuous maps: **(for the singular fiber extension)**

in the sense of Chambert-Loir and Ducros (2012). A precise reference:

Formes différentielles réelles et courants sur les espaces de Berkovich, Section (3.1.6)

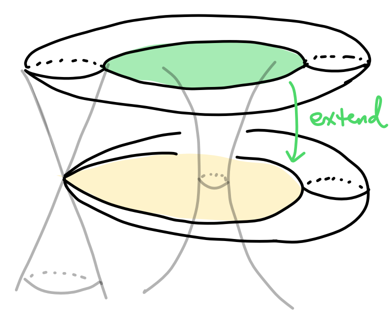
- A **tropically continuous map** F is locally in the following form:

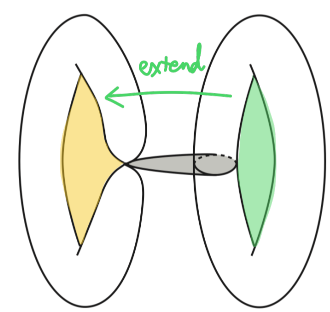
$$F|_{\mathcal{U}} = \varphi(\mathsf{v}(y_1), \ldots, \mathsf{v}(y_n))$$

  - $\mathcal{U}$ is an analytic open subset
  - $y_1, \ldots, y_n : \mathcal{U} \to \Lambda^*$ are invertible analytic functions.
  - $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is a **continuous** map for Euclidean topology eg. $\min\{\mathsf{v}(y_k)\}$

  Note: may similarly define *tropically* Piecewise-Linear/C^k/ L^p/measurable…

- It somehow reduces the analytic geometry to the real geometry on the base.
- Note: any affinoid torus fibration is tropically continuous (when φ is affine linear)
- Relation to quantum correction? Very intuitively, let's imagine $\mathsf{v}(y_1), \ldots, \mathsf{v}(y_n)$ are action coordinates from Lag fib π_0 , so they correspond to symplectic areas.
- The symplectic areas, as functions on B_0, can usually extend to the singular locus ***continuously*** and become piecewise-smooth on B. See the figures.
- Then, the topological extension from B_0 to B somehow "controls" the analytic extension above. (further use the second-type disks)





Slogan: there is an affine geometry coincidence without Floer theory. Maybe accessible to wider audience

A side: Let $X = \mathbb{C}^2 \backslash \{z_1 z_2 = 1\} = \{z_1 z_2 = 1 + w\}$ and consider the Lagrangian fibration $\pi : X \to \mathbb{R}^2$

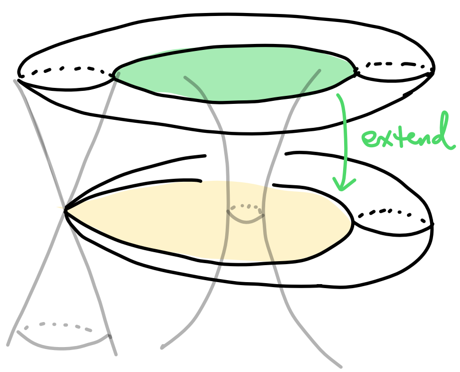
$$\pi(z_1, z_2) = \left(\tfrac{1}{2}(|z_1|^2 - |z_2|^2), \log|z_1 z_2 - 1|\right)$$

(This may be one of the most popular examples)

- We consider the symplectic areas of a family of holomorphic disks in $\mathbb{C}^2$ (yellow and green disks).
- This gives a continuous map (smooth in B_0)

$$\psi(q_1, q_2) : \mathbb{R}^2 \to \mathbb{R}_+$$

One can check that a set of action coordinates on B_0 can be locally given by $(\frac{1}{2}(|z_1|^2 - |z_2|^2), \psi)$.



Concrete configuration

B side: We will see the "SYZ mirror space" is the algebraic variety

$$Y = \{(x_0, x_1, y) \in \Lambda^2 \times \Lambda^* \mid x_0 x_1 = 1 + y\}$$

More precisely, a Zariski-dense analytic open domain $\mathscr{Y} = \{|x_1| < 1\}$ in Y^{an}

One explicit representation of the "dual analytic fibration" is

$$f = j^{-1} \circ F : \mathscr{Y} \to \mathbb{R}^2$$

where

$$j : \mathbb{R}^2 \to \mathbb{R}^3$$

is a topological embedding given by

$$j(q_1, q_2) = \big(\min\{-\psi(q_1, q_2), -\psi(q_1, 0)\} + \min\{0, q_1\}\ ,\quad \min\{\psi(q_1, q_2), \psi(q_1, 0)\}\ ,\quad q_1\big)$$

and

$$F : Y^{\mathrm{an}} \to \mathbb{R}^3$$

is a tropically continuous map given by

$$F(x_0, x_1, y) = \begin{pmatrix} \min\left\{ \mathsf{v}(x_0), -\psi(\mathsf{v}(y), 0) + \min\{0, \mathsf{v}(y)\} \right\} \\ \min\{\mathsf{v}(x_1), \psi(\mathsf{v}(y), 0))\} \\ \mathsf{v}(y) \end{pmatrix}$$

where v is the valuation on the Novikov field $\Lambda = \mathbb{C}((T^{\mathbb{R}}))$.

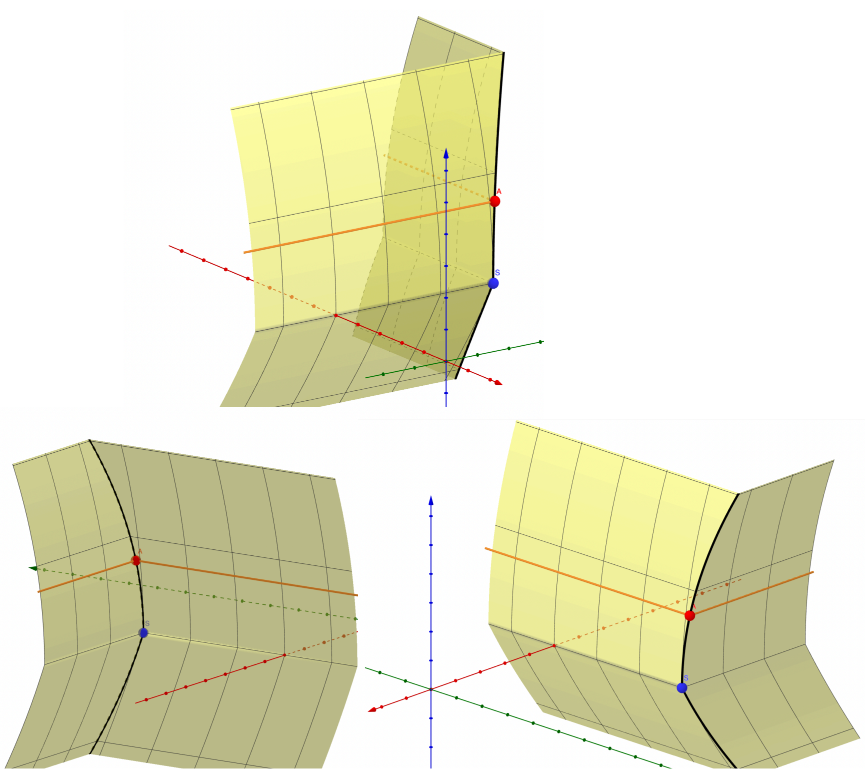
The $f = j^{-1} \circ F$ is well-defined, since we can check the domain $\mathscr{Y}$ satisfies

$$j(\mathbb{R}^2) = F(\mathscr{Y})$$

It's explicit, intrinsic (ψ is from A-side) and has singularity. No Floer theory so far

<u>Slogan</u>: there is an affine geometry coincidence without Floer theory. Maybe accessible to wider audience

- The figure of $j(\mathbb{R}^2) = F(\mathscr{Y})$ in $\mathbb{R}^3$ is as follows.
- It roughly visualizes the integral affine structure.
- The pair (j, F) is designed to explicitly describe f. The pair is not unique.
- May change (j, F) without affecting f. Then, the figure $j(\mathbb{R}^2) = F(\mathscr{Y})$ in $\mathbb{R}^3$ may change



<u>B side:</u> We will see the "SYZ mirror space" is the algebraic variety

$$Y = \{(x_0, x_1, y) \in \Lambda^2 \times \Lambda^* \mid x_0 x_1 = 1 + y\}$$

More precisely, a Zariski-dense analytic open domain $\mathscr{Y} = \{\,|x_1| < 1\}$ in Y^{an}

One explicit representation of the "dual analytic fibration" is

$$f = j^{-1} \circ F : \mathscr{Y} \to \mathbb{R}^2$$

where

$$j : \mathbb{R}^2 \to \mathbb{R}^3$$

is a topological embedding given by

$$j(q_1, q_2) = \big(\min\{-\psi(q_1, q_2), -\psi(q_1, 0)\} + \min\{0, q_1\}\ ,\ \ \min\{\psi(q_1, q_2), \psi(q_1, 0)\}\ ,\ \ q_1\big)$$

and

$$F : Y^{\mathrm{an}} \to \mathbb{R}^3$$

is a tropically continuous map given by

$$F(x_0, x_1, y) = \begin{pmatrix} \min\Big\{\mathsf{v}(x_0), -\psi(\mathsf{v}(y), 0) + \min\{0, \mathsf{v}(y)\}\Big\} \\ \min\{\mathsf{v}(x_1), \psi(\mathsf{v}(y), 0))\} \\ \mathsf{v}(y) \end{pmatrix}$$

where v is the valuation on the Novikov field $\Lambda = \mathbb{C}((T^{\mathbb{R}}))$.

The $f = j^{-1} \circ F$ is well-defined, since we can check the domain $\mathscr{Y}$ satisfies

$$j(\mathbb{R}^2) = F(\mathscr{Y})$$

It's explicit, intrinsic (ψ is from A-side) and has singularity. No Floer theory so far

<u>Slogan</u>: there is an affine geometry coincidence without Floer theory. Maybe accessible to wider audience

- In 2004, Kontsevich-Soibelman have a similar construction in NA context. It is our key motivation. (Tony Yue Yu suggests it). We also combine some ideas from Gross-Hacking-Keel-Siebert.
- But unlike KS, we must have extra data, like $\mathscr{Y}$ or ψ, in order to make connections with A-side Lagrangian fibration π, because we want to study "mirror partners" rather than each single side.
- The common smooth locus is $B_0 = \mathbb{R}^2 \setminus \{0\}$. We can read off an *integral affine structure* induced by the NA fibration f (B-side) Meanwhile, we can read off the other *integral affine structure* induced by the Lagrangian fibration π (A-side)
- Although very unmotivated now, we can surprisingly check a ***precise coincidence*** between the two ***integral affine structures*** (even codim-2 ***singular locus skeletons***, in higher dimensions)
- To check this, we don't need any Floer theory, we don't need any A_∞ structures, and we don't need any virtual techniques. The NA geometry we use is also standard.
- The affine structure matching should be very accessible to a wider audience without too specialized knowledge.
- In turn, the affine structure matching is also good evidence for: various A_∞ structures, virtual technique, family Floer homology program, SYZ proposal of T-duality, NA mirror symmetry …
- Everything presented so far is explicit and down-to-earth.

<u>B side:</u> We will see the "SYZ mirror space" is the algebraic variety

$$Y = \{(x_0, x_1, y) \in \Lambda^2 \times \Lambda^* \mid x_0 x_1 = 1 + y\}$$

More precisely, a Zariski-dense analytic open domain $\mathscr{Y} = \{ |x_1| < 1\}$ in Y^{an}

One explicit representation of the "dual analytic fibration" is

$$f = j^{-1} \circ F : \mathscr{Y} \to \mathbb{R}^2$$

where

$$j : \mathbb{R}^2 \to \mathbb{R}^3$$

is a topological embedding given by

$$j(q_1, q_2) = \big(\min\{-\psi(q_1, q_2), -\psi(q_1,0)\} + \min\{0,q_1\}\ ,\quad \min\{\psi(q_1, q_2), \psi(q_1,0)\}\ ,\quad q_1\big)$$

and

$$F : Y^{\mathrm{an}} \to \mathbb{R}^3$$

is a tropically continuous map given by

$$F(x_0, x_1, y) = \begin{pmatrix} \min\left\{ \mathsf{v}(x_0), -\psi(\mathsf{v}(y),0) + \min\{0,\mathsf{v}(y)\} \right\} \\ \min\{\mathsf{v}(x_1), \psi(\mathsf{v}(y),0))\} \\ \mathsf{v}(y) \end{pmatrix}$$

where v is the valuation on the Novikov field $\Lambda = \mathbb{C}((T^{\mathbb{R}}))$.

The $f = j^{-1} \circ F$ is well-defined, since we can check the domain $\mathscr{Y}$ satisfies

$$j(\mathbb{R}^2) = F(\mathscr{Y})$$

It's explicit, intrinsic (ψ is from A-side) and has singularity. No Floer theory so far

Slogan: there is an affine geometry coincidence without Floer theory. Maybe accessible to wider audience

- Next, we highlight the role of **mirror NA analytic topology**:

(1) If we use NA topology instead of complex one, there is no convergence issue and no "wall-crossing discontinuity".

- The NA convergence is ensured by Gromov's compactness.
- The formula of f does not involve J, and we cannot detect the walls of π (rely on J). But, the f is indeed from wall-crossing
- It largely justifies the uniqueness result in my thesis: the mirror NA analytic structure doesn't rely on J up to isomorphism.

(2) The singular fibers of f heavily rely on the NA properties.

- The NA triangle inequality tells (think higher dimensions) $\mathsf{v}(1+y_1+\cdots+y_{n-1})$ **(> or =)** $\min\{0,\mathsf{v}(y_1),\ldots,\mathsf{v}(y_n)\}$
- The ambiguous case ">" happen only if $\min\{0,\mathsf{v}(y_1),\ldots,\mathsf{v}(y_n)\}$ attains at least twice. This ambiguity induces the singularity of f.
- The singular locus of the A-side Lagrangian fibration π is also described by $\min\{0,q_1,\ldots,q_{n-1}\}$
- This tells why we can match the singular locus of both sides

◉ Now, we see that the mirror NA analytic topology matters a lot.

◉ Wish you might have some idea why the NA analytic topology is more than just uniting the Maurer-Cartan sets.

◉ Next, I will explain why my thesis is necessary for this.

B side: We will see the "SYZ mirror space" is the algebraic variety

$$Y=\{(x_0,x_1,y)\in\Lambda^2\times\Lambda^*\mid x_0x_1=1+y\}$$

More precisely, a Zariski-dense analytic open domain $\mathscr{Y}=\{\,|x_1|<1\}$ in Y^{an}

One explicit representation of the "dual analytic fibration" is

$$f=j^{-1}\circ F:\mathscr{Y}\to\mathbb{R}^2$$

where

$$j:\mathbb{R}^2\to\mathbb{R}^3$$

is a topological embedding given by

$$j(q_1,q_2)=\big(\min\{-\psi(q_1,q_2),-\psi(q_1,0)\}+\min\{0,q_1\}\ ,\quad \min\{\psi(q_1,q_2),\psi(q_1,0)\}\ ,\quad q_1\big)$$

and

$$F:Y^{\mathrm{an}}\to\mathbb{R}^3$$

is a tropically continuous map given by

$$F(x_0,x_1,y)=\begin{pmatrix}\min\Big\{\mathsf{v}(x_0),-\psi(\mathsf{v}(y),0)+\min\{0,\mathsf{v}(y)\}\Big\}\\ \min\{\mathsf{v}(x_1),\psi(\mathsf{v}(y),0))\}\\ \mathsf{v}(y)\end{pmatrix}$$

where v is the valuation on the Novikov field $\Lambda=\mathbb{C}((T^{\mathbb{R}}))$.

The $f=j^{-1}\circ F$ is well-defined, since we can check the domain $\mathscr{Y}$ satisfies

$$j(\mathbb{R}^2)=F(\mathscr{Y})$$

It's explicit, intrinsic (ψ is from A-side) and has singularity. No Floer theory so far

Why the conventional Maurer-Cartan idea is not enough for the global mirror analytic structure

Recall that the family Floer approach to mirror symmetry (initiated by Fukaya) has two main steps:

(1) **Family Floer mirror construction:** We planed to unite the *Maurer-Cartan sets* of A_∞ algebras associated to Lagrangian fibers. This should give a non-archimedean analytic mirror space in some way. (Let's call this *Maurer-Cartan picture*)

(2) **Family Floer functor**: Abouzaid used it to prove a version of HMS without correction. But, let's only focus on (1) today

- We know *local* analytic charts for long time (cf. Fukaya,Tu). But, for *local-to-global gluing,* we must go beyond the MC picture.

  *Can we obtain a local-to-global gluing for a non-archimedean **analytic space** by only applying the usual homotopy invariance of Maurer-Cartan **sets**?*

  - The MC picture is a great idea, but need more structures for NA
  - The MC only involves bijection of sets, and seems only give some local or set-theoretic approximations.
  - The NA transition maps are affinoid algebra morphisms, and the NA cocycle condition is to match affinoid algebra morphisms.

❖ A key **new idea** in my thesis: we should no longer think of *bounding cochains* in the MC sets. We should directly think of *formal power series* in the corresponding affinoid algebras. (Actually, this new idea also has some other application.)

❖ If not doing so, we can only achieve a ***set-theoretic gluing***.

❖ There are lots of new challenges. Let me mention two of them.

(1) The "improved" ***ud-homotopy theory*** for geometric A_∞ structures

- **u:** (strict+cyclic) unitality. **d:** divisor axiom: Seidel, Auroux, Fukaya, Tu, etc.
- Issue: *Individual* A_∞ maps with divisor axiom were quite not enough for the analytic gluing. We need much more properties for divisor axiom.
- Need divisor axiom for ***homtopies*** between A_∞ maps
  Need divisor axiom for ***homotopy inverses*** of A_∞ maps (very important but hard)
  They are necessary for both well-definedness and *analytic* cocycle condition

(2) The ***minimal model A_∞ algebras*** with Fukaya's trick, divisor axiom, etc

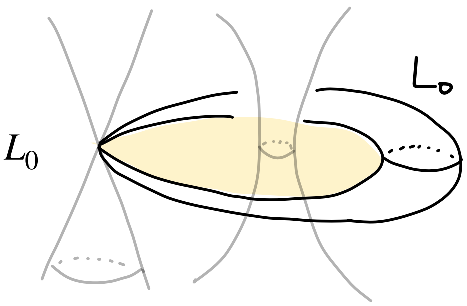
- It is necessary, roughly because the fiber is $H^1(L_q; U_\Lambda)$ (f.dim) not $\Omega^1(L_q; U_\Lambda)$.
  Or intuitively, the minimal model is like counting *holo pearly trees* (cf. Sheridan)
- Now, as we must use minimal models, I can explicitly indicate a fatal problem:
  We cannot make a 2-pseudo-isotopy of minimal model A_∞ algebras and keep Fukaya's trick simultaneously (let alone other ingredients like divisor axiom)
- The point is, even if every single ingredient is well-known and not difficult alone, naively putting them together can be *impossible*.
- Fortunately, pseudo-isotopy is stronger than homotopy. So, we still had a chance.

❖ Eventually, I solve all these delicate problems in my thesis. There're lots of twists and turns to finally achieve it.

❖ To sum up, we must go beyond the MC scope and study *more* structures. Only in a very comprehensive way, we can finally achieve the local-to-global ***analytic gluing*** for family Floer dual affinoid torus fibration.

Dual singular fiber is not a Maurer-Cartan set !

I just explained the logical/technical reasons for the issue of MC picture.
Moreover, the best evidence is probably by example.



Recall $X = \mathbb{C}^2 \backslash \{z_1 z_2 = 1\}$, $Y = \{x_0 x_1 = 1 + y \text{ in } \Lambda^2 \times \Lambda^*\}$. We have a singular pinched-sphere Lagrangian L_0
The dual analytic fibration f has an *explicit* formula.
After some computation, there is a natural decomposition of the dual singular fiber as follows:

$$f^{-1}(0) \cong S_1 \sqcup S_2$$

where

$$S_1 = \Lambda_0 \times \Lambda_+ \cup \Lambda_+ \times \Lambda_0 \qquad \text{if } 1 + y \in \Lambda_+$$

$$S_2 = \{(x_0, x_1) \in U_\Lambda^2 \mid \bar{x}_0 \bar{x}_1 \neq 1\} \cong U_\Lambda \times \left(\mathbb{C}^* \backslash \{-1\} \oplus \Lambda_+\right) \qquad \text{if } 1 + y \notin \Lambda_+$$

On the other hand, Hong, Kim, and Lau (2018) have proved that the Maurer-Cartan set for the singular Lagrangian L_0 is exactly given by the first part:

$$\mathscr{MC}(L_0) \cong S_1 \quad \subsetneq f^{-1}(0)$$

- Therefore, $f^{-1}(0) \supsetneq \mathscr{MC}(L_0)$. This is more interesting than just saying an inequality $f^{-1}(0) \neq \mathscr{MC}(L_0)$, since the bounding cochains in the MC set are still contained in the dual singular fiber $f^{-1}(0)$. But, there are extra points in S_2 beyond the MC scope.
- It may be interesting to note that a natural "*pair-of-pants*" $\mathbb{C}^* \backslash \{-1\}$ appears in the valuation-zero part of y in S_2
- The Maurer-Cartan picture is a great idea, but it may be only some approximation to the final NA analytic structure.

Further evidence: a folklore conjecture: (Kontsevich, Seidel, Auroux, …)
The critical values of the mirror Landau-Ginzburg superpotential on $X^\vee$ are the eigenvalues of the quantum multiplication by the first Chern class on X.

- There is other numerical evidence for our proposed new SYZ conjecture.
- The Lagrangian fibration π is placed in not only X but also in possibly a larger ambient compactification $\overline{X}$ such as $\mathbb{CP}^n$
- Having various ambient space $\overline{X}$ will produce various different Landau-Ginzburg superpotential W on Y
- Let's first only focus on the examples. Recall that $X = \mathbb{C}^n \backslash \{z_1 \cdots z_n = 1\}$ and $Y = \{(x, y) \in \Lambda^2 \times (\Lambda^*)^{n-1} \mid x_0 x_1 = 1 + y_1 + \cdots + y_{n-1}\}$
- Thanks to the previous SYZ examples, we can explicitly write down our computations.

$$\begin{tikzcd}
\overline{X} \arrow[d, phantom, sloped, "\subseteq"] & & & & \text{LG superpotential} \quad W \\
X \arrow[d, "\pi"] & X_0 \arrow[l, hook'] \arrow[d, "\pi_0"'] & \text{Family Floer} & X_0^\vee \arrow[r, hook, "g"] \arrow[d, "\pi_0^\vee"] & Y \arrow[d, "f"] \\
B & B_0 \arrow[l, hook'] & \text{T-duality} & B_0 \arrow[r, hook] & B
\end{tikzcd}$$

Further evidence: a folklore conjecture: (Kontsevich, Seidel, Auroux, …)
The critical values of the mirror Landau-Ginzburg superpotential on $X^\vee$ are the eigenvalues of the quantum multiplication by the first Chern class on X.

ambient space	$\overline{X} = \mathbb{CP}^n$ Let $H \in \pi_2(\overline{X})$ be the class of a complex line.
LG superpotential	$W = x_1 + \dfrac{T^{E(H)}\ x_0^n}{y_1 \cdots y_{n-1}}$ defined on $Y = \{x_0 x_1 = 1 + y_1 + \cdots + y_{n-1}\}$, where $E(H) = \frac{1}{2\pi}\omega \cap H$ Our superpotential is new and more global. It can retrieve the superpotential of Clifford (resp. Chekanov) tori by eliminating x_1 (resp. x_0) by the equation $x_0 x_1 = 1 + y_1 + \cdots + y_{n-1}$
Critical points	By direct computations, there are $n+1$ critical points $\begin{cases} x_0 = T^{-\frac{E(\mathcal{H})}{n+1}} e^{-\frac{2\pi i s}{n+1}} \\ x_1 = nT^{\frac{E(\mathcal{H})}{n+1}} e^{\frac{2\pi i s}{n+1}} \\ y_1 = \cdots = y_{n-1} = 1 \end{cases} \quad s \in \{0, 1, \ldots, n\}$
Critical values	$(n+1)T^{\frac{\omega(H)}{n+1}} e^{\frac{2\pi i}{n+1}s}$ for $s \in \{0,1,\ldots,n\}$ One can check the folklore conjecture holds

Further evidence: a folklore conjecture: (Kontsevich, Seidel, Auroux, …)

The critical values of the mirror Landau-Ginzburg superpotential on $X^\vee$ are the eigenvalues of the quantum multiplication by the first Chern class on X.

ambient space	$\overline{X} = \mathbb{CP}^m \times \mathbb{CP}^{n-m}$ for $0 < m < n$ This is also a compactification of $\mathbb{C}^n$ Let $H_1, H_2 \in \pi_2(\overline{X})$ be the classes of a complex line in $\mathbb{CP}^m \times pt$ and in $pt \times \mathbb{CP}^{n-m}$
LG superpotential	$W = x_1 + \dfrac{T^{E(H_1)}\ x_0^m}{y_1 \cdots y_m} + \dfrac{T^{E(H_2)}\ x_0^{n-m}}{y_{m+1} \cdots y_{n-1}}$ defined on the same $Y = \{x_0 x_1 = 1 + y_1 + \cdots + y_{n-1}\}$
Critical points	$\begin{cases} x_0 = \left(T^{\frac{E(\mathcal{H}_2)}{n-m+1}} e^{\frac{2\pi i s}{n-m+1}}\right)^{-1} \\ x_1 = T^{\frac{E(\mathcal{H}_2)}{n-m+1}} e^{\frac{2\pi i s}{n-m+1}} \cdot \left(m\, T^{\frac{E(\mathcal{H}_1)}{m+1}} e^{\frac{2\pi i r}{m+1}} \cdot \left(T^{\frac{E(\mathcal{H}_2)}{n-m+1}} e^{\frac{2\pi i s}{n-m+1}}\right)^{-1} + n - m\right) \\ y_k = T^{\frac{E(\mathcal{H}_1)}{m+1}} e^{\frac{2\pi i r}{m+1}} \cdot \left(T^{\frac{E(\mathcal{H}_2)}{n-m+1}} e^{\frac{2\pi i s}{n-m+1}}\right)^{-1} & 1 \leq k \leq m \\ y_\ell = 1 & m < \ell < n \end{cases}$
Critical values	$(m+1)T^{\frac{E(H_1)}{m+1}} e^{\frac{2\pi i}{m+1} r} + (n-m+1)T^{\frac{E(H_2)}{n-m+1}} e^{\frac{2\pi i}{n-m+1} s}$ for $r \in \{0,1,\ldots,m\}$ and $s \in \{0,1,\ldots,n-m\}$ One can also check the folklore conjecture.

Further evidence: a folklore conjecture: (Kontsevich, Seidel, Auroux, …)

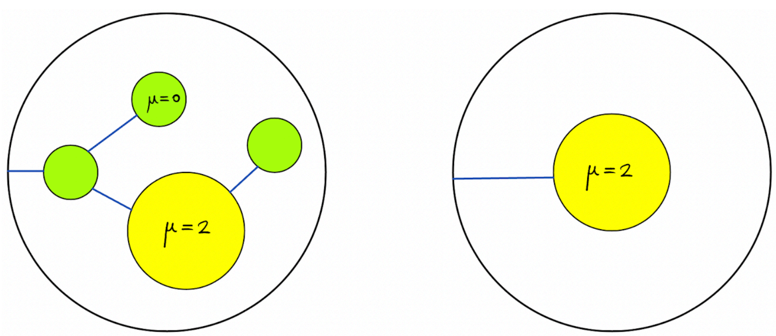
The critical values of the mirror Landau-Ginzburg superpotential on $X^\vee$ are the eigenvalues of the quantum multiplication by the first Chern class on X.

- With the Maslov-0 disks, I can still prove the folklore conjecture:

  Family Floer superpotential's critical values are eigenvalues of quantum product by c_1

✤ **Q**: How does the Maslov-0 disks cause troubles?

- Family Floer LG potential is only well-defined ***up to affinoid algebra isomorphism,*** or up to family Floer transition map
- New issue: must use *minimal model A_∞ algebras* (pearly trees). Otherwise only $\Omega^0(L)$-valued, not well-defined in any sense.
- With a <u>single</u> Maslov-0 disk, there may be <u>infinitely many</u> trees. The LG potential must be given by the sum of <u>all</u> these trees.
- If we perturb J, all Maslov-0 disks in the trees have wall-crossing. It is like a sort of "quantum fluctuation". It finally makes the LG potential only well-defined up to affinoid algebra isomorphisms.



contribution to LG potential **with / without** Maslov-0 disks

✤ <u>A main difficulty</u>: Hochschild cohomology is not functorial.

- a cochain A_∞ algebra $\check{\mathfrak{m}}$ is homotopy equivalent to its minimal model A_∞ alg $\mathfrak{m}$, so they have the same MC set. But, the HH* of $\check{\mathfrak{m}}$ and $\mathfrak{m}$ can be very different.
- Recall the LG potential must use the minimal model $\mathfrak{m}$ to be well-defined. But, the usual moduli space geometry can only do with HH* of the cochain A_∞ alg $\check{\mathfrak{m}}$
- A new operator I need to solve this issue is the following:
  $\Theta: \ \varphi \mapsto (\mathfrak{i}^{-1}\{\varphi\}) \diamond \mathfrak{i} \quad = \sum (-1)^* \mathfrak{i}^{-1}(\mathfrak{i},\dots,\mathfrak{i},\varphi(\mathfrak{i},\dots,\mathfrak{i}),\mathfrak{i},\dots,\mathfrak{i})$
  where $\mathfrak{i}^{-1}$ is **ud-**homotopy inverse of the A_∞ map $\mathfrak{i}: \mathfrak{m} \to \check{\mathfrak{m}}$ given by the homological perturbation. ($\diamond$ is Sheridan's notation, $\{\}$ is Gerstenhaber product)
- Θ is meaningless in pure algebra, but divisor axiom / involution from geometry. In fact, Θ uses ***<u>canceling trick</u>*** based on ud-homotopy theory: if $f \overset{ud}{\sim} g$ then

  $$\sum T^{E(\beta)} Y^{\partial\beta}(f_{0,\beta} - g_{0,\beta}) \equiv 0 \pmod{\mathfrak{a}} \qquad \text{For } \Theta\text{, we think } \mathfrak{i}^{-1} \diamond \mathfrak{i} \overset{ud}{\sim} \mathrm{id}$$

Here $\mathfrak{a}$ is an "obstruction ideal" in an affinoid algebra. And, $\mathfrak{a} = 0$, if L is preserved by an involution ($z_i \mapsto \bar{z}_i$ for Gross's Lag fib). We have lots of cancelling for the Θ.

✤other new ingredient is "*self-Floer cohomology with affinoid coefficients*"

- The coefficient is not $\mathbb{C}$, not Λ, but an affinoid algebra over Λ.
- It should be some preliminary version of "*Fukaya category with affinoid coefficients*" (Work in progress)
- The advantage of working over affinoid coefficients? Roughly speaking, we can disregard Maslov-0 holomorphic disks *up to affinoid algebra isomorphism on the coefficient*. (might use the similar canceling trick with ud-homotopy again)

Other examples of SYZ mirrors

Finally, recall that we really have a ***mathematically precise*** meaning of SYZ mirror.

Let $\Bbbk$ be a field, either $\mathbb{C}$ or $\Lambda = \mathbb{C}((T^{\mathbb{R}}))$. We define $X_{m,n} = X_{m,n}(\Bbbk) = \left\{ (x_0, x_1, \ldots, x_m, y_1, \ldots, y_n) \in \Bbbk^{m+1} \times (\Bbbk^*)^n : \prod_i x_i = 1 + \sum_j y_j \right\}$

The work of Abouzaid-Sylvan and Gammage prove the **HMS** (Homological Mirror Symmetry) between $X_{m,n}$ and $X_{n,m}$

In contrast, what we consider is the **SYZ** mirror between $X = \mathbb{C}^n \setminus \{z_1 \cdots z_n = 1\} \equiv X_{n-1,1}$ and $Y = \{x_0 x_1 = 1 + y_1 + \cdots + y_{n-1}\} \equiv X_{1,n-1}$

We can restate our basic example as follows:

Theorem: $X_{1,n-1}(\Lambda)$ is SYZ mirror to $X_{n-1,1}(\mathbb{C})$

Therefore, it is natural to propose:

(Interesting note: due to Chambert-Loir and Ducros, we do have the *partition of unity* for the non-archimedean analytic spaces)

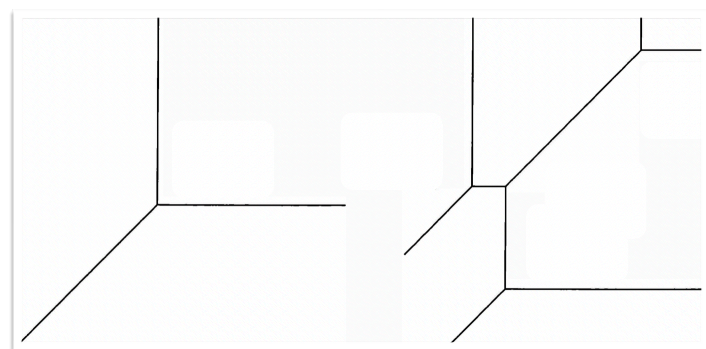
Conjecture: $X_{m,n}$ is SYZ mirror to $X_{n,m}$ (Hopefully, it would suggest good connections between HMS and SYZ)

Let X_P be a toric CY manifold associated to a polytope P and take a divisor $D = \{z^{m_0} = 1\}$.

One can use the A-side data to write down a Laurent polynomial h and the algebraic variety $Y_h = \{(x_0, x_1, y) \in \Lambda^2 \times (\Lambda^*)^{n-1} \mid x_0 x_1 = h(y)\}$. My paper also proves that:

Theorem: Y_h is SYZ mirror to $X_P \setminus D$ (give similar examples of the folklore conjecture)

One can recover from h to P (SYZ inverse). We get more models of singular locus skeleton



Prospect: *Our dual affinoid torus fibration $\pi_0^\vee$ does not have to use special Lagrangians.*

A graded (zero-Maslov class) Lagrangian fibration is enough to get $\pi_0^\vee$ and should be abundant

Hopefully, we might find more such Lagrangian fibration in the near future.

Thank you