Notes on the K-theory of Finite Fields

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The purpose of these notes is to give a complete proof, accessible to graduate students in algebraic topology, of Quillen's theorem on the K-theory of finite fields. Quillen's proof is long, intricate, and beautiful, and draws on a wide range of subjects and techniques. There is no way to avoid assuming a substantial background on the part of the reader—for example, the Serre spectral sequence, classifying spaces, K-theory, homological algebra, etc. However I *have* tried to avoid any assumption that the reader is an expert on these background topics. The arguments are therefore presented in substantially greater detail than would appear in a typical research paper. In addition, there are eight appendices elaborating on some of the methods and results used in the proof. For example, I have included complete proofs of "localization at the fixed point set" for $\mathbb{Z}/p$-actions and Nakaoka's theorem on the homology of extended powers. Many of these techniques will be useful to any aspiring algebraic topologist, whether specializing in K-theory or not. Quillen's theorem provides an opportunity to see the techniques "in action."

Here is the statement of Quillen's theorem in its simplest algebraic form. Let $\mathbb{F}_q$ denote the field with q elements, where $q = p^d$, p prime.

THEOREM. *$K_{2n-1}\mathbb{F}_q = \mathbb{Z}/(q^n - 1)$, and for $n > 0$ $K_{2n}\mathbb{F}_q = 0$.*

This computation is deduced from an even more remarkable theorem that explicitly identifies the space $BGL\mathbb{F}_q^+$. Let ψ^q denote the Adams operation in K-theory, and let $F\psi^q$ denote the homotopy fibre of $\psi^q - 1 : BU \to BU$.

MAIN THEOREM. *There is a homotopy equivalence $\theta : BGL\mathbb{F}_q^+ \xrightarrow[\simeq]{} F\psi^q$.*

Since the homotopy groups of $F\psi^q$ are easily computed from Bott periodicity, the above calculation is an immediate corollary. We proceed to outline the proof of the main theorem.

(0.1) LEMMA. *$BGL\mathbb{F}_q^+$ and $F\psi^q$ are homotopy associative and commutative H-spaces.*

In particular both spaces are simple, so by Whitehead's theorem it is enough to show:

(0.2). *There is a map $\theta : BGL\mathbb{F}_q^+ \to F\psi^q$ inducing an isomorphism on integral homology.*

The map θ is constructed by an ingenious application of representation theory (§2).

NOTE: *Throughout this paper. l denotes a prime distinct from the characteristic prime p.*

By elementary facts from homology theory, it is then enough to show:

(0.3). *θ induces an isomorphism on $H_*(-, R)$ for $R = \mathbb{Q}, \mathbb{Z}/p, \mathbb{Z}/\ell$.*

The cases $R = \mathbb{Q}, \mathbb{Z}/p$ are settled by the next two theorems.

THEOREM 1. $\widetilde{H}_*(F\psi^q; \mathbb{Q}) = 0 = \widetilde{H}_*(F\psi^q; \mathbb{Z}/p)$.

This is an easy consequence of Serre's "mod C" theory.

THEOREM 3.1. $\widetilde{H}_*(BGL\mathbb{F}_q^+; \mathbb{Q}) = 0 = \widetilde{H}_*(BGL\mathbb{F}_q^+; \mathbb{Z}/p)$.

The first equality is an easy application of the "transfer" in group cohomology. The second is much harder. The proof given in §3 is somewhat different from Quillen's original proof and is due to [Friedlander] (see also [Fiedoriwicz-Priddy]). The remaining and most difficult step is then:

THEOREM 4.1. *θ induces an isomorphism on $H_*(\ \ ; \mathbb{Z}/\ell)$.*

This step has its own outline in §4. The hardest parts are postponed to §5, 6, 7. The proof involves explicitly calculating the mod ℓ homology rings of both spaces. This calculation of the homology of $GL\mathbb{F}_q$ is of great interest in its own right.

Our approach to $H_*(F\psi^q; \mathbb{Z}/l)$ is to first reduce to the case $l \mid q-1$ and then use the Serre spectral sequence. The original argument was based on the Eilenberg-Moore spectral sequence.

The reader may wish to focus initially on §2, 4, 6; these sections contain the conceptual core of the argument. The lemma of §5 is a bit technical; §7 deals with the eccentricities of the prime 2. In §8 we give some further results on the cohomology of $GL_n\mathbb{F}_q$ (these are not required for the main theorem).

I do assume that the reader is familiar with Quillen's "plus" construction. However everything we need is contained in the following three theorems.

(0.4) THEOREM. *Let R be a ring. Then there is a CW-complex $BGLR^+$ and a map $\eta : BGLR \to BGLR^+$ such that*

(a) $\pi_1(\eta)$ is the abelianization:

$$\pi_1 BGLR = GLR \to GLR/[GLR, GLR] = \pi_1 BGLR^+$$

(b) For any local coefficient system $\mathcal{A}$ on $BGLR^+$, η induces an isomorphism $H_*(BGLR, \eta^*\mathcal{A}) \underset{\cong}{\to} H_*(BGLR^+, \mathcal{A})$.

(0.5) THEOREM (UNIVERSAL PROPERTY). *Let Y be a space with abelian fundamental group. Then for any map $f : BGLR \to Y$ there is a map g such that*

$$\begin{array}{ccc} BGLR & \xrightarrow{\eta} & BGLR^+ \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle g} & \\ Y & & \end{array}$$

commutes. Furthermore g is unique up to homotopy.

(0.6) THEOREM. *$BGLR^+$ is a homotopy associative and commutative H-space.*

For further information see Chapter I of [Loday] or its English reproduction in [Srinivas]. In fact we only need the case of 0.4b with $\mathcal{A}$ an ordinary coefficient group, and the case of (0.5) with Y a simple space. In (0.6), the multiplication map m has the property that the following diagrams are homotopy commutative:

$$\begin{array}{ccc} BGL_mR \times BGL_nR & \longrightarrow & BGL_{m+n}R \\ \downarrow & & \downarrow \\ BGLR^+ \times BGLR^+ & \xrightarrow[m]{} & BGLR^+ \end{array}$$

where the unlabelled maps are the obvious ones. See also Appendix 2.

We will frequently and without comment make use of Whitehead's theorem asserting that a weak equivalence of CW-spaces is a homotopy equivalence. This is justified because none of our constructions lead outside the category of CW-spaces (i.e., spaces of the homotopy type of a CW-complex). Alternatively, it is possible to take the viewpoint that a weak equivalence is good enough; for example, weak equivalences suffice for all homotopy and homology calculations.

A word on notation: to avoid a tedious proliferation of definitions and symbols, certain letters will be used generically in the following way. Maps labelled i, i', etc. are obvious inclusions or are induced by obvious inclusions. Maps labelled m, m', etc. are H-space multiplications. Maps which aren't labelled at all are supposed to be obvious.

Contents

1. The space $F\psi^q$.

Recall that BU is a homotopy associative and commutative H-space. In fact by Bott periodicity BU is a double-loop space. but the H-space structure can be obtained in a more elementary way as follows. For any connected finite complex X, $[X, BU] = \widetilde{K}X$. Write $BU = \bigcup_{n=1}^{\infty} X_n$, where the X_n are finite subcomplexes. Then there is an exact Milnor sequence

$$0 \to \lim\nolimits_n^1[X_n \times X_n, U] \to [BU \times BU, BU] \to \lim[X_n \times X_n, BU] \to 0.$$

The natural transformation $\oplus : \widetilde{K}X \times \widetilde{K}X \to \widetilde{K}X$ is therefore realized by a map $m : BU \times BU \to BU$. Furthermore, the X_n can be taken to have only even-dimensional cells, so $K^1(X_n \times X_n) = 0$ (see e.g. [Atiyah 1]). Hence the above $\lim^1$ term vanishes and m is unique. The same $\lim^1$-argument shows that m is homotopy associative and commutative.

Similarly, there are maps $\chi : BU \to BU$ and $\psi^q : BU \to BU$, unique up to homotopy, which realize the natural transformations $a \mapsto -a$ and $a \mapsto \psi^q a$, respectively. So we can form the difference $\psi^q - 1$:

$$BU \xrightarrow{\Delta} BU \times BU \xrightarrow{\psi^q \times \chi} BU \times BU \xrightarrow{m} BU$$

and $F\psi^q$ is the homotopy fibre of this map.

(1.1) PROPOSITION. *$F\psi^q$ is path-connected and*

$$\pi_{2n-1}F\psi^q = \mathbb{Z}/(q^n - 1)$$
$$\pi_{2n}F\psi^q = 0.$$

PROOF: Recall that ψ^q acts on $\pi_{2n}BU \cong \mathbb{Z}$ as multiplication by q^n. The proposition then follows immediately from the long exact homotopy sequence of the fibre sequence $F\psi^q \xrightarrow{j} BU \xrightarrow{\psi^q - 1} BU$.

(1.2) PROPOSITION. *$F\psi^q$ is a simple space.*

PROOF: Let $E \to B$ be any fibration with fibre F, where F, B path-connected. Then the action of $\pi_1 F$ on $\pi_n F$ is induced from an action of $\pi_1 E$ on $\pi_n F$, and hence is trivial if E is simply connected.

(1.3) PROPOSITION. $\widetilde{H}_*(F\psi^q; \mathbb{Q}) = 0 = \widetilde{H}_*(F\psi^q; \mathbb{Z}/p)$.

PROOF: The homotopy groups of $F\psi^q$ lie in the Serre class of S-torsion groups, where $S = \{\text{primes } \ell : \ell \neq p\}$. Since $F\psi^q$ is simple, Serre's theory implies that the integral homology groups are also S-torsion. Hence if $A = \widetilde{H}_*(F\psi^q; \mathbb{Z})$, we have $A \otimes_{\mathbb{Z}} \mathbb{Q} = A \otimes_{\mathbb{Z}} \mathbb{Z}/p = \mathrm{Tor}(A, \mathbb{Z}/p) = 0$, and the result follows from the universal coefficient theorem. □

(1.4) PROPOSITION. *$F\psi^q$ is a homotopy associative and commutative H-space. and $j : F\psi^q \to BU$ is an H-map.*

PROOF: To prove the first assertion we will show that $F\psi^q$ is in fact a double-loop space. For a simple space X, let $X \to LX$ denote the localization of X away from p, in the sense of Sullivan (A4). Thus $X \to LX$ induces isomorphisms $\pi_* X \otimes \mathbb{Z}\left[\frac{1}{p}\right] \cong \pi_* LX$ and $H_* X \otimes \mathbb{Z}\left[\frac{1}{p}\right] \cong H_* LX$. Consider the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc} F\psi^q & \longrightarrow & BU & \longrightarrow & BU \\ {\scriptstyle g}\downarrow & & \downarrow & & \downarrow \\ F & \xrightarrow[j']{} & LBU & \xrightarrow[h]{} & LBU \end{array}$$

where $h = L(\psi^q - 1)$ and g is chosen so that the diagram commutes up to homotopy. Since $(\pi_* F\psi^q) \otimes \mathbb{Z}\left[\frac{1}{p}\right] = \pi_* F\psi^q$, and $(-) \otimes \mathbb{Z}\left[\frac{1}{p}\right]$ is an exact functor, a simple argument with the 5-lemma shows that g is a homotopy equivalence. Thus it will be enough to show that F is a double-loop space, and for this it is enough to show that h is a double-loop map. (Unfortunately, $\psi^q - 1$ itself is not a double-loop map—that is why we are forced to localize.)

Observe that the Bott periodicity isomorphism $\beta : \widetilde{K}X \xrightarrow[\cong]{} \widetilde{K}(S^2 \wedge X)$ does not commute with the Adams operations. In fact we have the formula $\psi^q(\beta a) = q\beta(\psi^q a)$, because $\beta a = b \times a$ with $b \in \widetilde{K}S^2$ and ψ^q is multiplicative. But in the localization LBU, multiplication by q (in the H-space structure) is an equivalence $q : LBU \xrightarrow[\cong]{} LBU$. Hence we can form the diagram

$$\begin{array}{ccc} LBU & \xrightarrow{h} & LBU \\ {\scriptstyle L\beta}\downarrow & & \downarrow{\scriptstyle L\beta} \\ L\Omega_0^2 BU & \longrightarrow & L\Omega_0^2 BU \\ \wr\Vert & & \wr\Vert \\ \Omega_0^2 LBU & \xrightarrow[\Omega_0^2 h']{} & \Omega_0^2 LBU \end{array}$$

where $h' = (L\psi^q) \cdot q^{-1} - 1$. To see that the diagram is homotopy commutative, it is enough by another $\lim^1$ argument to check that the two ways around the diagram yield the same natural transformation $q^{-1}\widetilde{K}X \to q^{-1}\widetilde{K}(S^2 \wedge X)$, X a finite complex. But going around the right-hand side yields $a \mapsto \frac{1}{q}\psi^q \beta a - \beta a$, and around the left-hand side we get $\beta(\psi^q a - a)$. These agree by our earlier formula. Since $L\beta$ is an equivalence, this shows h is a double-loop map, as desired.

So $F\psi^q$ is an H-space as claimed, and it remains to show that j is an H-map. Let h_1, h_2 denote the two ways around the diagram

$$\begin{array}{ccc} F\psi^q \times F\psi^q & \xrightarrow{j \times j} & BU \times BU \\ \downarrow & & \downarrow \\ F\psi^q & \xrightarrow[j]{} & BU \end{array}$$

where the vertical maps are the multiplications. We want to show that $h_1 - h_2 \sim 0$. Now clearly g and j' are H-maps. so $h_1 - h_2$ becomes null in LBU and so lifts to the homotopy-fibre Y of $BU \to LBU$. But Y is a simple space whose homotopy groups are all p-torsion, and it then follows from (1.3) that $\widetilde{H}^*(F\psi^q \times F\psi^q; \pi_n Y) = 0$ for all n. So the lift is null by obstruction theory, and $h_1 - h_2 \sim 0$. This completes the proof of (1.4). □

We conclude this section with an alternate description of $F\psi^q$ that is enlightening and will be used in §5. Let X be any homotopy associative and commutative H-space, $f : X \to X$ a self-map. Let F be the homotopy fibre of $1 - f$. Then it is natural to think of F as a sort of "fixed-point set" of the map f, by analogy with the situation where X is an abelian group and F is the kernel of $1 - f$. The actual fixed point set of f is precisely the pullback of the diagram

$$\begin{array}{ccc} & & X \\ & & \big\downarrow \Delta \\ X & \xrightarrow{(1,f)} & X \times X \end{array}$$

In practice however the map f is only defined up to homotopy, so it makes no sense to talk about the actual fixed point set. Instead we consider the *homotopy fixed-point set* F' defined as the homotopy pullback of the above diagram—that is, convert Δ to a fibration and then form the pullback. Now F' is at least well-defined up to homotopy equivalence.

(1.5) PROPOSITION. *Given X, f as above, F is homotopy equivalent to F'.*

PROOF: Given any pullback diagram or "fibre square"

$$\begin{array}{ccc} E' & \longrightarrow & E \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ B' & \xrightarrow[g]{} & B \end{array}$$

in which π (and hence also π') is a fibration, there is an exact "Mayer-Vietoris" sequence

$$\to \pi_n E' \to \pi_n B' \oplus \pi_n E \xrightarrow{g-\pi} \pi_n B \to \pi_{n-1} E' \to \cdots$$

To see this form the ladder of exact sequences given by the fibrations π', π; any such ladder yields a Mayer-Vietoris sequence by a trivial diagram chase. Now F is the homotopy pullback of the diagram

$$\begin{array}{ccc} & & * \\ & & \big\downarrow \\ X & \xrightarrow{1-f} & X \end{array}$$

and if $d : X \times X \to X$ is the difference map for the H-space structure, the diagrams

$$\begin{array}{ccc} X & \xrightarrow{=} & * \\ {\scriptstyle\Delta}\downarrow & & \downarrow \\ X \times X & \xrightarrow[d]{} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{(1,f)} & X \times X \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle d} \\ X & \xrightarrow[1-f]{} & X \end{array}$$

are homotopy commutative. So, after (implicitly) converting Δ and $* \to X$ to fibrations we get a map of fibre squares

$$\begin{array}{ccc} F' & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle\Delta} \\ X & \xrightarrow[(1,f)]{} & X \times X \end{array} \quad \longrightarrow \quad \begin{array}{ccc} F & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \xrightarrow[1-f]{} & X \end{array}$$

where by "map" we mean a cube of maps with the squares above as front and back faces, and with the four remaining sides homotopy commutative. In particular we obtain a ladder of Mayer-Vietoris sequences

$$\begin{array}{ccccccccc} \longrightarrow & \pi_n F' & \longrightarrow & \pi_n X \oplus \pi_n X & \xrightarrow{k} & \pi_n X \oplus \pi_n X & \longrightarrow & \pi_{n-1} F' & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & \pi_n F & \longrightarrow & \pi_n X & \longrightarrow & \pi_n X & \longrightarrow & \pi_{n-1} F' & \longrightarrow \end{array}$$

It is easy to check that k is an isomorphism on the kernels of the vertical maps. It follows (in effect, by the 5-lemma) that $\pi_n F' \to \pi_n F$ is an isomorphism for all n. □

(1.6) COROLLARY. *$F\psi^q$ is the homotopy fixed-point set of ψ^q. That is, there is a homotopy pullback diagram*

$$\begin{array}{ccc} F\psi^q & \longrightarrow & BU \\ \downarrow & & \downarrow{\scriptstyle\Delta} \\ BU & \xrightarrow[(1,\psi^q)]{} & BU \times BU \end{array}$$

2. The Brauer lift.

We begin with a general discussion of K^*BG, G a finite group. Here we stress that for infinite complexes X, K^0X is *defined* to be $[X, BU \times \mathbb{Z}]$; similarly $K^1X = [X, U]$.

Suppose given a finite-dimensional complex representation V of G. Then the Borel construction yields a complex vector bundle $\xi_V = EG \times_G V$ over the classifying space BG.

It is easy to check that this construction takes isomorphic representations to isomorphic bundles, and that it commutes with direct sums, tensor products, and exterior powers. In particular, letting $\operatorname{Rep}(G, \mathbb{C})$ denote the set of isomorphism classes of such representations, we get a homomorphism of semirings

$$\operatorname{Rep}(G, \mathbb{C}) \to K^0 BG.$$

The *representation ring* of G over $\mathbb{C}$, denoted $\mathcal{R}_{\mathbb{C}} G$, is the group completion of $\operatorname{Rep}(G, \mathbb{C})$. Thus the above map factors uniquely through a ring homomorphism

$$\mathcal{R}_{\mathbb{C}} G \xrightarrow{\varphi} K^0 BG = [BG, BU \times \mathbb{Z}].$$

The map φ is compatible with the evident augmentations:

$$\begin{array}{ccc} \mathcal{R}_{\mathbb{C}} G & \longrightarrow & K^0 BG \\ & {\scriptstyle \epsilon'} \searrow \quad \swarrow {\scriptstyle \epsilon} & \\ & \mathbb{Z} & \end{array}$$

(e.g. $\epsilon'([v]) = \dim V$) and so induces a map

$$IG \to \widetilde{K} BG = [BG, BU]$$

where $IG \equiv \operatorname{Ker} \epsilon'$ is the "augmentation ideal." We may also view IG as $\mathcal{R}_{\mathbb{C}} G / \mathbb{Z}$, where $\mathbb{Z} \subseteq \mathcal{R}_{\mathbb{C}} G$ is generated by the trivial representation—provided we bear in mind that this only makes sense as groups, not rings.

It is natural to ask how far φ is from being an isomorphism. The striking answer is the following deep Theorem of Atiyah, which the reader should be aware of even though *only part (b)* is ever used in these notes.

(2.1) THEOREM [ATIYAH], SEE ALSO [ATIYAH-SEGAL].

(a) φ *factors through an isomorphism* $(\mathcal{R}_{\mathbb{C}} G)^\wedge \cong K^\circ BG$, *where* $(\mathcal{R}_{\mathbb{C}} G)^\wedge$ *is the* IG*-adic completion* $\varprojlim \mathcal{R}_{\mathbb{C}} G / (IG)^n$.

(b) $K^1 BG = 0$.

Let us now turn to the problem of constructing a good map $\theta : BGL\mathbb{F}_q^+ \to F\psi^q$. We will do this in two steps.

Step 1. Construction of a map $\bar{\theta} : BGL\mathbb{F}_q^+ \to BU$

Step 2. $(\psi^q - 1) \circ \theta$ is nullhomotopic, so $\bar{\theta}$ lifts to a map $\theta : BGL\mathbb{F}_q^+ \to F\psi^q$. (We will see shortly that the lift is necessarily unique).

For Step 1 we first observe:

(2.2) LEMMA. $[BGL\mathbb{F}_q^+, BU] = \varprojlim [BGL_n\mathbb{F}_q, BU]$.

PROOF: Since BU is simple, the universal property of the plus construction shows that $[BGL\mathbb{F}_q^+, BU] = [BGL\mathbb{F}_q, BU]$. Since $BGL\mathbb{F}_q = \cup BGL_n\mathbb{F}_q$ and $[BGL_n\mathbb{F}_q, U] = 0$ by Atiyah's theorem, the lemma follows from Milnor's exact sequence. □

Now suppose we had a sequence V_n of representations of $GL_n\mathbb{F}_q$, compatible under restriction. The construction $V \mapsto \xi_V$ clearly commutes with restriction to subgroups, so this would yield an element of $\varprojlim[BGL_n\mathbb{F}_q, BU]$ and hence a map $BGL\mathbb{F}_q \to BU$. However the only obvious representations in sight are the natural representations of $GL_n\mathbb{F}_q$ on $(\mathbb{F}_q)^n$. We are going to "lift" these representations to virtual representations over $\mathbb{C}$.

Recall that the character χ_V of a complex representation V is defined by $\chi_V(g) = \text{trace}(\varphi_V(g))$ where $\varphi_V : G \to \text{Aut}\, V$. Representations are determined up to isomorphism by their characters, and $V \mapsto \chi_V$ defines an injective ring homomorphism from $\mathcal{R}_{\mathbb{C}}G$ to the ring $C(G)$ of complex-valued class functions on G. One can define characters of representations over $\mathbb{F}_q$ in the same way, but they are not so useful—for example, the $(n \times n)$ identity matrix over $\mathbb{F}_q$ has trace zero if $p \mid n$. Instead we proceed as follows. Let $\bar{\mathbb{F}}_q$ denote the algebraic closure of $\mathbb{F}_q$, and fix, once and for all, an injective homomorphism $\bar{\mathbb{F}}_q^\times \xrightarrow{i} \mathbb{C}^\times$. (Recall that $\mathbb{F}_{q^n}^\times = \mathbb{Z}/(q^n - 1)$, so $\bar{\mathbb{F}}_q^\times$ is isomorphic to the group of roots of unity in $\mathbb{C}^\times$ which have order prime to p). Suppose W is a representation of G over $\mathbb{F}_q$. The *Brauer character* χ_W is defined by $\chi_W(g) = \sum_{\alpha \in S_g} i(\alpha)$, where $S_g \subseteq \bar{\mathbb{F}}_q^\times$ is the set of eigenvalues (with multiplicity) of $\varphi_W(g) \in \text{Aut}\, W$. Then χ_W is a class function on G.

(2.3) THEOREM (GREEN). *For any finite group G and representation W over $\mathbb{F}_q$, the Brauer character is a virtual complex character of G—that is, it is the character of a virtual complex representation of G.* □

For a proof of Green's theorem from the Brauer induction theorem, see A8.

We may now define the "Brauer lift" $\bar{\theta} : BGL\mathbb{F}_q^+ \to BU$. Let χ_n denote the Brauer character of the standard representation of $GL_n\mathbb{F}_q$ on $\mathbb{F}_q^n$. Clearly $\chi_n|_{GL_{n-1}} = \chi_{n-1}$, so by Green's theorem the χ_n define a compatible family of virtual complex representations. As remarked earlier, this yields an element of $\varprojlim[BGL_n\mathbb{F}_q, BU]$ and hence a map $BGL\mathbb{F}_q^+ \xrightarrow{\bar{\theta}} BU$. We reiterate that $\bar{\theta}$ depends on the choice of embedding $\bar{\mathbb{F}}_q^\times \rightarrowtail \mathbb{C}^\times$, but that this choice is fixed once and for all.

For Step 2, we first observe that the lift θ is unique (up to homotopy) if it exists:

(2.4) PROPOSITION. $[BGL\mathbb{F}_q^+, U] = 0$.

PROOF: By Atiyah's Theorem (b) and the Milnor exact sequence, it is enough to show $\lim^1[BGL_n\mathbb{F}_q, BU] = 0$. This follows from a general fact proved in Appendix 3. □

Now observe that since $\mathcal{R}_{\mathbb{C}}G \xrightarrow{\varphi} K^0 BG$ is a ring homomorphism that commutes with exterior powers, it automatically commutes with Adams operations.

Here the Adams operations are defined on $\mathcal{R}_{\mathbb{C}}G$ exactly as for K-theory, using exterior powers. In particular ψ^k is a ring homomorphism, and if L is a one-dimensional representation then $\psi^k L$ is the k-fold tensor power L^k.

Identifying $\mathcal{R}_{\mathbb{C}}G$ with the ring of virtual characters χ, we can define ψ^k on such characters. Here there is a very simple formula.

(2.5) PROPOSITION. *Let χ be a virtual character of G. Then $(\psi^k\chi)(g) = \chi(g^k)$ for all $g \in G$.*

PROOF: Let $\theta^k\chi$ be defined by $(\theta^k\chi)(g) = \chi(g^k)$. If $\chi = \chi_L$ with L one-dimensional, then χ is actually a homomorphism $G \to \mathbb{C}^\times$, so

$$(\psi^k\chi)(g) = \chi_{L^k}(g) = (\chi(g))^k = \chi(g^k) = \theta^k\chi(g).$$

Since ψ^k and θ^k are additive, it follows that $\psi^k\chi_V = \theta^k\chi_V$ whenever V is a direct sum of one-dimensional representations. But if G is cyclic, any V has this form, so ψ^k and θ^k agree for cyclic groups. The general case follows immediately by restricting ψ^k, θ^k to the various cyclic subgroups $\langle g\rangle$. □

(2.6) COROLLARY. *Suppose χ is the Brauer character of a representation W over $\mathbb{F}_q$. Then $\psi^q\chi = \chi$.*

PROOF: Fix $g \in G$. Then $S_g = S_{g^q}$. To see this, recall that the eigenvalues of any linear transformation lie in a finite Galois extension K and are permuted by the Galois group (cf. A8). Here $\mathrm{Gal}(K/\mathbb{F}_q)$ will be cyclic, generated by the Frobenius $\alpha \mapsto \alpha^q$. Hence $S_{g^q} = \{\alpha^q : \alpha \in S_g\} = S_g$. The corollary then follows immediately from the definition of the Brauer character. □

We may now complete Step 2 and construct the desired map $\theta : BGL\mathbb{F}_q^+ \to F\psi^q$. Consider the composite

$$BGL_n\mathbb{F}_q \xrightarrow{j} BU \xrightarrow{\psi^q - 1} BU.$$

By (2.6), $\psi^q j \sim j$ so $(\psi^q - 1)j$ is nullhomotopic. Since $\lim^1[BGL_n\mathbb{F}_q, U] = 0$, this implies the composite

$$BGL\mathbb{F}_q \to BGL\mathbb{F}_q^+ \xrightarrow{\bar\theta} BU \xrightarrow{\psi^q - 1} BU$$

is null. By the universal property of the plus construction, it follows that $(\psi^q - 1)\circ\bar\theta$ is null. Hence there is a unique lift θ, as desired.

(2.7) PROPOSITION. *The Brauer lift is an H-map.*

PROOF: We must show the following diagram is homotopy commutative:

$$\begin{array}{ccc} BGL\mathbb{F}_q^+ \times BGL\mathbb{F}_q^+ & \xrightarrow{\theta \times \theta} & F\psi^q \times F\psi^q \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\ BGL\mathbb{F}_q^+ & \xrightarrow[\theta]{\qquad} & F\psi^q \end{array}$$

Now $[BGL\mathbb{F}_q^+ \times BGL\mathbb{F}_q^+, U] = 0$, by an argument identical to the proof of (2.4). So it is enough to show $jm'(\theta \times \theta) \sim j\theta m$ where $j : F\psi^q \to BU$. In other words, it is enough to show $\bar{\theta}$ is an H-map. Using an argument which by now should be routine ($\lim^1$, universal property ...), this in turn reduces to showing the following homotopy commutes:

$$\begin{array}{ccc} BGL_m \times BGL_n\mathbb{F}_q & \longrightarrow & BU \times BU \\ \downarrow & & \downarrow \\ BGL_{m\times n}\mathbb{F}_q & \longrightarrow & BU \end{array}$$

But it is clear on inspection that each composite is given by the Brauer lift of the natural representation of $GL_m \times GL_n$ on $\mathbb{F}_q^{m+n}$. □

We conclude this section by recording for future reference the behaviour of θ with respect to field extension. We assume that at this point the reader can fill in the part of the argument involving $\lim^1 = 0$, the universal property of the plus construction, etc. We will refer to this as "our usual argument."

Consider the diagram

$$\begin{array}{ccccccc} BGL\mathbb{F}_q^+ & \xrightarrow{\theta} & F\psi^q & \xrightarrow{j} & BU & \xrightarrow{\psi^q - 1} & BU \\ {\scriptstyle i}\downarrow & \text{I} & {\scriptstyle h}\downarrow & \text{II} & {\scriptstyle =}\downarrow & \text{III} & \downarrow{\scriptstyle N} \\ BGL\mathbb{F}_{q^r}^+ & \xrightarrow[\theta']{} & F\psi^{q^r} & \xrightarrow[j']{} & BU & \xrightarrow[\psi^{q^r} - 1]{} & BU \end{array}$$

where i is induced by the inclusion $\mathbb{F}_q \subset \mathbb{F}_{q^r}$ and $N = 1 + \psi^q + \cdots + \psi^{q^{r-1}}$ ("+" with respect to the H-space structure on BU, of course). Since $\psi^j \circ \psi^k = \psi^{jk}$, the square III homotopy commutes and hence there exists a map h such that the square II homotopy commutes.

(2.8) PROPOSITION. *For any choice of h, the square I homotopy commutes.*

PROOF: By our usual argument, it is enough to show $j'\theta' i \sim j\theta$ after restriction to $BGL_n\mathbb{F}_q$. Thus we have two maps $BGL_n\mathbb{F}_q \to BU$, each of which is obtained from the Brauer lift of some representation over $\bar{\mathbb{F}}_q$. But it is clear that the two representations are the same—namely, each is given by $GL_n\mathbb{F}_q \subset GL_n\bar{\mathbb{F}}_q$. □

Now consider the diagram

$$\begin{array}{ccccccc}
BGL\mathbb{F}_{q^r}^{+} & \xrightarrow{\theta'} & F\psi^{q^r} & \xrightarrow{j'} & BU & \xrightarrow{\psi^{q^r}-1} & BU \\
\tau\downarrow & \text{I} & t\,\vdots & \text{II} & N\downarrow & \text{III} & \downarrow = \\
BGL\mathbb{F}_{q}^{+} & \xrightarrow[\theta]{} & F\psi^{q} & \xrightarrow[j]{} & BU & \xrightarrow[\psi^{q}-1]{} & BU
\end{array}$$

where τ is the transfer (Appendix 2). As before, III homotopy commutes, so there exists a map t such that II homotopy commutes.

(2.9) Proposition. *For any choice of t. the square I homotopy commutes.*

Proof: By our usual argument, it is enough to show $j\theta\tau \sim Nj'\theta'$ after restriction to $BGL_n\mathbb{F}_{q^r}$. As in (2.8), this amounts to showing two representations agree, but in this case it's not quite so clear what the representations are. Let χ denote the Brauer character of the standard representation V of $GL_n\mathbb{F}_{q^r}$. Then $Nj'\theta'$ corresponds to the virtual character $\chi' = \chi + \psi^q\chi + \cdots + \psi^{q^{r-1}}\chi$ by definition of θ' and N. On the other hand $j\theta\tau$ corresponds to the Brauer character χ'' of the representation $V \otimes_{F_q} \mathbb{F}_{q^r}$; this follows by inspecting the definition of τ given in Appendix 2. By A8.2, $V \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r} = V \oplus V^{\sigma} \oplus \cdots \oplus V^{\sigma^{r-1}}$, where σ is the Frobenius. But the Brauer character χ_i of V^{σ^i} is given by $\chi_i(g) = \chi(\sigma^i(g)) = \chi(g^{q^i}) = (\psi^{q^i}\chi)(g)$. Hence $\chi' = \chi''$, as desired. □

Remark: Once the main theorem is proved, we will know that $[F\psi^q, U] = 0$ for all q. Hence the maps h and t above are in fact unique.

3. Rational and mod p homology of $BGL\mathbb{F}_q$.

(3.1) Proposition. $\widetilde{H}_*(GL\mathbb{F}_q : \mathbb{Q}) = 0$.

Proof: Since homology commutes with direct limits, it's enough to show $\widetilde{H}_*(GL_n\mathbb{F}_q, \mathbb{Q}) = 0$. But this is true for any finite group, by a simple argument with the transfer (Appendix 1). □

Recall that p is the characteristic of $\mathbb{F}_q : q = p^d$.

(3.2) Theorem. $\widetilde{H}_*(BGL\mathbb{F}_q; \mathbb{Z}/p) = 0$.

(3.3) Lemma. $\widetilde{H}_i(BGL_n\mathbb{F}_q; \mathbb{Z}/p) = 0$ *for $i < d(p-1)$, all n.*

PROOF OF (3.2): Choose r prime to p and consider

$$BGL\mathbb{F}_q^+ \xrightarrow{i} BGL\mathbb{F}_{q^r}^+ \xrightarrow{\tau} BGL\mathbb{F}_q^+$$

where τ is the transfer in algebraic K-theory (Appendix 2). Now let $(\quad)_{(p)}$ denote Sullivan localization at p. Then $(\tau i)_{(p)}$ is an equivalence, since $\pi_*(\tau i)_{(p)}$ is multiplication by r and r is prime to p. Hence $H_*(\tau i; \mathbb{Z}_{(p)})$ is an isomorphism and therefore also $H_*(\tau i; \mathbb{Z}/p)$ is an isomorphism. But applying (3.3) to $\mathbb{F}_{q^r}$ shows that $\widetilde{H}_n(\tau i, \mathbb{Z}/p)$ is the zero map for $n < dr(p-1)$. So $\widetilde{H}_n(BGL\mathbb{F}_q; \mathbb{Z}/p) = 0$ for $n < dr(p-1)$, and since r was arbitrary this completes the proof. □

It remains to prove the lemma. Let $B_n \subseteq GL_n\mathbb{F}_q$ denote the subgroup of upper triangular matrices and let $H_n = \{b \in B_n : b_{ii} = 1.1 \leq i \leq n\}$.

(3.4) LEMMA. *H_n is a p-Sylow subgroup of $GL_n\mathbb{F}_q$.*

PROOF: Clearly $|H_n| = q^{\binom{n}{2}}$, so H_n is a p-subgroup. On the other hand it is easy to show

$$|GL_n\mathbb{F}_q| = \prod_{i=0}^{n-1}(q^n - q^i) = q^{\binom{n}{2}}\prod_{i=1}^{n}(q^i - 1).$$ □

(3.5) COROLLARY. *The restriction map $H^*(GL_n\mathbb{F}_q; \mathbb{Z}/p) \to H^*(B_n; \mathbb{Z}/p)$ is injective. (See Appendix 1).*

Thus lemma (3.3) follows from

$$\widetilde{H}_i B_n = 0 \quad \text{for} \quad i < d(p-1). \tag{3.6}$$

We proceed by induction on n. For $n = 1$ we have $B_1 = \mathbb{F}_q^\times$. Hence $p \not| |B_1|$ so $\widetilde{H}_i B_1 = 0$ for all i. At the inductive step consider the evident group extension

$$A_n \to B_n \to B_{n-1}$$

where A_n is the "top row" subgroup.

(3.7) LEMMA. *$\widetilde{H}_i A_n = 0$ for $i < d(p-1)$.*

Assuming this, it follows that the terms $E^2_{p,q}$ of the Hochschild-Serre spectral sequence of the above group extension vanish for $0 < p + q < d(p-1)$. Hence the same is true for $E^{p,q}_\infty$, proving (3.6).

PROOF OF (3.7): It is enough to prove the analogous statement in cohomology. The group A_n is a semidirect product of the form

$$V \to A_n \to \mathbb{F}_q^\times$$

where V is the additive group of a vector space over $\mathbb{F}_q$, and $\mathbb{F}_q^\times$ acts on V by scalar multiplication. Since $p \nmid |\mathbb{F}_q^\times|$, the cohomology spectral sequence of this extension collapses to its vertical axis (see Appendix 1), so H^*A_n is the ring of invariants $(H^*V)^{\mathbb{F}_q^\times}$. Fix $\alpha \in \mathbb{F}_q^\times$ and regard α as an $\mathbb{F}_p$-linear endomorphism of V. Then the eigenvalues of α are the Galois conjugates of α—namely $\alpha, \alpha^p, \ldots, \alpha^{p^{d-1}}$. (Each eigenvalue occurs with multiplicity $\dim_{\mathbb{F}_q} V = n-1$, but we won't use this.) Now recall that by the Künneth theorem H^*V is the symmetric algebra $S(V^*)$ if $p=2$ or $S(V^* \oplus \Sigma V^*)$ if p odd (cf. A5). Hence if $W = V^* \otimes_{\mathbb{F}_p} \mathbb{F}_q$ we have $H^*(V;\mathbb{F}_q) = S(W)$ or $S(W \oplus \Sigma W)$. Now suppose p odd; the case $p=2$ is similar. Choose bases x_i, y_i of eigenvectors for the action of α on $W, \Sigma W$ respectively. Then $H^*(V, \mathbb{F}_q) = \mathbb{F}_q[y_1, \ldots, y_m] \otimes \mathbb{F}_q\langle x_1, \ldots, x_m\rangle$ where $m = d(n-1)$. Hence $H^*(V, \mathbb{F}_q)$ itself has a basis of eigenvectors, namely the monomials in the y_i, x_i. It is then clear that the minimal dimension s such that $(\widetilde{H}^*V)^{\mathbb{F}_q^\times}$ is nonzero can be described as follows: let $A = \{(a_0, \ldots, a_{d-1}) : a_i \geq 0 \text{ and } \Sigma a_i p^i = 0 = \operatorname{mod} p^d - 1\}$. Then s is the minimal value of the function Σa_i on the set A. Note $(p-1, \ldots, p-1) \in A$, so $s \leq d(p-1)$. Now suppose $(a_0, \ldots, a_{d-1})$ has Σa_i minimal. If $a_i \geq p$ for some i, replace a_i by $a_i - p$ and a_{i+1} by $a_{i+1}+1$ (where $a_d = a_0$). This decreases Σa_i, contradicting the assumption of minimality. So $a_i \leq p-1$ for all i, and this trivially forces $a_i = p-1$ for all i. Hence $s = d(p-1)$. This completes the proof of (3.7). □

4. Mod l homology.

In this section we outline the proofs of the following theorems 4.1 and 4.2 below, postponing the three hardest steps to later sections. All homology and cohomology groups have $\mathbb{Z}/l$ coefficients, l fixed. For unexplained terms involving symmetric algebras, Hopf algebras, etc., see Appendix 5.

(4.1) THEOREM. *$\theta : BGL\mathbb{F}_q^+ \to F\psi^q$ induces an isomorphism on $H_*(\ \ ;\mathbb{Z}/l)$.*

As explained in the introduction, this will complete the proof of the main theorem.

Let r be minimal such that $l \mid q^r - 1$. Thus r is the order of q in the group $\mathbb{Z}/l^\times$, so $r \mid l-1$. Let a be maximal such that $l^a \mid q^r - 1$. Then if μ_{l^n} denotes the l^n-th roots of unity, $\mathbb{F}_q(\mu_l) = \mathbb{F}_{q^r}$ and $\mu \equiv \mu_{l^a}$ is the l-torsion subgroup of $\mathbb{F}_{q^r}^\times$. Now let C_q denote the subgroup of $GL_r\mathbb{F}_q$ generated by μ and the Galois group $G(\mathbb{F}_{q^r}/\mathbb{F}_q)$. Here we have identified $(\mathbb{F}_q)^r$ with $\mathbb{F}_{q^r}$. The Galois group is cyclic of order r, generated by the Frobenius σ. Thus C_q fits into a split extension

$$\mu \xrightarrow{i} C_q \to \mathbb{Z}/r$$

with σ acting on μ by $\sigma(\alpha) = \alpha^q$. Recall that $H^*\mu = \mathbb{Z}/l[y] \otimes \mathbb{Z}/l\langle x \rangle$ with $|y| = 2$, $|x| = 1$, except in the case $l = 2$ $a = 1$ when $H^*\mathbb{Z}/2 = \mathbb{Z}/2[x]$. However for the next lemma we need only consider l odd.

LEMMA. *$H^*C_q \underset{\cong}{\xrightarrow{i^*}} H^*(\mu)^{\mathbb{Z}/r} = \mathbb{Z}/l[y^r] \otimes \mathbb{Z}/l\langle xy^{r-1}\rangle$ if l odd or $a > 1$. If $l = 2$ and $a = 1$, $H^*\mathbb{Z}/2 = \mathbb{Z}/2[x]$.*

PROOF: We can assume l odd, since for $l = 2$ C_q is just μ. Since r is prime to l, the Hochschild-Serre spectral sequence of the above extension collapses to its vertical edge and the first isomorphism is immediate. On the other hand $H^1\mu = H^2\mu = \mathbb{Z}/l$ with σ acting as multiplication by q. This is clear for $H^1\mu = \mathrm{Hom}(\mu, \mathbb{Z}/l)$. For $H^2\mu$, use the Bockstein isomorphism $\beta : H^1\mu \underset{\cong}{\rightarrow} H^2\mu$ defined by $\beta(a) = r(b)$, where $l^{a-1}b = \delta a$, $b \in H^2(\mu; \mathbb{Z})$, r is reduction mod l and δ is the coboundary associated to the coefficient sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/l \to 0$. The naturality of β shows in particular that β commutes with the σ-action. Then for example $\sigma(y^n) = q^n y^n$ so $\sigma(y^n) = y^n$ if and only if r divides n. Similarly $\sigma(xy^n) = xy^n$ if and only if r divides $n + 1$. □

Now the inclusion $C_q \subset GL_r\mathbb{F}_q$ induces a map $BC_q \to BGL\mathbb{F}_q^+$ and hence a map $H_*BC_q \to H_*BGL\mathbb{F}_q^+$. Since $BGL\mathbb{F}_q^+$ is a homotopy associative and commutative H-space, this map in turn extends to a ring homomorphism $S(\widetilde{H}_*BC_q) \to H_*BGL\mathbb{F}_q^+$, where $S(-)$ denotes the symmetric algebra. Let $S'(-)$ denote the *strict* symmetric algebra—that is, the quotient of the symmetric algebra obtained by factoring out the ideal generated by all a^2 with $|a|$ odd. This refinement is only relevant when $l = 2$, since for l odd $S = S'$.

(4.2) THEOREM. *The natural map $S(\widetilde{H}_*BC_q) \to H_*BGL\mathbb{F}_q^+$ induces an isomorphism $S'(\widetilde{H}_*BC_q) \underset{\cong}{\rightarrow} H_*BCL\mathbb{F}_q^+$.*

It is convenient to begin by reducing to the case $l \mid q - 1$. Among other things, this simplifies the notation in the computational part of the argument. Note also that C_q is just μ_{l^a}.

(4.3) LEMMA. *Suppose Theorems (4.1) and (4.2) are true when $l \mid (q-1)$. Then they are true for all q.*

PROOF: Of course we can assume l odd. Now consider the commutative diagrams of §2. Applying $\pi_*(-) \otimes \mathbb{Z}_{(l)}$ to the diagrams of 2.8, 2.9 and using A2.3 shows that if θ_r is an isomorphism on $\pi_*(-) \otimes \mathbb{Z}_{(l)}$ so is θ. Hence Theorem (4.1) in the case $l \mid (q-1)$ implies the general case (using Appendix 4). Next consider Theorem 4.2. We have a commutative diagram

$$\begin{array}{ccc} B\mu & \xrightarrow[j']{} & BGL\mathbb{F}_{q^r}^+ \\ {\scriptstyle \tau'}\downarrow & & \downarrow{\scriptstyle \tau} \\ BC_q & \xrightarrow[j]{} & BGL\mathbb{F}_q^+ \end{array}$$

The map τ' is the obvious inclusion, and induces a surjection on homology. Hence j induces a surjection $S(\widetilde{H}_*BC_q) \underset{S(j)}{\to} H_*BGL\mathbb{F}_q^+$ (since τ is an H-map and j' is a generating complex by hypothesis). To see that $S(j)$ is injective, consider the composite map k

$$B\mu \xrightarrow{\tau} BC_q \xrightarrow{j} BGL\mathbb{F}_q^+ \xrightarrow{i} BGL\mathbb{F}_{q^r}^+.$$

Now the map j' is induced by a one-dimensional representation λ of μ over $\mathbb{F}_{q^r}$. It follows from the definition of the transfer that k is induced by the r-dimensional representation $\lambda \oplus \lambda^\sigma \oplus \cdots \oplus \lambda^{\sigma^{r-1}}$ (see Appendix 2 and Appendix 8). This in turn implies that if $x \in H_n B\mu$ is a generator,

$$k_*x = j_*(x + \sigma x + \cdots + \sigma^{r-1}x) \quad \text{mod decomposables.}$$

So if $\sigma x = x$, $k_*x = rj_*x \neq 0$ mod decomposables. It follows that $\widetilde{H}_*(ij)$ is injective mod decomposables and hence $S(j)$ is injective. □

For the rest of this section, we assume l divides $q-1$.

The first step is to compute the homology of $F\psi^q$.

(4.4) LEMMA. *The homology spectral sequence of the fibre sequence $U \to F\psi^q \to BU$ is a spectral sequence of commutative $\mathbb{Z}/l$-algebras.*

PROOF: By §1 there is a map of fibre sequences

$$\begin{array}{ccccc} U & \longrightarrow & F\psi^q & \longrightarrow & BU \\ \downarrow & & \downarrow & & \downarrow \\ U_{(l)} & \longrightarrow & F & \longrightarrow & BU_{(l)} \end{array}$$

The vertical maps all induce isomorphisms on mod l-homology (and the base spaces are simply-connected), so we get an isomorphism of spectral sequences. On the other hand the bottom row, as we've seen, can be "delooped"—i.e. has the form $\Omega A \xrightarrow{\Omega f} \Omega B \xrightarrow{\Omega g} \Omega C$ for some fibre sequence $A \xrightarrow{f} B \xrightarrow{g} C$. This proves the lemma. □

(4.5) LEMMA. *If $l \mid (q-1)$, $H_*F\psi^q \cong H_*U \otimes H_*BU$ as algebras.*

PROOF: We have a commutative diagram (using $l \mid q-1$)

$$\begin{array}{ccc} B\mu & \xrightarrow{\bar{j}} & F\psi^q \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathbb{C}P^\infty & \longrightarrow & BU \end{array}$$

where $\bar{j}$ is the composite $B\mu \to BGL\mathbb{F}_q^+ \to F\psi^q$ and f is induced by our fixed inclusion $\mu \subset S^1 \subset \mathbb{C}^\times$. It follows that the homology generators b_n of H_*BU are permanent cycles, because the map $B\mu \xrightarrow{f} \mathbb{C}P^\infty$ is an isomorphism on homology in even degrees. So (4.4) the spectral sequence collapses and $E^\infty \cong H_*U \otimes H_*BU$. Here $H_*U \cong \mathbb{Z}/l\langle x_1, x_2, \dots \rangle$ with $|x_i| = 2i-1$. But there is no extension problem when $l = 2$, because the map $H_*U \to H_*F\psi^q$ is a ring homomorphism by (4.4). □

REMARK: If $l \not| (q-1)$, $B\mu$ must be replaced by BC in the above diagram. Then $H_n f$ is no longer surjective, and the spectral sequence does not in fact collapse.

The result of (4.5) looks rather complete, but it is not good enough. We need to know that $B\mu$ is a generating complex.

(4.6) THEOREM. *The map j induces an isomorphism*

$$S'(\widetilde{H}_*B\mu) \underset{\cong}{\to} H_*F\psi^q.$$

Write b_i (resp. e_i) for generators of $H_{2i}B\mu$ (resp. $H_{2i-1}B\mu$), and for simplicity write b_i also for $\bar{j}_*b_i$, etc. The next lemma is crucial. Its rather technical proof is deferred to §5.

(4.7) LEMMA. *e_i is indecomposable in $H_*F\psi^q$.*

PROOF OF 4.6: By Lemma (4.5), the two rings in (4.6) are abstractly isomorphic. Moreover each has finite type over $\mathbb{Z}/l$, so it is enough to show the given map is surjective. For this, we need only show $\widetilde{H}_*B\mu = QS'\widetilde{H}_*B\mu \to QH_*F\psi^q$ is surjective. This amounts to showing the elements b_n, e_n are indecomposable in $H_*F\psi^q$. For b_n this assertion is obvious from the diagram in (4.5) (b_n is indecomposable in H_*BU). For e_n this is (4.7). □

The next result is perhaps the key insight of the entire proof. It will be proved in §6.

(4.8) THEOREM. *Let $D_n \overset{i}{\subset} GL_n\mathbb{F}_q$ denote the subgroup of diagonal matrices. Then i^* : $H^*GL_n\mathbb{F}_q \to H^*D_n$ is injective.*

REMARK: We are still assuming $l|q-1$—otherwise, D_n has order prime to l and (4.8) would be absurd.

(4.9) COROLLARY. *$B\mu \to BGL\mathbb{F}_q^+$ induces a surjection*

$$S(\widetilde{H}_*B\mu) \xrightarrow{\varphi} H_*BGL\mathbb{F}_q^+.$$

PROOF OF (4.9): We have a commutative diagram

$$\begin{array}{ccc}
\prod_{i=1}^{n} B\mu & \longrightarrow & \prod_{i=1}^{n} BGL\mathbb{F}_q^+ \\
\downarrow f & & \downarrow m \\
BGL_n\mathbb{F}_q & \longrightarrow & BGL\mathbb{F}_q^+
\end{array}$$

where f is induced by the composite $\Pi\mu \xrightarrow{i'} \Pi\mathbb{F}_q^\times = D_n \subset GL_n\mathbb{F}_q$. So it is enough, clearly, to show H_*f is surjective. But H_*i' is an isomorphism, so this is immediate by dualizing (4.8).

We may now prove Theorems 4.1, 4.2 when l is odd (the point being that $S = S'$ for l odd). Consider the maps

$$S(\widetilde{H}_*B\mu) \xrightarrow{\varphi} H_*BGL\mathbb{F}_q^+ \xrightarrow{\theta_*} H_*F\psi^q.$$

By (4.6) $\theta_*\varphi$ is an isomorphism, so φ is injective. By (4.9) φ is then an isomorphism, proving (4.2). Then also θ_* is an isomorphism, proving (4.1).

For $l = 2$ we need one more lemma, which will be proved in §7.

(4.10) LEMMA. *$\epsilon_i^2 = 0$ in $H_*BGL\mathbb{F}_q^+$, $i \geq 1$.*

Of course the lemma is trivial for l odd. For $l = 2$ the result of (4.10) is that the surjection of (4.9) factors through the strict symmetric algebra $S'(\widetilde{H}_*B\mu)$. The argument for l odd then goes through verbatim for $l = 2$. This completes the proof of Theorems 4.1 and 4.2.

5. Proof of Lemma 4.7.

In this section we will prove:

(5.1) THEOREM. *Suppose l divides $q - 1$. Then the map $\varphi : BC \to F\psi^q$ induces an isomorphism $\widetilde{H}_*BC \to QH_*F\psi^q$.*

Here $Q(-)$ denotes the quotient $\mathbb{F}_l$-vector space of indecomposable elements (see Appendix 5). It is clear that (5.1) implies 4.7. We will prove the dual form of (5.1), namely:

$$PH^*F\psi^q \to \widetilde{H}^*BC \tag{5.2}$$

is an isomorphism, where $P(\quad)$ denotes the subspace of primitives.[1]

In even degrees (5.1) is obvious, so we may restrict our attention to odd degrees. Now by (4.5) the cohomology Serre spectral sequence of the fibre sequence $U \to F\psi^q \to BU$ collapses. Hence $H^*F\psi^q$ has an associated graded algebra of the form $\mathbb{Z}/l[c_1, c_2, \dots] \otimes \mathbb{Z}/l\langle f_1, f_2, \dots\rangle$. Here the c_i's are the Chern classes of the map $F\psi^q \to BU$, and so are actually elements of $H^*F\psi^q$. However if we choose elements $e_i \in H^*F\psi^q$ that reduce to the f_i modulo Serre filtration one, there is no guarantee that $e_i^2 = 0$ unless l is odd. Furthermore even for l odd there is little hope of "identifying" the primitives in $H^*F\psi^q$ without some canonical choice of the e_i. The next lemma is the key to (5.1).

(5.3) LEMMA. *There are classes $e_i \in H^{2i-1}F\psi^q$ satisfying*

(a) $m^*e_n = \sum_{i+j=n} e_i \otimes c_j + c_i \otimes e_j$

(b) φ^*e_n *is nonzero if and only if* $n = 1$.

Assuming this, we may easily prove (5.2) as follows. The formula (a) can be written compactly in the form $m^*e = e \otimes c + c \otimes e$, where $e = e_1 + e_2 + \cdots$ and $c = 1 + c_1 + c_2 + \cdots$. So $m^*\left(\frac{e}{c}\right) = (e \otimes c + c \otimes e)/c \otimes c = \frac{e}{c} \otimes 1 + 1 \otimes \frac{e}{c}$. Hence, writing $\frac{e}{c} = p_1 + p_2 + \cdots$, the elements p_i are primitive. Since $\varphi^*c_i = 0$ for $i > 1$, and $\varphi^*e_i = 0$ for $i > 1$ by 5.3b, we have $\varphi^*\frac{e}{c} = \varphi^*\left(\frac{e_1}{1+c_1}\right)$. Hence $\varphi^*p_n = (-1)^{n-1}c_1^{n-1}\varphi^*e_1$. By 5.3b, φ^*p_n is nonzero for all n. This shows $\varphi^* : PH^*F\psi^q \to H^*BC$ is nonzero in odd degrees and hence an isomorphism, as desired.

We now turn to the construction of the e_n. Here we follow [Quillen 2] essentially verbatim. We use the following general device. By a *square* Γ we mean a commutative diagram of spaces

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow[\ f'\ ]{} & Y' \end{array}$$

[1] This section has its own notation as we are working exclusively in cohomology. In particular the cohomology classes e_i defined here should not be confused with the homology classes e_i occuring in (4.7).

that is, a morphism $f \mapsto f'$ in the category of maps. Given two squares Γ_1, Γ_2, a *morphism* of squares $\Gamma_1 \to \Gamma_2$ consists of maps $X \to X_1$, $Y \to Y_1$, etc. such that the four new faces of the resulting cube are homotopy commutative (picture Γ_1, Γ_2 as the front and back faces of the cube). We may also form the product $\Gamma \times \Gamma'$ of two squares by taking cartesian products of spaces and maps. Now for any map $f : X \to Y$, let $H^*[f]$ denote the cohomology of the pair (I_f, X), where I_f is the mapping cylinder. Given a square Γ we then get a commutative diagram with exact rows

$$\begin{CD}
H^{i-1}X' @>{\delta'}>> H^i[f'] @>{j'}>> H^i(Y') @>{(f')^*}>> H^i(X') \\
@VV{g^*}V @VV{\bar{g}^*}V @VV{h^*}V @VV{g^*}V \\
H^{i-1}X @>>{\delta}> H^i[f] @>>{j}> H^i(Y) @>>{f^*}> H^i(X)
\end{CD}$$

Now let $M^i_\Gamma = \mathrm{Ker}(H^iY' \xrightarrow{((f')^*,h^*)} H^iX' \oplus H^iY)$ and let $N^{i-1}_\Gamma = \mathrm{Coker}(H^{i-1}Y \oplus H^{i-1}X' \xrightarrow{f^*+g^*} H^{i-1}X)$. Then there is a homomorphism $D_\Gamma : M^i_\Gamma \to N^{i-1}_\Gamma$ obtained by a diagram chase: Given $u \in M^i_\Gamma$, choose $w \in H^i[f']$ with $j'w = u$. Then $\bar{g}^*w = \delta v$ for some v and we set $D_\Gamma(u) = [v]$. It is clear that D_Γ is natural with respect to morphisms of squares, and that D_Γ is a map of H^*Y'-modules (since all the maps in the diagram are maps of H^*Y'-modules).

Now take Γ to be the square giving our alternate definition of $F\psi^q$:

$$\begin{CD}
F\psi^q @>{j}>> BU \\
@V{j}VV @VV{(1,\psi^q)}V \\
BU @>>{\Delta}> BU \times BU
\end{CD}$$

(5.4) LEMMA. *Let $\tilde{c}_n \in H^{2n}(BU;\mathbb{Z})$ denote the integral Chern class. Then $(\psi^q)^*\tilde{c}_n = q^n\tilde{c}_n$.*

PROOF: The total characteristic classes $\theta^q = 1 + qc_1 + q^2c_2 + \cdots$ and $c(\psi^q)$ are both multiplicative. By the splitting principle it is then enough to show that $\theta^q(L) = c(\psi^q L)$ for line bundles L. But $c(\psi^q L) = c(L^q) = 1 + c_1(L^q) = 1 + qc_1(L)$. □

Thus if $u_i = c_i \otimes 1 - 1 \otimes c_i$, $(1,\psi^q)^*(u) = (q^n - 1)c_i = 0$ since $l \mid q-1$. Obviously $\Delta^*(u_i) = 0$, so $u_i \in M^{2i}_\Gamma$. Furthermore N^{2i-1}_Γ is just $H^{2i-1}F\psi^q$, since $H^{\mathrm{odd}}BU = 0$. So we define $e_i = D_\Gamma(u_i)$.

PROOF OF 5.3A: All of the maps in the square Γ are maps of H-spaces. It follows that there is a morphism of squares $\Gamma^2 = \Gamma \times \Gamma \xrightarrow{m} \Gamma$, using the H-space multiplications. By naturality of the operation D we have

$$m^*(e_n) = D_{\Gamma^2} m^* u_n = D_{\Gamma^2}\left(\sum_{i+j=n} a_{ij} - b_{ij}\right)$$

where:

$$a_{ij} = c_i' c_j'' \otimes 1, b_{ij} = 1 \otimes c_i' c_i' \quad \text{in} \quad H^*(BU^2 \times BU^2)$$

and primes (resp. double primes) refer to inverse images under the first (resp. second) projection $\Gamma^2 \to \Gamma$. Here we have identified $(BU \times BU)^2 = BU^2 \times BU^2$. Now write

$$a_{ij} - b_{ij} = (c_i' \otimes 1 - 1 \otimes c_i')(c_j'' \otimes 1) + (1 \otimes c_i')(c_j'' \otimes 1 - 1 \otimes c_j'').$$

Then, using naturality of D with respect to the two projections, as well as the fact that D is linear over $H^*(BU^2 \times BU^2)$, we obtain

$$D_{\Gamma^2}(a_{ij} - b_{ij}) = D_\Gamma(u_i) \otimes c_j + c_i \otimes D_\Gamma u_j = e_i \otimes c_j + c_i \otimes e_j. \qquad \square$$

Before proving 5.3b we need two more lemmas.

(5.5) LEMMA. *Let $f : X \to Y$ be a map, and consider the long exact sequence*

$$\to H^{i-1}(X;\mathbb{Z}) \xrightarrow{\delta} H^i[f;\mathbb{Z}] \xrightarrow{j} H^i(Y;\mathbb{Z}) \to$$

Suppose $u \in H^i[f;\mathbb{Z}]$ and $ju = dv$, $d \in \mathbb{Z}$. Choose $w \in H^{i-1}(X;\mathbb{Z}/d)$ such that $\delta w = \bar{u}$, where $(\bar{\ })$ denotes reduction $\bmod d$. Then $\beta w = -f^* v \bmod \beta f^* H^{i-1}(Y;\mathbb{Z}/d)$, where $\beta : H^{i-1}(\ \ ,\mathbb{Z}/d) \to H^i(\ \ ,\mathbb{Z})$ is the Bockstein.

PROOF: Note w exists since $ju = 0 \bmod d$. Clearly $\beta w \bmod \beta f^* H^{i-1}(Y;\mathbb{Z}/d)$ is independent of the choice of w; we'll choose w so that $\beta w = -f^* v$ precisely. Replacing Y by the mapping cylinder, we may assume $f : X \hookrightarrow Y$ is an embedding. Represent u by a cocycle x that vanishes on X and let y be a cocycle representing v. Then $x = dy + \delta' z$ for some $(i-1)$-cochain z on Y, where δ' is the coboundary on cochains. By the definition of δ, we may then take $w = f^* \bar{z}$. Then by the definition of β we have

$$\beta w = \frac{1}{d}(\delta' f^* z) = \frac{1}{d}(f^*(x - dy)) = -f^* y. \qquad \square$$

Let β_n denote the Bockstein associated to $\mathbb{Z}/(q^n - 1)$, and let $r_n : \mathbb{Z}/q^n - 1 \to \mathbb{Z}/l$ denote the natural reduction.

(5.6) LEMMA. *There are classes* $\widetilde{e}_i \in H^{2i-1}(F\psi^q; \mathbb{Z}/q^i - 1)$ *such that* $r_i(\widetilde{e}_i) = e_i$ *and* $\beta_i\widetilde{e}_i = \widetilde{c}_i$.

PROOF: Consider once again the diagram used to define D_Γ, but using $\mathbb{Z}$-coefficients:

$$\begin{array}{ccccccc}
 & & 0 & \longrightarrow & H^{2i}[\Delta;\mathbb{Z}] & \xrightarrow{j'} & H^{2i}(BU \times BU;\mathbb{Z}) \xrightarrow{\Delta^*} \\
 & & \vdots & & \downarrow g & & \downarrow (1,\psi^q)^* \\
0 & \longrightarrow & H^{2i-1}(F\psi^q;\mathbb{Z}) & \longrightarrow & H^{2i}[\ \ ,\mathbb{Z}] & \xrightarrow[j]{} & H^{2i}(BU,\mathbb{Z})
\end{array}$$

Again we have $\Delta^*(\widetilde{c}_i \otimes 1 - 1 \otimes \widetilde{c}_i) = 0$, but now

$$(1, \psi^q)^*(\widetilde{c}_i \otimes 1 - 1 \otimes \widetilde{c}_i) = (1 - q^i)\widetilde{c}_i = -(q^i - 1)\widetilde{c}_i.$$

It follows that there is a unique class $\widetilde{e}_i \in H^{2i-1}(F\psi^q; \mathbb{Z}/q^i - 1)$ such that $\delta\widetilde{e}_i = gz$ where $j'z = \widetilde{c}_i \otimes 1 - 1 \otimes \widetilde{c}_i$. It is immediate from the definitions that $r_i(\widetilde{e}_i) = e_i$, and by Lemma 5.5 $\beta_i\widetilde{e}_i = \widetilde{c}_i$. □

Since $l \mid q-1$, $\beta_n : H^{2n-1}(C, \mathbb{Z}/(q^n - 1)) \to H^{2n}(C; \mathbb{Z})$ is an isomorphism for all n; both groups are isomorphic to $\mathbb{Z}/l^a$. Since $\varphi^*(\widetilde{c}_n)$ is zero for $n > 1$ and is a generator for $n = 1$, we conclude the same for $\varphi^*\widetilde{e}_n$ by (5.6). Hence $\varphi^* e_n$ is nonzero if and only if $n = 1$. □

6. Quillen's Detection Theorem

We assume throughout this section that $l \mid q-1$, and $v_l(q-1) = a$. All cohomology groups have $\mathbb{Z}/l$ coefficients. Our goal is to prove:

(4.8) THEOREM. *Let D_n denote the subgroup of diagonal matrices in $GL_n\mathbb{F}_q$. Then (if $l \mid q-1$), $i^* : H^*GL_n\mathbb{F}_q \to H^*D_n$ is injective.*

We have already noted that it is often necessary to separate the cases l odd and $l = 2$. Occasionally it is also necessary to isolate the case $l = 2$ and $q = 3 \bmod 4$. We call this the ***exceptional case***. The other case, l odd or $l = 2$ and $q = 1 \bmod 4$ is the ***typical case***. The following elementary lemma, which arises frequently in number theory and homotopy theory, is the main source of this distinction. Let $v_l(n)$ denote the number of powers of l in the integer n.

(6.1) LEMMA. *Let x be an integer such that $x = 1 \bmod l$, and let $a = v_l(x-1)$. Then if either l is odd or $l = 2$ and $a > 1$, $v_l(x^n - 1) = v_l(n) + a$.*

The problem when $l = 2$ and $a = 1$ is that $v_l(x^n - 1)$ can jump by more than $v_l(n)$. For example, $3 \neq 1 \bmod 4$ but $3^2 = 1 \bmod 8$.

The argument will read more smoothly if we focus on the typical case first. The modifications required for the exceptional case will be discussed later. So assume that we are in the typical case, and let N denote the subgroup of $GL_n\mathbb{F}_q$ generated by D_n and the permutation matrices Σ_n. Thus N is the semi-direct product $\Sigma_n \tilde{\times} D_n$, where Σ_n acts on $D_n = D_1^n$ by permuting the factors.

(6.2) LEMMA (TYPICAL CASE, $l \mid q-1$). *N contains an l-Sylow subgroup of $GL_n\mathbb{F}_q$. Hence (see A1) $H^*GL_n\mathbb{F}_q \to H^*N$ is injective.*

PROOF: We must show $v_l|N| = v_l|GL_n\mathbb{F}_q|$. We have $v_l|N| = v_l(n!) + na$ and $v_l|GL_n| = \sum_{i=1}^n v_l(q^i - 1)$. Since by assumption either l is odd or $a > 1$, $v_l(q^i - 1) = v_l(i) + a$ by (6.1) and the lemma follows. □

Now for any finite group G and family of subgroups $\{H_\alpha\}$, we say that $\{H_\alpha\}$ is a *detecting family*, or that $\{H_\alpha\}$ *detects* H^*G, if the evident map $H^*G \to \Pi H^*H_\alpha$ is injective. We call G *l^a-good* if the family consisting of abelian subgroups of exponent l^a is a detecting family.

(6.3) LEMMA. *N is l^a-good.*

Assuming this, we may now prove Theorem 4.5. Consider the inclusions

$$D_n \xrightarrow{i_1} N \xrightarrow{i_2} GL_n\mathbb{F}_q$$

with $i = i_2 i_1$, and suppose $i^*x = 0$, $x \in H^*GL_n\mathbb{F}_q$. Since $l^a \mid q-1$, any abelian subgroup A of exponent l^a in $GL_n\mathbb{F}_q$ is conjugate to a subgroup of D_n (see Appendix 8). Since inner automorphisms of a group induce the identity map on group cohomology, it follows that x restricts to zero on every such subgroup A. So $i_2^*x = 0$ by Lemma 6.3, and since N contains an l-Sylow subgroup $x = 0$ as desired.

REMARK: Note that this argument does ***not*** imply that i_1^* is injective (it isn't).

It remains to prove Lemma 6.3, and this is the hardest step. The key ingredient is Theorem (6.7) below, which will require a lengthy digression to explain.

Let X be a space with $\mathbb{Z}/l$-action. Then $H^*(E\mathbb{Z}/l \times_{\mathbb{Z}/l} X)$ is a module over $H^*B\mathbb{Z}/l$. If $y \in H^2B\mathbb{Z}/l$ is a generator, then for any graded $H^*B\mathbb{Z}/l$-module M we can form the localized module $y^{-1}M$. Finally, note that for any G-space X there is an equivariant inclusion $X^G \to X$, which induces a map $\eta : BG \times X^G = EG \times_G X^G \to EG \times_G X$.

(6.4) THEOREM ("LOCALIZATION AT THE FIXED POINT SET"). *Let X be a compact Hausdorff $\mathbb{Z}/l$-space. Then η induces an isomorphism*

$$y^{-1}H^*(E\mathbb{Z}/l\times_{\mathbb{Z}/l}X)\underset{\cong}{\to}y^{-1}H^*(B\mathbb{Z}/l\times X^{\mathbb{Z}/l}).$$

This is a special case of Theorem 4.2 of [Quillen 3] (the reference given in [Quillen 1] is a misprint). A proof of 6.4 (under an additional hypothesis on X that suffices for Theorem 6.7) is given in Appendix 6. However one can give a quick intuitive explanation of (6.4) as follows. There are two key observations. First, if the graded $H^*B\mathbb{Z}/l$-module M is bounded above (i.e. $M_k = 0$ for $k >> 0$) then obviously $y^{-1}M = 0$, for $y^{-1}M = \varinjlim(M \xrightarrow{y} M \xrightarrow{y} M \to \cdots)$. Second, if Y is a finite complex and a principal G-space, then $H^*(EG\times_G Y)$ is bounded above. Here "principal" means the action is free and $Y \to Y/G$ is a principal G-bundle, hence a covering map. It follows that the natural projection $EG\times_G Y \to Y/G$ is a fibre bundle with contractible fibre EG. Since Y/G is a finite complex, $H^*(EG\times_G Y)$ is bounded above as claimed. Returning to $G = \mathbb{Z}/l$, this shows that $y^{-1}H^*(E\mathbb{Z}/l\times_{\mathbb{Z}/l}Y)$ is zero if Y is a free $\mathbb{Z}/l$-space as above. Since $X - X^{\mathbb{Z}/l}$ is always a free $\mathbb{Z}/l$-space, this gives a strong plausibility argument for (6.4).

Now let X be any CW-complex and let $\mathbb{Z}/l$ act on X^l by permuting the factors. Then (with a fixed l understood) ΓX denotes the *l-th extended power* $E\mathbb{Z}/l\times_{\mathbb{Z}/l}X^l$.

(6.5) THEOREM (NAKAOKA; SEE A7). *There is a natural isomorphism* $H^*\Gamma X\cong H^*(\mathbb{Z}/l, H^*X^l)$.

The right-hand side of (6.5) is precisely the E_2-term of the Serre spectral sequence of the fibre sequence

$$X^l\xrightarrow{i}\Gamma X\xrightarrow{\pi}B\mathbb{Z}/l.$$

So we obtain as a *corollary*:

(6.6) COROLLARY. *The Serre spectral sequence of ΓX collapses.*

PROOF: It is enough to show the homology spectral sequence collapses, since the cohomology spectral sequence is dual. By a direct limit argument we then reduce to the case X a finite complex. If the spectral sequence does not collapse, then

$$\dim_{\mathbb{Z}/l}\Big(\bigoplus_{p+q=n}F_\infty^{p,q}\Big)<\Big(\dim_{\mathbb{Z}/l}\Big(\bigoplus_{p+q=n}H^p(\mathbb{Z}/l;H^qX^l)\Big)\Big)\text{ for some }n.$$

contradicting (6.5). □

REMARK: The reader may be wondering why (6.6) is a corollary of (6.5) rather than vice versa. Our proof of (6.5) *does* use an algebraic version of the Serre spectral sequence.

However (6.5) is much stronger than (6.6), for it asserts a natural isomorphism *on the nose* as opposed to merely an isomorphism of associated graded objects. This subtle distinction is not important for our present purposes, but it plays a role in other applications (see the last paragraph of A7).

Let us pause to compute $H^*(\mathbb{Z}/l; H^*X^l)$. To avoid confusing the permutation group $\mathbb{Z}/l$ with the coefficient ring $\mathbb{Z}/l$, we temporarily let G denote the former. Fix a basis $\{e_\alpha\}$ for H^*X, so $\otimes^l H^*X$ has basis $\{e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_l}\}$. The Künneth isomorphism $H^*X^l \cong \otimes^l H^*X$ is G-equivariant, and G acts on $\otimes^l H^*X$ by cyclically permuting the factors. So G permutes the basis vectors $\{e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_l}\}$, and since G has prime order every orbit of this action is either fixed or free. Hence $H^*X^l = M \oplus N$, where M is a free $\mathbb{Z}/l[G]$-module and N is a trivial module (spanned by the $e_\alpha \otimes e_\alpha \cdots \otimes e_\alpha$). Finally

$$H^*(G, H^*X^l) = H^*(G, M) \oplus H^*(G, N),$$

where $H^p(G; M) = 0$ for $p > 0$ and $H^*(G, N) = (H^*G) \otimes N$. In particular $H^*(G; M)$ is a trivial H^*G-module and $H^*(G; N)$ is a free H^*G-module. Note also that $H^*(G; M) = M^G$ injects into H^*X^l under the edge homomorphism $H^*\Gamma X \to H^*X^l$.

We now arrive at a key result.

(6.7) THEOREM [QUILLEN 1]. *The restriction map*

$$\eta^* \times i^* : H^*\Gamma X \to H^*(B\mathbb{Z}/l \times X) \times H^*(X^l)$$

is injective.

PROOF: As in (6.6), we easily reduce to the case X finite. Suppose $x \in H^*\Gamma X$ and $i^*x = 0$. Then by definition $x \in F^1$, where $F^0 \supseteq F^1 \supseteq F^2 \supseteq \cdots$ is the Serre filtration. On the other hand the above description of $H^*(\mathbb{Z}/l; H^*X^l)$ shows that the associated graded module of F^1—i.e. $\bigoplus_{p>0} E_\infty^{p,q}$—is a torsion-free $\mathbb{Z}/l[y]$-module. So F^1 is also torsion-free and $\eta^*|_{F^1}$ is injective by (6.4). Hence if $\eta^*x = 0 = i^*x$, $x = 0$. □

We will need the following special case. Let G be a subgroup of Σ_n, H any group. Then the *wreath product* $G \int H$ is the semidirect product $G \tilde{\times} H^n$, where G acts on H^n by permuting the factors. There is an inclusion $G \times H \to G \int H$ obtained by embedding H in H^n via the diagonal map. Now observe that if H is discrete then $EG \times_G (BH)^n$, where G acts on $(BH)^n$ by permuting the factors, is a $K(G \int H, 1)$ and hence a model for $B(G \int H)$. This yields the next corollary.

(6.8) COROLLARY. *If H is a finite group, $\{\mathbb{Z}/l \times H, H^l\}$ is a detecting family for $\mathbb{Z}/l \int H$.*

We return at last to (6.3).

(6.9) LEMMA. *If G and H are l^a-good, so is $G \times H$.*

PROOF: This is an exercise with the Künneth theorem (hint—consider first the case when one of the factors is itself abelian of exponent l^a).

(6.10) LEMMA. *If H is l^a-good, so is $\mathbb{Z}/l \int H$.*

PROOF: This follows at once from (6.8) and (6.9).

(6.11) LEMMA. *If H is l^a-good, so is $\Sigma_n \int H$.*

PROOF: We use induction on n, the case $n = 1$ being trivial. At the inductive step there are two cases.

Case 1: l is prime to n. Consider the subgroup $\Sigma_{n-1} \tilde{\times} H = (\Sigma_{n-1} \int H) \times H$ in $\Sigma_n \int H$. It has index n so the restriction map in cohomology is injective. Hence this case follows by inductive hypothesis and (6.9).

Case 2: $n = lm$. In this case Σ_n contains a subgroup $G = \Sigma_m \int \mathbb{Z}/l$ defined as follows. Divide the set $\{1, 2, \dots, n\}$ into m blocks of l elements each. The subgroup G' of all permutations which permute the blocks is $\Sigma_m \int \Sigma_l$, and we take G to be the subgroup which permutes the elements of an individual block cyclically. Now by a trivial calculation the index of G in Σ_n is prime to l, and this is the index of $G \int H$ in $\Sigma_n \int H$. So it is enough to show $G \int H$ is l^a-good. But $G \int H = \Sigma_m \int (\mathbb{Z}/l \int H)$. So $\mathbb{Z}/l \int H$ is l^a-good by (6.10) and $G \int H$ is l^a-good by inductive hypothesis. □

Since N is precisely $\Sigma_n \int \mathbb{F}_q^\times$, this completes the proof of (4.8) in the typical case. In the exceptional case—i.e. $l = 2$, $q = 3 \bmod 4$—some parts of the argument require a slight modification. The main point is that the definition of N given in (6.2) doesn't work. Instead we define $N = \Sigma_m \int GL_2\mathbb{F}_q$ if $n = 2m$ and $N = (\Sigma_m \int GL_2\mathbb{F}_q) \times \mathbb{Z}/2$ if $n = 2m+1$. Here when $n = 2m$, N is embedded as the group of (2×2) block matrices generated by Σ_m (permuting coordinates in pairs e_{2i-1}, e_{2i}) and the block diagonal matrices $\prod_{i=1}^m GL_2$. If $n = 2m + 1$ there is an extra $\mathbb{Z}/2$-factor on the diagonal in the (n, n) position.

(6.12) LEMMA (EXCEPTIONAL CASE). *N contains a 2-Sylow subgroup of $GL_n\mathbb{F}_q$.*

PROOF: Suppose $n = 2m$ and let $b = v_2(q^2 - 1)$. Then $v_2|N| = v_2(m!) + m(b + 1)$ and $v_2|GL_n\mathbb{F}_q| = \sum_{i=1}^n v_2(q^i - 1) = m + \sum_{j=1}^m v_2(q^{2j} - 1) = m + mb + v_2 m!$. The case $n = 2m+1$ follows immediately. □

(6.13) LEMMA. *N is 2-good[2]—i.e. H^*N is detected by subgroups of the form $(\mathbb{Z}/2)^k$.*

[2]This may seem 2-good to be true.

Assuming this, Theorem 4.8 follows from 6.13 exactly as before. The final lemma requires special treatment; its proof is deferred to §7.

(6.14) LEMMA (EXCEPTIONAL CASE). *The restriction map* $H^*GL_2\mathbb{F}_q \to H^*(\mathbb{Z}/2)^2$, *where* $(\mathbb{Z}/2)^2 \subset D_2$ *as* $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$, *is injective.*

In particular $GL_2\mathbb{F}_q$ is 2-good. Then (6.13) follows from (6.11). This completes the proof in the exceptional case. modulo (6.14).

7. Mod 2 **Cohomology of** $GL_2\mathbb{F}_q$

In this section, $l = 2$ and all homology and cohomology groups have $\mathbb{Z}/2$-coefficients. Our goal is to prove:

LEMMA 4.10. $e_k^2 = 0$ *in* $H_*BGL\mathbb{F}_q$.

LEMMA 6.14. $H^*GL_2\mathbb{F}_q \to H^*D_2$ *is injective.*

The proofs will be based on the following theorem.

THEOREM 7.1. $H^*SL_2\mathbb{F}_q \cong \mathbb{Z}/2[y] \otimes \mathbb{Z}/2\langle x\rangle$, *where* $|y| = 4$, $|x| = 3$.

Let V denote the second symmetric power of the unreduced homology $H_*B\mathbb{Z}/2^a$. Thus V has a basis consisting of b_ib_j $(0 \le i \le j)$, b_ie_j $(0 \le i, 1 \le j)$, e_ie_j $(1 \le i \le j)$, where $b_0 = 1 \in H_0$. Let W denote the subspace spanned by this basis with the e_i^2 removed. Since V is isomorphic as graded vector space to a polynomial algebra on two generators of degrees 1, 2, the Poincaré series of V is $P(V,t) = \frac{1}{(1-t)(1-t^2)}$. Hence W has Poincaré series

$$P(W,t) = \frac{1}{(1-t)(1-t^2)} - \frac{t^2}{1-t^4} = \frac{1+t^3}{(1-t)(1-t^4)}.$$

Now consider the commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{\ f\ } & H_*BGL_2\mathbb{F}_q & \xrightarrow{\ i_*\ } & H_*BGL\mathbb{F}_q \\ \downarrow{\scriptstyle \pi} & & & & \downarrow{\scriptstyle \theta_*} \\ V/\langle e_i^2\rangle & & \xrightarrow{\qquad\quad h \quad\qquad} & & H_*F\psi^q \end{array}$$

where the maps are as usual and the bottom arrow exists by (4.5). Let $U = H_*BGL_2\mathbb{F}_q$.

(7.2) LEMMA. $P(U,t) \le \frac{1+t^3}{(1-t)(1-t^4)}$

PROOF: Consider the Serre spectral sequence of the group extension

$$SL_2\mathbb{F}_q \to GL_2\mathbb{F}_q \to \mathbb{F}_q^\times .$$

Note that by Theorem (7.1), the action of $\mathbb{F}_q^\times$ on $H^*SL_2\mathbb{F}_q$ is necessarily trivial and hence (by 7.1 again) the E_2-term has Poincaré series exactly $\frac{1+t^3}{(1-t)(1-t^4)}$. Since $P(U,t) = P(E_\infty, t) \le P(E_2, t)$, the lemma follows. □

On the other hand the map h in the above diagram is injective by (4.6). This forces $f|_W$ injective and hence $P(U,t) \ge \frac{1+t^3}{(1-t)(1-t^4)}$. We conclude that equality holds in (7.2) and that f is onto. In particular this proves Lemma 6.14 (by dualizing).

Finally we prove Lemma 4.10. As we have just seen, $f|_W$ is an isomorphism. Hence if $f(e_i^2) \neq 0$ there is a nonzero $w \in W$ such that $f(w) = f(e_i^2)$. But then the above diagram shows $h\pi(w) = 0$, contradicting the fact that $h\pi|_W$ is injective.

This completes the proof of 4.10 and 6.14. It remains to prove Theorem 7.1. We begin by determining the 2-Sylow subgroups of $SL_2\mathbb{F}_q$.

(7.2) LEMMA. *The 2-Sylow subgroups of $SL_2\mathbb{F}_q$ are generalized quaternion groups Q_n, where $n+1 = v_2(q^2-1)$.*

PROOF: By definition Q_n has presentation

$$\{x, y : x^{2^n} = 1, y^4 = 1, yxy^{-1}x = 1\} \qquad (n \ge 2)$$

and fits into a nonsplit extension $\mathbb{Z}/2^n \to Q_n \to \mathbb{Z}/2$.

Case 1: $q = 1 \bmod 4$. Let H be the subgroup generated by matrices $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, with α a 2-primary root of unity, and the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then H is generalized quaternion and has the correct order, hence is a 2-Sylow subgroup.

Case 2: $q = 3 \bmod 4$. We first construct a 2-Sylow subgroup of $GL_2\mathbb{F}_q$. Identify $(\mathbb{F}_q)^2$ with the field $\mathbb{F}_{q^2} = \mathbb{F}_q(\sqrt{-1})$. This defines an inclusion $\mathbb{F}_{q^2}^\times \subset GL_2\mathbb{F}_q$, and in particular we get a cyclic subgroup A of order 2^{n+1} (note $n \ge 2$ here). Let H' denote the subgroup generated by A and the Frobenius σ. Then H' fits into a split extension

$$A \to H' \to \mathbb{Z}/2$$

in which σ generates the $\mathbb{Z}/2$ and σ acts on A by $\alpha \mapsto \alpha^q$. Since $n = v_2(q+1)$, it follows that $q = 2^n - 1 \bmod 2^{n+1}$, so we can write the action as $\alpha \mapsto \alpha^{2^n-1}$ (this group is usually called the *semi-dihedral group*). Now $\det\sigma = -1$ (use the basis $1, \sqrt{-1}$) and $\det\alpha = \alpha \cdot \sigma\alpha = \alpha^{q+1}$. Hence if α generates A, $\det\alpha = -1$ and the subgroup H of H' generated by $\beta = \alpha^2$ and $\tau = \alpha\sigma$ lies in $SL_2\mathbb{F}_q$. It is clear on inspection that H is generalized quaternion (e.g. $\tau\beta\tau^{-1} = \beta^{2^n-1} = \beta^{-1}$, etc.), and since H has the correct order 2^{n+1} this completes the proof. □

The cohomology of Q_n can be computed by purely algebraic means, using the Hochschild-Serre spectral sequence of a suitable extension. Here we'll compute it by a more topological method that is amusing and instructive. Recall that the 3-sphere S^3 can be identified with the group of unit quaternions. The group Q_n can be identified with the subgroup generated by the 2^n-th roots of unity in $S^1 \subset S^3$ and the element j. In particular we get a canonical map $BQ_n \xrightarrow{f} BS^3$.

(7.4) LEMMA. $H^*BS^3 \cong \mathbb{Z}/2[y]$, $|y| = 4$.

PROOF: This follows immediately from the Serre spectral sequence of the fibre sequence $S^3 \to ES^3 \to BS^3$ (in fact one could use $\mathbb{Z}$-coefficients here). □

By a general fact about classifying spaces, there is a fibre sequence

$$S^3/Q_n \to BQ_n \xrightarrow{f} BS^3.$$

We will compute H^*BQ_n from the associated Serre spectral sequence.

(7.5) LEMMA. $\widetilde{H}^k(S^3/Q_n) = \begin{cases} (\mathbb{Z}/2)^2 & k = 1, 2 \\ \mathbb{Z}/2 & k = 3 \\ 0 & k > 3 \end{cases}$

PROOF: Since $\pi_1(S^3/Q_n) = Q_n$, $H^1(S^3/Q_n) = \mathrm{Hom}(Q_n, \mathbb{Z}/2) = (\mathbb{Z}/2)^2$. The rest follows by Poincaré duality, since S^3/Q_n is 3-manifold. □

Thus the only possible nonzero differential is d_4, but:

(7.6) LEMMA. $f^*y \neq 0$.

PROOF: Consider the group inclusions $\mathbb{Z}/2 \subset Q_n \subset S^3 \subset O(4)$, where $\mathbb{Z}/2 \subset S^3$ as ± 1 and the last inclusion comes from the natural action of S^3 on $\mathbb{H} = \mathbb{R}^4$. Take classifying spaces to get

$$B\mathbb{Z}/2 \to BQ_n \to BS^3 \to BO(4).$$

The composite map $g : B\mathbb{Z}/2 \to BO(4)$ classifies $\lambda \oplus \lambda \oplus \lambda \oplus \lambda$, where λ is the canonical line bundle. Hence by the Whitney sum formula, $g^*(w_4) \neq 0$ (w_4 = 4-th universal Stiefel-Whitney class). Clearly this forces $f^*y \neq 0$. □

Thus the spectral sequence collapses. Identifying y with f^*y, and using the fact that the spectral sequence is a spectral sequence of H^*BS^3-modules, we arrive at the following result:

(7.7) PROPOSITION. *As* $\mathbb{F}_2[y]$*-module,* H^*BQ_n *is isomorphic to* $\mathbb{F}_2[y] \otimes V$*, where* $V = H^*(S^3/Q_n)$ *has Poincaré polynomial* $1 + 2t + 2t^2 + t^3$.

REMARK: It is possible to determine the complete ring structure, but we won't need that here. Note that in particular multiplication by y gives an isomorphism

$$H^k BQ_n \underset{\cong}{\rightarrow} H^{k+4} BQ_n, \qquad k \geq 0.$$

One says that Q_n has *periodic cohomology with period 4*.

(7.8) LEMMA. *Let* G *be any finite group with 2-Sylow subgroup a generalized quaternion group* Q_n*. Then* H^*G *is periodic with period four, and* $H^3G = H^4G = \mathbb{Z}/2$.

PROOF: By the stable element theorem (see A1), H^*G can be identified with the subring of stable elements in H^*Q_n. Now if K is any subgroup of Q_n, then K is either cyclic or generalized quaternion. In particular if K is nontrivial then $H^3K = H^4K = \mathbb{Z}/2$. Hence $y \in H^4Q_n$ and the generator $x \in H^3Q_n$ are necessarily stable. Finally we must check that multiplication by $y : H^nG \to H^{n+4}G$ is bijective for all $n \geq 0$. Since y is not a zero-divisor in H^*Q_n, the injectivity is clear. Similarly the definition of stable element shows that if yz is stable then so is z, proving the surjectivity. □

PROOF OF THEOREM 7.1: By Lemma 7.8, it only remains to show that $H^1SL_2\mathbb{F}_q = 0 = H^2SL_2\mathbb{F}_q$. If K is any field, an exercise in linear algebra shows that SL_2K is generated by the elements $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, $a \in K$. When $K = \mathbb{F}_q$ these elements have order p. Since p is odd, we conclude $H^1SL_2\mathbb{F}_q = \mathrm{Hom}(SL_2\mathbb{F}_q, \mathbb{Z}/2) = 0$.

To see that $H^2 = 0$, we consider the exact sequences

$$0 \to H^n(X;\mathbb{Z})/2 \xrightarrow{r} H^n(X;\mathbb{Z}/2) \xrightarrow{\delta} {}_2H^{n+1}(X;\mathbb{Z}) \to 0$$

arising from the long exact coefficient sequence associated to $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$. Here ${}_2A = \{a \in A : 2a = 0\}$ for an abelian group A. Now clearly $y \in \mathrm{Im}\, r$, since this is true in H^*BS^3. In particular the finite 2-group $H^4(SL_2\mathbb{F}_q;\mathbb{Z})$ is nonzero. Taking $n = 3$ in the above sequence then forces $H^3(SL_2\mathbb{F}_q;\mathbb{Z}) = 0$. Taking $n = 1$ forces $H^2(SL_2\mathbb{F}_q;\mathbb{Z}) = 0$, and then taking $n = 2$ shows $H^2SL_2\mathbb{F}_q = 0$ as desired.[3]

8. Further results

Our homology and cohomology calculations were aimed directly at the proof of the main theorem. In this section we derive some further results of Quillen on the homology

[3]The argument showing $H^2 = 0$ can be done more systematically using the "Bockstein spectral sequence."

and cohomology of $GL_n\mathbb{F}_q$. The proofs are occasionally much sketchier than in previous sections. All cohomology groups have $\mathbb{Z}/l$-coefficients until further notice.

(8.1) THEOREM.

(a) In the typical case

$$H^*BGL\mathbb{F}_q \cong \mathbb{Z}/l[c_r, c_{2r}, \dots] \otimes \mathbb{Z}/l\langle z_1, z_2, \dots \rangle$$

where $|z_i| = 2ri - 1$.

(b) In the exceptional case

$$H^*BGL\mathbb{F}_q \cong \mathbb{Z}/2[c_1, c_2, \dots, d_1, d_2, \dots]/I$$

where $|d_i| = 2i - 1$ *and* I *is the ideal generated by the relations* $d_i^2 = c_{2i-1} + c_1c_{2i-2} + \cdots + c_{i-1}c_i$.

NOTE: Various choices of the z_i in (a) will be discussed below. The d_i in (b) are unique.

PROOF:

(a) Suppose first that $l \mid q-1$, and consider the isomorphism of Hopf algebras

$$H_*BGL\mathbb{F}_q \cong S'(\widetilde{H}_*BC).$$

The coproduct is determined by the formulas $\Delta_*b = b \otimes b$, $\Delta_*e = e \otimes b + b \otimes e$. Here $b = 1 + b_1 + b_2 + \cdots$ as usual and $e = e_1 + e_2 + \cdots$. Note that our assertion $\Delta_*b = b \otimes b$ uses the fact that we are in the typical case—in the exceptional case the even-dimensional part of H_*BC is not a sub-coalgebra. As in §5 we have that the terms p_i of $\frac{e}{b}$ are primitive, and $p_i = e_i$ mod decomposables. Hence

$$H_*BGL\mathbb{F}_q = S(\widetilde{H}_{\text{ev}}BC) \otimes \mathbb{Z}/l\langle p_1, p_2, \dots \rangle$$

as Hopf algebras. Dualizing yields (a) when $l \mid q-1$. The general case follows easily by considering the map

$$S(\widetilde{H}_*BC_{q^r}) \to S(\widetilde{H}_*BC_q)$$

which clearly splits as algebras (and also as Hopf algebras, but we don't need that).

(b) Let S denote the Hopf algebra $S(\widetilde{H}_*\mathbb{R}P^\infty)$ and let $a_n \in H_n\mathbb{R}P^\infty$ be a generator, $n \geq 1$. Thus $S = \mathbb{Z}/2[a_1, a_2, \dots]$ with diagonal $\Delta a_n = \sum_{i+j=n} a_i \otimes a_j$ or $\Delta a = a \otimes a$, $a = 1 + a_1 + \cdots$. (In our earlier notation, $a_{2n} = b_n$ and $a_{2n-1} = e_n$.) Let I denote the ideal generated by the a_n^2, n odd. Then we have shown that $S/I \cong H_*BGL\mathbb{F}_q$ as Hopf algebras. Hence if $R \subset S^*$ is the sub-Holf algebra with annihilator I, we have an isomorphism $H^*BGL\mathbb{F}_q \cong R$. So it is now a purely algebraic problem to identify the subalgebra R.

Note $S^* \cong H^*BO = \mathbb{Z}/2[w_1, w_2, \ldots]$ with diagonal map $m^*w = w \otimes w$, where m is the multiplication $S \otimes S \to S$. Let $c_n = w_n^2$ and $d_n = w_{2n-1} + w_1 w_{2n-2} + \cdots + w_{n-1}w_n$. Let R' denote subalgebra of S^* generated by c_n, d_n, $n \geq 1$. Since $d_n^2 = c_{2n-1} + c_1 c_{2n-2} + \cdots + c_{n-1}c_n$ it is enough to show

(8.2) LEMMA. *$R = R'$. For this we first observe:*

(8.3) LEMMA. *I is a Hopf ideal—that is, an ideal such that $x \in I$ implies $\Delta x = \Sigma x_i' \otimes x_i''$ with either x_i' or x_i'' in I for each i.*

PROOF: This is obvious since $\Delta b_n^2 = \sum_{i+j=n} b_i^2 \otimes b_j^2$.

(8.4) LEMMA. *$m^*d_n = \sum_{i+j=n} c_i \otimes d_j + d_i \otimes c_j$. In particular R' is a sub-Hopf algebra of S^*.*

PROOF: We may formally define a Hopf algebra A over $\mathbb{Z}$ that reduces mod 2 to S^*. Thus $A = \mathbb{Z}[\widetilde{w}_1, \widetilde{w}_2, \ldots]$ with $\Delta\widetilde{w} = \widetilde{w} \otimes \widetilde{w}$, etc. Then $\widetilde{w}^2 = \widetilde{c} + 2\widetilde{V}$ where $c = \Sigma c_i$, etc. and $\widetilde{V}_{\text{odd}} = \widetilde{d}\ \widetilde{m}\widetilde{w}^2 = (\widetilde{c} + 2\widetilde{v}) \otimes (\widetilde{c} + 2\widetilde{v})$ and $\widetilde{m}\widetilde{V} = \widetilde{c} \otimes \widetilde{V} + \widetilde{V} \otimes \widetilde{c} + 2\widetilde{v} \otimes \widetilde{v}$. Reducing mod 2 and picking out the odd degree terms yields the desired result. □

(8.5) LEMMA. *Suppose S, S^* are any two dual Hopf algebras, $I \subset S$ a Hopf ideal and $R' \subset S^*$ a sub-Hopf algebra. Suppose $\langle x, y\rangle = 0$ for all x, y ranging over a set of ideal generators of I and algebra generators of R' respectively. Then $\langle x, y\rangle = 0$ for all $x \in I$, $y \in R'$.*

PROOF: This is a trivial exercise, exploiting formulae such as $\langle xy, z\rangle = \langle x \otimes y, m^*z\rangle$, etc.

PROOF OF 8.2: We have equalities of Poincaré series $f(R,t) = f(S/I, t) = f(R', t)$, where the first equality holds by definition of R and the second by direct inspection. So it is enough to show $R' \subset R$—i.e. I annihilates R'. By the preceeding lemmas we have only to check $\langle b_n^2, x\rangle = 0$ when n is odd and x is c_m or d_m for some m. This is trivial unless $x = c_n$, and then $\langle b_n^2, c_n\rangle = \langle b_n \otimes b_n, \Sigma w_i^2 \otimes w_i^2\rangle = 0$ since n odd. □

This completes the proof of 8.1.b.

There are two canonical choices of the exterior generators z_i in 8.1(a). Consider first the case $l \mid q-1$. Recall that the Chern classes c_n are dual to the elements b_1^n, with respect to the monomial basis in the b_i's. Similarly one may define d_n as dual to $b_1^{n-1}e_1$, with respect to the monomial basis in the b_i, e_i. In other words $\langle d_n, b_1^{n-1}e_1\rangle = 1$ and $\langle d_n, b^I e^J\rangle = 0$ for all other multi-indices I, J. Note d_n is indecomposable because

$\langle d_n, p_n\rangle = (-1)^{n-1} \neq 0$ (use the formula $p = \frac{e}{b}$ from the proof of (a)). Hence the d_n provide one choice of exterior generators. An easy and purely algebraic exercise then shows that the coproduct is given by $m^*d = c \otimes d + d \otimes c$. Hence if $f = \frac{d}{c}$, the elements f_n are primitive and indecomposable, providing another choice of generators. Finally if $l \not| q-1$ we can use the elements d_{rn} or f_{rn} obtained by restriction from the natural map $BGL\mathbb{F}_q \to BGL\mathbb{F}_{q^r}$.

We next turn to the calculation of $H^*GL_n\mathbb{F}_q$ for finite n. If l is odd, write $n = rm + \epsilon$ with $0 \le \epsilon < r$. Let $T_n = (C_q)^m$, embedded in $GL_n\mathbb{F}_q$ in the obvious way: $(C_q)^m \subset (GL_r\mathbb{F}_q)^m \subset GL_{rm}\mathbb{F}_q \subset GL_n\mathbb{F}_q$.

(8.6) THEOREM. *If l odd, $H^*GL_n\mathbb{F}_q \cong \mathbb{Z}/l[c_r, c_{2r}, \dots, c_{mr}] \otimes \mathbb{Z}/l\langle d_r, d_{2r}, \dots, d_{mr}\rangle$. Furthermore the restriction map $H^*GL_n\mathbb{F}_q \xrightarrow{i^*} H^*T_n$ is an isomorphism onto $(H^*T_n)^{\Sigma_m}$.*

PROOF: First of all i^* is injective. When $l \mid q-1$ this is Theorem 4.8. The general case is proved by an identical argument that will be left to the reader. Then the injection $i^* : H^*GL_n\mathbb{F}_q \to (H^*T_n)^{\Sigma_m}$ is surjective because the dual map is injective by Theorem 4.2. It follows that each inclusion $GL_n \subset GL_{n+1}$ induces a surjection on cohomology and hence the restriction $H^*GL\mathbb{F}_q \to H^*GL_n\mathbb{F}_q$ is surjective. On the other hand it is easily checked that c_{jr} and d_{jr} restrict to zero if $j > m$. Hence we have a surjective map

$$\mathbb{Z}/l[c_r, \dots, c_{mr}] \otimes \mathbb{Z}/l\langle d_r, \dots, d_{mr}\rangle \to H^*GL_n\mathbb{F}_q.$$

Comparing Poincaré series (using the isomorphism of graded vector spaces $H^*GL_n\mathbb{F}_q = (\otimes^m H_*C_q)/\Sigma_m$) shows this map is an isomorphism. □

(8.7) REMARK: Implicit in the proof of (8.2) is a partly topological proof of the following purely algebraic fact: let $S = \mathbb{Z}/l[y_1, \dots . y_m] \otimes \mathbb{Z}/l\langle x_1, \dots, x_m\rangle$, where $|y_i| = 2$, $|x_i| = 1$, and let Σ_m act on S by permuting the generators. Then

$$S^{\Sigma_m} = \mathbb{Z}/l[\sigma_1, \dots . \sigma_m] \otimes \mathbb{Z}/l\langle \tau_q, \dots, \tau_m\rangle$$

where σ_i is the i-th elementary symmetric function in the y's and τ_i is the symmetrization of the monomial $y_1 \cdots y_{m-1}x_m$. The point is that when $l \mid q-1$, $S = H^*T_n$ and $i^*d_i = \tau_i$. Of couse the argument can also be carried out purely algebraically.

(8.8) THEOREM. *Let $l = 2$. Then*
(a) If $q = 1 \bmod 4$, $H^*GL_n\mathbb{F}_q \cong \mathbb{Z}/2[c_1, \dots, c_n] \otimes \mathbb{Z}/2\langle d_1, \dots, d_n\rangle$
(b) If $q = 3 \bmod 4$, $H^*GL_n\mathbb{F}_q \cong \mathbb{Z}/2[c_1, \dots, c_n, d_1, \dots, d_n]/I$, where I is the ideal generated by the relations $d_i^2 = c_{2i-1} + c_1c_{2i-2} + \dots + c_{i-1}c_i$.

The proof is similar to that of (8.6). Here, however, the map $i^* : H^*GL_n\mathbb{F}_q \to (H^*T_n)^{\Sigma_n}$ is no longer onto. In case (a), i^* is an isomorphism onto $\mathbb{Z}/2[\sigma_1, \dots, \sigma_n] \otimes \mathbb{Z}/2\langle\tau_1, \dots, \tau_n\rangle$ as in Remark 8.7. But this is not the full ring of invariants—the problem is that when $l = 2$ elements such as $x_1 \cdots x_n$ are fixed by Σ_n. In case (b), i^* maps c_i to the i-th elementary symmetric function in the x_i^2 and maps d_i to the symmetrization of the monomial $x_1^2 \cdots x_{i-1}^2 x_i$. This latter assertion is easily proved using the coproduct formula $m^*d = c \otimes d + d \otimes c$.

Next we consider what happens when we pass to the algebraic closure $\bar{\mathbb{F}}_q$. Note that $\bar{\mathbb{F}}_q = \bigcup_n \mathbb{F}_{q^n}$, but the lattice of finite subfields $\mathbb{F}_{q^n}$ is ordered by divisibility: $\mathbb{F}_{q^m} \subset \mathbb{F}_{q^n}$ if $m \mid n$. To get a linear ordering we write $\bar{\mathbb{F}}_q = \cup \mathbb{F}_{q(n!)}$. Then $BGL\bar{\mathbb{F}}_q = \bigcup_n BGL\mathbb{F}_{q(n!)}$ and we can use the Milnor exact sequence to assemble the various Brauer lifts into a single map $\bar{\theta} : BGL\bar{\mathbb{F}}_q \to BU$.

(8.9) THEOREM. *$\bar{\theta}$ induces an isomorphism on* mod l *homology: $H_*BGL\bar{\mathbb{F}}_q \cong H_*BU$.*

PROOF: Clearly $H_*(\bar{\theta})$ is onto since it is onto when restricted to $BGL\mathbb{F}_{q^r}$. Since $H_*BGl\bar{\mathbb{F}}_q = \varinjlim H_*BGL\mathbb{F}_{q^n}$, to show $H_*(\bar{\theta})$ is injective it will be enough to show the odd generators e_i vanish in the limit. This in turn reduces to the fact that the inclusions $\mathbb{Z}/l^a \subset \mathbb{Z}/l^{a+1}$ induce the zero map on H_{odd}. □

Of course $\bar{\theta}$ is not an isomorphism on $H_*(-; R)$ for $R = \mathbb{Z}/p$ or $\mathbb{Q}$, since $\widetilde{H}_*(BU; R) \neq 0$.

Observe that we have now computed $H^*(GL_n\mathbb{F}_q; R)$ for $R = \mathbb{Q}$, $\mathbb{Z}/p$, or $\mathbb{Z}/l$ and $n \leq \infty$, with one exception: $H^*(GL_n\mathbb{F}_q; \mathbb{Z}/p)$ when $n < \infty$. When $n = \infty$, we showed this cohomology is trivial (§3). It also follows immediately from Lemma 3.3 that $\widetilde{H}^*(GL_n\bar{\mathbb{F}}_q; \mathbb{Z}/p) = 0$. However the computation of $H^*(GL_n\mathbb{F}_q; \mathbb{Z}/p)$ is an open problem. A few qualitative facts can be deduced from [Quillen 3], and some explicit computations have been done for $n \leq 3$. Beyond that, almost nothing is known.

A1. The transfer in group cohomology

Let G be a discrete group. Recall that if M is a $\mathbb{Z}G$-module, $H^n(G;M) \equiv \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M)$. So we may regard $H^*(G,M)$ as the right derived functors of the left exact functor $M \mapsto \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) = M^G$. If H is a subgroup of G, there is a natural restriction map $i^* : H^*(G;M) \to H^*(H;M)$ defined as follows: choose an injective resolution of $\mathbb{Z}G$-modules $M \to I^\cdot$, so by definition $H^*(G;M)$ is the cohomology of the chain complex $(I^\cdot)^G$. Now any injective $\mathbb{Z}G$-module is also injective as $\mathbb{Z}H$-module, because the restriction functor $(\mathbb{Z}G - \mathrm{Mod}) \to (\mathbb{Z}H - \mathrm{mod})$ has an exact left adjoint—namely, $N \mapsto \mathbb{Z}G \otimes_{\mathbb{Z}H} N$. So $I^\cdot$ is also an injective resolution as $\mathbb{Z}H$-modules, and i^* is simply the map on cohomology induced by the inclusion of chain complexes $(I^\cdot)^G \subset (I^\cdot)^H$.

Now suppose H has finite index in G. Then there is also a map τ, called the "transfer", which goes the "wrong" way: $H^*(H;M) \xrightarrow{\tau} H^*(G,M)$. To define it, let $g_1, \ldots, g_n$ be a set of left coset representatives of H in G. Then we have a natural transformation $\tau : M^H \to M^G$ given by $\tau(m) = \Sigma g_i m$. It is obvious that τ is independent of the choice of coset representatives. To extend this definition to the derived functors, we simply apply it to the terms of a resolution. Explicitly, τ is the map induced on cohomology by the map of cochain complexes $\tau : (I^\cdot)^H \to (I^\cdot)^G$.

This simple construction is surprisingly powerful.

(A1.1) PROPOSITION. *Suppose H has finite index in G. Then the composite*

$$H^*(G;M) \xrightarrow{i^*} H^*(H;M) \xrightarrow{\tau} H^*(G;M)$$

is multiplication by $[G : H]$.

PROOF: τi^* is the map on cohomology induced by $(I^\cdot)^G \subset (I^\cdot)^H \xrightarrow{\tau} (I^\cdot)^G$. But clearly for any G-module N, $N^G \subset N^H \xrightarrow{\tau} N^G$ is multiplication by $[G : H]$. □

(A1.2) COROLLARY. *Suppose G is finite and M is a an RG-module, where $|G|$ is a unit in R. Then $H^n(G,M) = 0$ for all $n > 0$.*

PROOF: By assumption multiplication by $|G|$ is an isomorphism (of $\mathbb{Z}G$-modules) from M to itself. It follows that multiplication by $|G|$ is an isomorphism on $H^n(G;M)$ for all n. On the other hand, taking H to be the trivial subgroup in (1.1) shows multiplication by $|G|$ is zero on H^n when $n > 0$. □

EXAMPLES: (a) For any finite group G, $\widetilde{H}^*(G;\mathbb{Q}) = 0$. (b) If G is finite and $p \nmid |G|$, $\widetilde{H}^*(G;\mathbb{Z}/p) = 0$.

This corollary can be generalized:

A1.3) COROLLARY. *Suppose H has finite index d in G and M is a $\mathbb{Z}\left[\frac{1}{d}\right]$ G-module. Then $i^* : H^*(G,M) \to H^*(H;M)$ is split injective.*

PROOF: The splitting map is $\frac{1}{d}\cdot\tau$. □

EXAMPLE: Suppose G is finite and H is a subgroup which contains a p-Sylow subgroup of G. Then $i^* : H^*(G;\mathbb{Z}/p) \to H^*(H;\mathbb{Z}/p)$ is injective.

Of course there is also a transfer in homology. Recall that $H_*(G;M) = \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z};M)$. These are the left-derived functors of $M \mapsto \mathbb{Z}\otimes_{\mathbb{Z}G} M = M_G$. Here M_G is the group of coinvariants $M/\langle gm - m\rangle$. Using a projective resolution $P. \to M$, we get maps $i_* : H_*(H;M) \to H_*(G;M)$ and $\tau : H_*(G;M) \to H_*(H;M)$ as before.

Reflecting on A1.1, it is natural to ask for a formula for $i^*\tau$ as well. This is the "double coset formula". It looks somewhat complicated at first glance, but in fact it is easy to prove and easy to use. Write $i(G,H)$ for the restriction $H^*(G;M) \to H^*(H;M)$ and $\tau(H,G)$ for the transfer. If $x \in G$ write c_x for the map $H^*(H;M) \to H^*(xHx^{-1},M)$ induced by conjugation by x^{-1} $xHx^{-1} \to H$. On the level of cochain complexes, c_x is represented by multiplication by $x : I^H \to I^{xHx^{-1}}$. Finally, if K is another subgroup of G, the $K-H$ ***double cosets*** in G are the orbits of the left K-action on G/H. These double cosets can be expressed in a more symmetrical form as $K\backslash G/H$ (mod out by H on the right, K on the left), but for present purposes the first description is more relevant.

(A1.4) PROPOSITION (DOUBLE COSET FORMULA).

$$i(G,K)\circ\tau(H,G) = \Sigma_x \tau(K\cap xHx^{-1},K)\circ i(xHx^{-1},K\cap xHx^{-1})\circ c_x$$

where x ranges over a set of representatives of the K-orbits of G/H.

PROOF: We work on the cochain level. The lefthand side is just $I^H \underset{\tau}{\to} I^G \subset I^K$, where $\tau(\alpha) = \Sigma_{g\in G/H} g\alpha$. A typical term on the right is given by the composite

$$I^H \underset{x}{\to} I^{xHx^{-1}} \subset I^{K\cap xHx^{-1}} \underset{\tau_x}{\to} I^K$$

where $\tau_x(\alpha) = \Sigma_k k\alpha$ and the sum is over $k \in K/(K\cap xHx^{-1})$. Hence the composite is given by $\alpha \mapsto \Sigma(kx)\alpha$. But $K\cap xHx^{-1}$ is precisely the isotropy group in K of xH. So in this last sum kx is ranging over the elements of the orbit KxH/H. Since G/H is the disjoint union of the K-orbits, the result follows. □

As an application, fix a prime p and suppose $[G:H]$ prime to p. By A1.1 the restriction map $i^* : H^*(G;\mathbb{Z}/p) \to H^*(H;\mathbb{Z}/p)$ is injective and we wish to describe the image. Call $\alpha \in H^*(H;\mathbb{Z}/p)$ ***stable*** if for all $x \in G$, $i(H,H\cap xHx^{-1})\alpha = i(xHx^{-1},H\cap xHx^{-1})\circ c_x\alpha$.

COROLLARY (STABLE ELEMENT THEOREM). *Suppose $p \not| [G:H]$. Then $\alpha \in H^*(H;\mathbb{Z}/p)$ is in the image of i^* if and only if α is stable.*

PROOF: Suppose α is stable and let $\beta = \tau(H,G)\alpha$. Take $H = K$ in the double coset formula. Then

$$\begin{aligned} i^*\beta &= \sum_x \tau(H \cap xHx^{-1}, H) \circ i(H, H \cap xHx^{-1})\alpha \\ &= \sum_x [H : H \cap xHx^{-1}]\alpha \\ &= [G:H]\alpha \end{aligned}$$

Hence $\alpha = i^*([G:H]^{-1}\beta)$. The converse is a trivial consequence of the fact that conjugation by x induces the identity map on H^*G. □

A2. The transfer in algebraic K-theory

Any ring homomorphism $R \xrightarrow{\varphi} S$ induces a map on algebraic K-theory $K_nR \to K_nS$, $n \geq 0$. If $n = 0$ this map takes a finitely-generated projective R-module M to $S \otimes_R M$. For $n > 0$, φ induces a group homomorphism $GLR \to GLS$, hence a map of spaces $BGLR \to BGLS$ and hence a homotopy class of maps $\varphi^+ : BGLR^+ \to BGLS^+$. Then $K_nR \to K_nS$ is just the induced map on homotopy groups, $n > 0$.

If $R \subset S$ as a subring, then under certain conditions one can define a "wrong-way" homomorphism $K_*S \to K_*R$. This map is called the "transfer" because of a strong analogy with the transfer in group homology (for example, compare the homology version of A1.1 with A2.3 below).

Our goal here is only to give a quick and cheap construction of the transfer, sufficient for the purposes at hand. In particular we will only discuss the transfer for fields; we will leave it to the interested reader to imagine possible generalizations.

So suppose $F \subset E$ is a *finite* field extension, of degree d. Then it is obvious how to define a transfer $\tau : K_0E \to K_0F$—if V is a finite-dimensional E-vector space, set $\tau([V]) = [V_F]$, where V_F is V regarded as F-vector space. It is equally obvious that $\tau \circ i_* : K_0F \to K_0F$ is multiplication by d.

A moment's reflection suggests an extension to higher K-groups. Fixing a basis for E over F determines compatible homomorphisms $GL_nE \to GL_{dn}F$ and hence a homomorphism $GLE \to GLF$. This yields the *transfer* $\tau : BGLE^+ \to BGLF^+$. Of course we want to show the homotopy class of τ is independent of the choice of basis made above. This follows immediately from a special case of a lemma from [Loday]:

(A2.1) LEMMA. *Let $u : \mathbb{N} \to \mathbb{N}$ be any permutation (which we regard as a permutation of the standard basis vectors in R^{∞}). Then conjugation by u defines an automorphism of GLR, and the induced map $u^{+} : BGLR^{+} \to BGLR^{+}$ is homotopic to the identity.*

For any homotopy associative H-space X, let $[d]$ denote the d-th power map:

$$X \xrightarrow{\Delta} \prod_{i=1}^{d} X \xrightarrow{m} X.$$

(A2.2) PROPOSITION. *The composite $BGLF^{+} \xrightarrow{i} BGLE^{+} \xrightarrow{\tau} BGLF^{+}$ is homotopic to $[d]$.*

PROOF: τi is induced by a certain group homomorphism $GLF \to GLE \to GLF$, described above. On the other hand $[d]$ is also induced by a group homomorphism. Namely, one can use the composite $GLF \xrightarrow{\Delta} (GLF)^{d} \xrightarrow{\eta} GLF$ where η may be defined using any partition of the basis $\mathbb{N}$ into d infinite subsets (using 2.1 again). For a suitable choice of partition, the two homomorphisms are actually equal. □

(A2.3) COROLLARY. *$\pi_{*}(\tau i)$ is multiplication by d.*

PROOF: For any H-space X, the group structure on $\pi_{*}X$ inherited from the H-space structure on X agrees with the usual one. So $\pi_{*}[d]$ is multiplication by d.

(A2.4) PROPOSITION. *The transfer is a map of H-spaces.*

PROOF: We must show the diagram

$$\begin{array}{ccc} BGLE^{+} \times BGLE^{+} & \xrightarrow{\tau \times \tau} & BGLF^{+} \times BGLF^{+} \\ \downarrow m_E & & \downarrow m_F \\ BGLE^{+} & \xrightarrow[\tau]{} & BGLF^{+} \end{array}$$

is homotopy commutative. But both $m_F \circ (\tau \times \tau)$ and $\tau \circ m_E$ are induced by group homomorphisms $GLE \times GLE \to GLF$. These homomorphisms differ by an infinite permutation as in (2.1). □

For completeness we record the behaviour of $i\tau$.

(A2.5) PROPOSITION. *Suppose E is Galois over F with group G. Then the composite*

$$BGLE^{+} \xrightarrow{\tau} BGLF^{+} \xrightarrow{i} BGLE^{+}$$

is homotopic to $\sum_{\sigma \in G} \sigma$ (sum in the H-space structure of $BGLE^{+}$).

It follows immediately from A8.2 that A2.5 holds after restriction to any BGL_nE, $n < \infty$. The complete result requires the technique of Loday's lemma A2.1.

A3. Milnor's $\lim^1$ sequence.

Consider an inverse system of abelian groups

$$A_0 \xleftarrow{f_1} A_1 \xleftarrow{f_2} A_2 \leftarrow \cdots$$

Let $T : \Pi A_i \to \Pi A_i$ denote the "shift" map

$$(a_0, a_1, a_2, \dots) \mapsto (f_1(a_1), f_2(a_2), \dots)$$

Then $\mathrm{Ker}(1-T)$ is by definition $\varprojlim A_i$, which we will write as $\lim^0 A_i$. We define $\lim^1$ by $\lim^1 A_i = \mathrm{Coker}(1-T)$.

The notation is justified by the fact that $\lim^0$ is a left exact functor {inverse systems} $\to$ {abelian groups} and $\lim^1$ is the first right derived functor (it turns out that the higher derived functors are always zero). However we will not use this viewpoint here; we simply define $\lim^1$ as above.

(A3.1) PROPOSITION. *Suppose $0 \to A_i \to B_i \to C_i \to 0$ is a short exact sequence of inverse systems. Then there is an exact sequence*

$$\begin{aligned} 0 \to \lim{}^0 A_i &\to \lim{}^0 B_i \to \lim{}^0 C_i \\ &\to \lim{}^1 A_i \to \lim{}^1 B_i \to \lim{}^1 C_i \to 0 \end{aligned}$$

PROOF: Map the exact sequence

$$0 \to \Pi A_i \to \Pi B_i \to \Pi C_i \to 0$$

to itself by $1-T$ and apply the snake lemma. □

The next theorem is Milnor's exact sequence. It can be generalized somewhat, but we will leave such generalizations to the interested reader.

(A3.2) THEOREM. *Let X be a pointed CW-complex, $X_0 \underset{i_1}{\subset} X_1 \underset{i_2}{\subset} \cdots \subset X = \cup X_n$ a filtration by pointed subcomplexes. Let Y be a homotopy associative H-space. Then there is a natural exact sequence of groups*

$$0 \to \lim{}^1[\Sigma X_n, Y]_* \to [X, Y]_* \to \lim{}^0[X_n, Y]_* \to 0$$

NOTE: $[\ ,\ ]_*$ denotes pointed maps, and $[\Sigma X_n, Y]_*$ is an abelian group so the $\lim^1$ makes sense. In all examples considered in this paper, Y is homotopy commutative so all three groups are abelian.

The proof is simple and may be sketched as follows. Let X' denote the reduced mapping telescope of the sequence X_n. Thus $X' = \coprod(X_n \times I/ * \times I)/\sim$ where the equivalence relation is $(x_n, 1) \sim (i_{n+1}(x_n), 0)$. One can show that X' is naturally homotopy equivalent to X, so we may replace X by X'. There is an obvious inclusion $\bigvee X_n \to T$ whose cofibre is $\bigvee \Sigma X_n$. So we get a cofibre sequence

$$\bigvee X_n \to T \to \bigvee \Sigma X_n \to \bigvee \Sigma X_n$$

and then applying $[\ , Y]_*$ we get an exact sequence

$$\prod [X_n, Y]_* \underset{\varphi}{\leftarrow} [X', Y]_* \leftarrow \prod [\Sigma X_n, Y]_* \underset{\psi}{\leftarrow} \prod [\Sigma X_n, Y]_*$$

One can check that Im φ is precisely $\lim^0 [X_n, Y]_*$, and that ψ is precisely $1 - T$, proving the theorem.

It can be very difficult to determine whether the $\lim^1$-term in Milnor's sequence is zero or not (unless the groups $[\Sigma X_n, Y]$ are all zero!). However there is one criterion that is easy to apply in practice. It is based on:

(A3.3) PROPOSITION. *Suppose A_i is an inverse system of compact Hausdorff topological groups, and the maps f_i are continuous. Then* $\lim^1 A_i = 0$.

PROOF: By definition $\lim^1 = 0$ if and only if $1 - T$ is onto. This amounts to the assertion that for any $\{a_i\} \in \prod A_i$, the infinite sequence of equations

$$a_0 = x_0 - f_1(x_1), \ldots, a_n = x_n - f_{n+1}(x_{n+1}), \ldots$$

can always be solved for the x_i. Now clearly we can always solve the first $(n+1)$ equations by setting $x_{n+1} = 0$ and working backwards. In other words (for any inverse system) the composites $\prod_{i=0}^{\infty} A_i \xrightarrow{1-T} \prod_{i=0}^{\infty} A_i \to \prod_{i=0}^{n} A_i$ are onto for all n. If we give $\prod A_i$ the product topology, this trivially implies that Im$(1 - T)$ is dense. But by Tychonoff's theorem, $\prod_{i=1}^{\infty} A_i$ is compact. So Im$(1 - T)$ is both closed and dense, hence $1 - T$ is onto. □

Note the proposition applies to any inverse system of *finite* abelian groups (with the discrete topology).

We will use the following hypotheses:

(*) X is a connected CW-complex with finitely many cells in each dimension.

(**) Y is a connected homotopy associative and commutative H-space with finitely-generated homotopy groups.

(A3.4) PROPOSITION. *Suppose X satisfies (*) and Y satisfies (**). Assume in addition that either (a) $H_n(X;\mathbb{Z})$ is finite for all $n > 0$ or (b) $\pi_n Y$ is finite for all n. Then $[X, Y]$ is profinite, and the profinite topology is natural in both variables.*

PROOF: Suppose (a) holds.

LEMMA. *If X is a finite complex, $[X, Y]$ is finite.*

PROOF: For $n \geq 1$ define subcomplexes X_n, with $X^n \subset X_n \subset X^{n+1}$ (X^n = n-skeleton) and $\widetilde{H}_* X_n$ finite, as follows: $H_k X^n$ is finite for $k < n$ but $H_n X^n$ is free abelian of finite rank d, say. Since $H_n X$ is finite, the boundary map $C_{n+1}X \to C_n X$ of the cellular chain complex must have rank exactly d. Hence by elementary linear algebra there are $(n+1)$-cells $e_{\alpha_1}, \dots, e_{\alpha_d}$ such that $X_n = X^n \cup e_{\alpha_1} \cup \cdots \cup e_{\alpha_d}$ has the desired property.

Now the cofibre sequences $X_n \to X_{n+1} \to X_{n+1}/X_n$ yield exact sequences

$$[X_n, Y] \leftarrow [X_{n+1}, Y] \leftarrow [X_{n+1}/X_n, Y]$$

so by induction we reduce to showing $[X_{n+1}/X_n, Y]$ is finite. Since X_{n+1}/X_n has only $(n+1)$-cells and $(n+2)$-cells (plus the zero-cell), there is a cofibre sequence

$$\bigvee^{d} S^{n+1} \xrightarrow{f} \bigvee^{d} S^{n+12} \to X_{n+1}/X_n \to \bigvee^{d+1} S^{n+2} \xrightarrow{g} \bigvee^{d+1} S^{n+2}$$

with $g = \Sigma f$. To see this, note the long exact homology sequence of the pair (X_{n+1}, X_n) shows $\widetilde{H}_*(X_{n+1}/X_n)$ is torsion, which forces the number of $(n+1)$-cells. Furthermore $H_{n+1}(f)$ is given by a $(d \times d)$-matrix A over $\mathbb{Z}$ of rank d. Finally $[\bigvee^d S^{n+1}, Y] = \oplus^d \pi_{n+1} Y$ and $f^* : [\bigvee^d S^{n+1}, Y] \to [\bigvee^d S^{n+1}, Y]$ is also given by the matrix A (and similarly for Σf). Hence (using (**)) $\operatorname{Ker} f^*$ and $\operatorname{Coker}(\Sigma f)^*$ are finite, which completes the proof of the lemma. □

Returning to an arbitrary X satisfying (a), the lemma and its proof show we can always filter X by finite subcomplexes X_n such that $[X_n, Y]$ is finite for all n. Similarly $[\Sigma X_n, Y]$ is finite for all n, so $\lim^1 [\Sigma X_n, Y] = 0$ by (A3.3). Hence $[X, Y] = \lim^0 [X_n, Y]$ is profinite.

If X' is another such complex and $f : X' \to X$ a map, the naturality assertion is that $f^* : [X, Y] \to [X', Y]$ is continuous with respect to the profinite topologies obtained from filtrations X_n, X'_n. To see this, it is enough to check that $(f^*)^{-1}(U)$ is open when U ranges over a neighborhood base at zero in $[X', Y]$. But such a base is given by the subgroups $U'_n = \{g \in [X', Y] : g|_{X'_n} = 0\}$, and by the cellular approximation theorem we can assume $f(X'_n) \subset X_{n+1}$. Hence if $h \in (f^*)^{-1} U'_n$—i.e., $hf|_{X'_n} = 0$—the coset neighborhood $h + U_{n+1}$

of h is contained in $(f^*)^{-1}U'_n$. [Note that taking f to be the identity map of X shows that the profinite topology obtained is independent of the choice of filtration X_n]. The naturality in Y is similar.

The proof under assumption (b) is similar but much easier; the point is that in the lemma we can just take $X_n = X^n$. □

REMARK: In most cases of interest Y is an infinite loop space. For such Y, the reader who knows the Atiyah-Hirzebruch spectral sequence can prove the lemma instantly.

EXAMPLE: $X = BG$, G finite, $Y = BU$ or U. To see that BG satisfies (*), consider the Milnor construction for EG (using the weak topology on EG). It is the union of closed subspaces $(EG)_n = n$-fold join $G * G \cdots * G$. From this it is easy to show that EG is a free G-CW-complex (see A6) with n-skeleton $(EG)_n$. Hence $BG = EG/G$ is a CW-complex satisfying (*). Of course BG satisfies condition (a) of (A3.4) by A1 (example (a) following A1.2).

(A3.5) COROLLARY. *Suppose X is a CW-complex filtered by subcomplexes X_n such that each X_n satisfies (*). Suppose Y satisfies (**) and either each X_n has finite homology groups or $\pi_n Y$ is finite for all n. Then $\lim^1[X_n, Y] = 0$ and $[X, Y]$ is profinite.*

PROOF: By (A3.4) each $[X_n, Y]$ is a compact topological group and the restriction maps are continuous, so we can apply (A3.3). □

EXAMPLE: Let G be a discrete group, filtered by finite subgroups G_n (e.g. $GL\mathbb{F}_q$). Take $X = BG$, $Y = U$ or BU.

A4. Localization of spaces.

Let S be a set of primes, and let $S^{-1}\mathbb{Z}$ denote the subring of $\mathbb{Q}$ given by $\{\frac{a}{b} : p \mid b \Rightarrow p \in S\}$. For any abelian group A, $S^{-1}A \equiv A \otimes S^{-1}\mathbb{Z}$ will be called the *localization of A away from S*. Note the special cases $S = \{p\}$ (localization away from p), $S = \{\text{primes } q \neq p\}$ (localization at p, also denoted $A_{(p)}$) and $S =$ all primes (rationalization—$S^{-1}\mathbb{Z} = \mathbb{Q}$). It is easy to check that $S^{-1}(-)$ is an exact functor and that $S^{-1}A = 0$ if and only if A is a torsion group having p-primary torsion only for the primes in S. It is also easy to see:

(A4.1) $$S^{-1}A = \varinjlim(A \xrightarrow{f_1} A \xrightarrow{f_2} A \to \cdots),$$

where if S is finite $f_n = \prod_{p \in S} p$ for all n and if $S = \{p_1, p_2, \dots\}$ is infinite $f_n = p_1 \cdots p_n$.

It is often convenient to localize homology or homotopy groups in this way. But even better is to somehow localize the underlying space (!); this idea is due to Dennis

Sullivan. Although Sullivan's localization functor has been refined and generalized over the years, we will present it in essentially its original form. *All spaces will be assumed to be path-connected simple CW-spaces.* For the rest of this section, a set of primes S as above is fixed. A map $f : X \to Y$ is said to *localize homotopy* (resp. *localize homology*) if $\pi_n(f)$ induces an isomorphism $S^{-1}\pi_n X \to \pi_n Y$ (resp. $H_n(f)$ induces an isomorphism $S^{-1}H_n(X;\mathbb{Z}) \to H_n(Y,\mathbb{Z})$). We say that X is *local* if the identity map localizes homotopy—in other words, $\pi_* X$ is already a module over $S^{-1}\mathbb{Z}$.

(A4.2) THEOREM (SULLIVAN). *The following are equivalent for a map $f : X \to Y$:*

(a) f localizes homotopy

(b) f localizes homology

(c) f has the universal property: Given any map $g : X \to Z$ with Z local, there is a map h, unique up to homotopy, such that the following diagram is homotopy commutative:

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle g}\big\downarrow & \swarrow_{h} & \\ Z & & \end{array}$$

In particular Y is unique up to a canonical equivalence.

(A4.3) THEOREM (SULLIVAN). *For any simple X as above, there exists a space Y and a map $f : X \to Y$ satisfying the conditions of A4.2.*

We call Y the *localization of X away from S*, denoted $S^{-1}X$. If S is the complement of a single prime p, we also write this as $X_{(p)}$—the localization of X at p. Property A4.2c shows $S^{-1}X$ is unique up to a canonical homotopy equivalence. If X is a homotopy associative H-space, there is a simple construction of $S^{-1}X$, imitating A4.1: take $S^{-1}X$ to be the mapping telescope of the sequence

$$X \xrightarrow{f_1} X \xrightarrow{f_2} X \to \cdots$$

with f_n as in A4.1, using the H-space structure. It's easy to check that the natural map into the telescope localizes homotopy. Similarly if $X = \Sigma X'$ with X' path-connected, one gets an easy construction by taking a mapping telescope as above, using the co-multiplication to define the f_n. In this case it is easy to check that f localizes homology.

Localization is a functor on the appropriate homotopy category, and has many pleasant properties. For example, it is not hard to see (using only A4.2, 3) that the functor $S^{-1}X$ commutes with fibre sequences and loop spaces. The precise statement and proof is left to the reader. Another sample property, used in these notes, is:

(A4.4) PROPOSITION. *If $p \notin S$. $X \to S^{-1}X$ induces an isomorphism on* mod p *homology.*

The proof is a trivial exercise.

A5. Graded algebras, co-algebras and Hopf algebras

Let K be a field, $V = \bigoplus_{n \geq 0} V_n$ a graded vector space over K. We assume throughout this section that all such V have *finite type*—i.e. each V_n is finite-dimensional. Then the dual $V^* = \oplus V_n^*$ also has finite type. $V^{**} = V$, and $(V \otimes W)^* = V^* \otimes W^*$. This also allows us to define the Poincaré series $f(V, t) = \Sigma(\dim V_n)t^n$. This formal power series is an extremely useful counting device. It has the obvious properties $f(V \oplus W, t) = f(V, t) + f(W, t)$ and $f(V \otimes W, t) = f(V, t)f(W, t)$.

Now suppose A is a graded algebra over K. We will always assume A is *connected*—i.e. $A_0 = K$. We say that A is *commutative* if $ba = (-1)^{|a||b|}ab$ for all homogeneous $a, b \in A$, where $|\ \ |$ denotes the grade. The *suspension* ΣV is defined by $(\Sigma V)_n = V_{n-1}$. We set $V^{\text{ev}} = \bigoplus_{n \text{ even}} V_n$, $V^{\text{odd}} = \bigoplus_{n \text{ odd}} V_n$, and $\bar{V} = \bigoplus_{n > 0} V_n$.

(A5.1) EXAMPLE: Let V be a graded vector space, with $V_0 = 0$. The *tensor algebra* $T(V)$ is defined by $T(V) = \bigoplus_{i \geq 0} \otimes^i V$. with multiplication given by juxtaposition of tensors. Note $T(V)$ is in fact bigraded—it inherits an "internal" grading from V but also has a grading by the "length" i. However for our purposes the internal grade is the primary one; thus $|v_1 \otimes \cdots \otimes v_i| = |v_1| + \cdots + |v_i|$.

(A5.2) EXAMPLE: The *symmetric algebra* $S(V)$ is the quotient of $T(V)$ obtained by factoring out the two-sided ideal generated by all $ab - (-1)^{|a||b|}ba$; a, b homogeneous elements of V. It is easy to see that $S(V \oplus W) = S(V) \otimes S(W)$; in particular $S(V) = S(V^{\text{ev}}) \otimes S(V^{\text{odd}})$, respectively then $S(V^{\text{ev}})$ is just the polynomial algebra $K[a_1, a_2, \ldots]$ and, if char $K \neq 2$, $S(V^{\text{odd}})$ is just the exterior algebra $K\langle b_1, b_2, \ldots \rangle$. If char $K = 2$, $S(V)$ is a polynomial algebra.

(A5.3) EXAMPLE: The *strict symmetric algebra* $S'(V)$ is the quotient of $S(V)$ obtained by factoring out the ideal generated by all a^2 with $|a|$ odd, or equivalently just such elements with $a \in V$. If char $K \neq 2$ then $S(V) = S'(V)$, so $S'(V)$ is only of interest when char $K = 2$.

All three examples are functorial in V; it is often advisable to avoid choosing a basis as in the second example.

(A5.4) EXAMPLE: Let V be a vector space over $\mathbb{F}_p$ of dimension n. Grade V by declaring all elements to have grade one. Regarding V is an abelian group, we have the classifying space BV with $H_1(BV;\mathbb{Z}/p)$ naturally isomorphic to V. Hence $H^1(BV;\mathbb{Z}/P)$ is naturally isomorphic to V^*. Then $H^*(BV;\mathbb{Z}/p)$ is naturally isomorphic to $S(V^*)$ for $p = 2$ and to $S(V^* \oplus \Sigma V^*)$ for p odd. In the latter case ΣV^* is identified with $\beta V^* \subseteq H^2 V$, where β is the Bockstein operation (i.e. the coboundary in the coefficient sequence associated to $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$). The naturality implies in particular that the action of the general linear group $GL(V)$ on BV induces the evident algebraic action on $S(V^*)$ or $S(V^* \oplus \Sigma V^*)$.

(A5.5) REMARK: The functors $T(-)$, $S(-)$, and $S'(-)$ are all left adjoints: we have for $F = T, S, S'$ that $\mathrm{Hom}(F(V), B) = \mathrm{Hom}_K(V, B)$ where the first Hom is in the category of graded K-algebras, commutative graded K-algebras, or strictly commutative graded K-algebras, respectively, and the second Hom is in the category of graded vector spaces.

If A is a graded K-algebra then $\bar{A}$ is a two-sided ideal. If A is connected, as we are assuming, the elements of $\bar{A}^2$ are called the *decomposable* elements. Thus $a \in A$ is decomposable if $a \in \Sigma a_i b_i$ with $|a_i|$, $|b_i| > 0$. The space of *indecomposable* elements is $QA = \bar{A}/\bar{A}^2$. Note the terminology is somewhat misleading since QA is a *quotient* of $\bar{A}$, *not* a subspace. Since we are working over a field, the projection $\bar{A} \to QA$ always splits—but there are usually many choices of such a splitting. In examples (1)–(3) above there is a canonical (but certainly not unique) choice: for example, the composite $V \to \overline{S(V)} \to QS(V)$ is an isomorphism.

Now observe that the notion of a K-algebra A can be defined entirely in terms of diagrams. That is, a K-algebra consists of a (graded) vector space A together with multiplication and unit maps $m : A \otimes A \to A$, $\eta : K \to A_0 \subset A$ such that the following diagrams commute:

(i)

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes 1} & A \otimes A \\ {\scriptstyle 1 \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow[m]{} & A \end{array}$$

(ii)

$$\begin{array}{ccccc} K \otimes A & \xrightarrow{\eta \otimes 1} & A \otimes A & \xleftarrow{1 \otimes \eta} & A \otimes K \\ & {\scriptstyle =}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle =} & \\ & & A & & \end{array}$$

A is commutative if the following commutes:

(iii)

$$\begin{array}{ccc} A \otimes A & \xrightarrow{T} & A \otimes A \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\ & A & \end{array}$$

where $T(a \otimes b) = (-1)^{|a||b|} b \otimes a$.

This suggests the obvious dualization. A *co-algebra* is a graded vector space C equipped with comultiplication (or "diagonal") and co-unit maps

$$\Delta : C \to C \otimes C, \epsilon : C \to K$$

such that the analogues of diagrams (i) and (ii) (reverse all arrows!) commute. Similarly a co-algebra is co-commutative if $T\Delta = \Delta$. Again, we will only consider co-algebras which are connected—i.e. $\epsilon|_{C_0}$ is an isomorphism—and of finite type. Thus if A is an algebra, A^* is a co-algebra and conversely. Now given a co-algebra C, we have $\Delta : C_n \to \bigoplus_{i+j=n} C_i \otimes C_j$. Hence if $x \in C_n$, $\Delta x = \Sigma x_{ij}$ with each x_{ij} of the form $\Sigma a_k \otimes b_k$, $a_k \in C_i$, $b_k \in C_j$. For brevity we often write $\Delta x = \Sigma x'_k \otimes x''_k$, with the x'_k, x''_k understood to be homogeneous. The co-unit axiom implies that for all x, $\Delta x = 1 \otimes x + x \otimes 1+$ other terms, and our assumption C connected implies the other terms lie in $C_i \otimes C_j$ with i, $j \neq 0$. We call x *primitive* if there are no other terms—that is, $\Delta x = 1 \otimes x + x \otimes 1$. The primitives form a K-subspace $P(C)$.

(A5.6) PROPOSITION. *If A is a graded K-algebra, QA and $P(A^*)$ are dual vector spaces.*

PROOF: We must show that the annihilator in A^* of $\bar{A}^2$ is precisely $P(A^*)$. This is immediate from the definitions, since the annihilator of $\bar{A} \otimes \bar{A}$ is $K \otimes A^* \oplus A^* \otimes K \subseteq A^* \otimes A^* = (A \otimes A)^*$. ❑

(A5.7) EXAMPLE: If X is a path-connected space, then $H_*(X; K)$ is a connected co-commutative co-algebra (possibly not of finite type) with diagonal map

$$\Delta = \Delta_* : H_*X \to H_*(X \times X) = H_*X \otimes H_*X.$$

For example if $X = \mathbb{C}P^\infty$, let $b_n \in H_{2n}\mathbb{C}P^\infty$ be defined by $\langle c_1^n, b_n \rangle = 1$. Then $\Delta b_n = \sum_{i+j=n} b_i \otimes b_j$. Similarly if $X = \mathbb{R}P^\infty$ and $K = \mathbb{Z}/2$, and $a_n \in H_n\mathbb{R}P^\infty$ is the non-zero element, $\Delta a_n = \sum_{i+j=n} a_i \otimes a_j$. Now suppose $X = B\mathbb{Z}/l^a$, with $K = \mathbb{Z}/l$; if $l = 2$ assume $a > 1$. Then we may similarly define generators $b_n \in H_{2n}B\mathbb{Z}/l^a$ and $e_n \in H_{2n-1}B\mathbb{Z}/l^a$ so that $\Delta b_n = \sum_{i+j=n} b_i \otimes b_j$ and $\Delta e_n = \sum_{i+j=n} e_i \otimes b_j + b_i \otimes e_j$. All of these formulae are easily proved by dualizing. Note that b_n, a_n, e_n are primitive only for $n = 1$.

A *Hopf algebra* is a graded algebra A which is also a co-algebra, such that the diagonal $A \xrightarrow{\Delta} A \otimes A$ is an algebra. This is equivalent to requiring that the multiplication $A \otimes A \xrightarrow{m} A$ be a co-algebra map (the definition of the co-algebra structure on $A \otimes A$ is left to the reader). Thus A^* is again a Hopf algebra, with multiplication Δ^* and diagonal m^*.

(A5.8) EXAMPLE: If X is a path-connected homotopy associative H-space, $H_*(X, K)$ is a connected co-commutative Hopf algebra (but possibly not of finite type).

(A5.9) EXAMPLE: Suppose V is a co-algebra over K. Then the diagonal map on V extends in a natural way to $T(\bar{V}), S(\bar{V})$ and $S'(\bar{V})$ (see Remark A5.5), so that these algebras are all Hopf algebras over K. The last two are ***bicommutative***—that is, both commutative and co-commutative.

Now suppose X is an H-space as in A5.8, and $W \xrightarrow{f} X$ a map. Then $H_*(f) : H_*W \to H_*X$ induces a map of Hopf algebras $T(f) : T(\widetilde{H}_*W) \to H_*X$. If this map is surjective we say that W is a *generating complex* for X (the map f and the coefficients K being understood here). If X is homotopy commutative, or more generally if H_*X is a commutative ring, $T(f)$ factors through a map of Hopf algebras $S(f) : S(\widetilde{H}_*W) \to H_*X$.

(A5.10) EXAMPLE: $\mathbb{C}P^\infty$ is a generating complex for BU, and $S(\widetilde{H}_*\mathbb{C}P^\infty) \cong H_*BU$. Here we could even use $\mathbb{Z}$-coefficients. Similarly if $K = \mathbb{Z}/2$, $\mathbb{R}P^\infty$ is a generating complex for BO, and $S(\widetilde{H}_*\mathbb{R}P^\infty) \cong H_*BO$.

Let X be a connected homotopy associative and commutative H-space, so that for any Y, $[Y, X]$ is an abelian group.[4] Given $f, g : Y \to X$, $\pi_*(f+g) = \pi_*f + \pi_*g$. However this is certainly false for homology. Instead we have, in the notation used earlier:

(A5.11) PROPOSITION. *If $a \in H_nY$, $(f+g)_*a = \Sigma(f_*a_i')(g_*a_i'')$, where $\Delta_*a = \Sigma a_i' \otimes a_i''$.*

PROOF: By definition. $f+g$ is the composite

$$Y \xrightarrow{\Delta} Y \times Y \xrightarrow{f \times g} X \times X \xrightarrow{m} X. \qquad \square$$

(A5.12) COROLLARY. *$(f+g)_*a = f_*a + g_*a$ mod decomposables. Also if a is primitive. $(f+g)_*a = f_*a + g_*a$.*

A6. Localization at the Fixed Point Set

Let X be a G-space, G a topological group. The *Borel construction* is the balanced product $EG \times_G X$. The *equivariant cohomology* of X is $H_G^*X \equiv H^*(EG \times_G X)$. (All cohomology in this section is with $\mathbb{Z}/p$-coefficients.) Note that $H_G^*(\text{point}) = H^*BG$, and that $H_G^*(-)$ is a functor (G-spaces)$\to$(H^*BG-modules).

[4] A theorem of James (see e.g. [Srinivas]) says that any connected homotopy associative H-space has a homotopy inverse; that is why $[Y, X]$ is a group.

(A6.1) PROPOSITION. *Suppose $f : X \to Y$ is both a G-map and a homotopy equivalence. Then $EG \times_G X \xrightarrow{1\times f} EG \times_G Y$ is a homotopy equivalence. In particular $H^*_G Y \underset{\cong}{\xrightarrow{f^*}} H^*_G X$.*

NOTE: We do not assume that the homotopy inverse is a G-map.

PROOF: The 5-lemma shows that $1 \times f$ is an isomorphism on homotopy groups. □

If $A \subset X$ is a G-subspace, we define $H^*_G(X, A) = H^*(EG \times_G X, EG \times_G A)$. So there is a long exact sequence of the usual sort for the pair (X, A), and it is a sequence of H^*BG-modules.

Now suppose G is discrete. There is a very simple and very convenient generalization of the notion of CW-complex to the equivariant setting. Briefly, we simply take the usual definition of a CW-complex and substitute "G-cell" for cell; of course we require that the attaching maps are equivariant, etc. Here a "G-cell" means $G/H \times e^n$, where e^n is an n-cell in the usual sense, H is a subgroup of G, and G acts on $G/H \times e^n$ by left translation on the discrete set G/H. Such a G-space X will be called a G-CW-complex. It is in particular a CW-complex in the ordinary sense, with $[G : H]$ ordinary n-cells for every G-cell $G/H \times e^n$. It is built up inductively by attaching G-cells $G/H \times D^n$ via equivariant attaching maps $G/H \times S^{n-1} \to X^{(n-1)}$. Note that if X is a G-CW-complex, X^G is a subcomplex and X/G is a CW-complex.

EXAMPLES: (a) Let X be any CW-complex. Then the universal cover $\widetilde{X}$ receives a G-CW-structure in an obvious way, where $G = \pi_1 X$. In this case every G-cell is a free G-cell—i.e. of the form $G \times e^n$.

(b) Let $X = |S|$ be a simplicial complex with vertex set S. If G acts on S and preserves the subsets which span simplices, then X becomes a G-space which one might wish to call a "simplicial G-complex". However X need not be a G-CW-complex, because X^G may not be a subcomplex. For example if $G = \mathbb{Z}/3$ acts on $X = \Delta^2$ by permuting the vertices, X^G is precisely the barycenter, which is not a vertex. But the solution to this difficulty is clear: simply pass to the barycentric subdivision. Thus the reader can easily prove that if $|S|$ is a simplicial G-complex in the above sense, then so is the barycentric subdivision $|S^{(1)}|$, and $|S^{(1)}|$ is a G-CW-complex.

(A6.2) THEOREM. *Let X be a finite $\mathbb{Z}/p$-CW-complex. Then the inclusion $X^{\mathbb{Z}/p} \to X$ induces an isomorphism $y^{-1} H^*_{\mathbb{Z}/p} X \to y^{-1} H^*_{\mathbb{Z}/p} X^{\mathbb{Z}/p}$.*

PROOF: We will proceed by induction over the number of free $\mathbb{Z}/p$-cells of X. If there are no free cells, then $X = X^{\mathbb{Z}/p}$ and the theorem is trivial. Now suppose the theorem is true for X', and X is obtained from X' by attaching a free cell $\mathbb{Z}/p \times D^n$. Then it is enough to show that $y^{-1} H^*_{\mathbb{Z}/p}(X, X') = 0$ (apply the exact functor $y^{-1}(\ \ )$ to the long exact sequence of the pair). Now

$$(E\mathbb{Z}/p \times_{\mathbb{Z}/p} X)/(E\mathbb{Z}/p \times_{\mathbb{Z}/p} X') = E\mathbb{Z}/p \times_{\mathbb{Z}/p} (\mathbb{Z}/p \times D^n)/E\mathbb{Z}/p \times_{\mathbb{Z}/p} (\mathbb{Z}/p \times S^{n-1})$$
$$= (E\mathbb{Z}/p \times D^n)/(E\mathbb{Z}/p \times S^{n-1}) = E\mathbb{Z}/p_+ \wedge S^n.$$

It follows that $H^*_{\mathbb{Z}/p}(X, X')$ is in particular bounded above, so $y^{-1}H^*_{\mathbb{Z}/p}(X, X') = 0$. □

To apply this to the extended power construction we need to ensure that X^p (with permutation action) is a $\mathbb{Z}/p$-CW-complex. One way to do this is as follows. Any finite CW-complex X has the homotopy type of a finite simplicial complex (see e.g. [Maunder], p. Exercise). Choose an equivalence $X \xrightarrow{f} Y$ with Y a simplicial complex; then $\Gamma X \to \Gamma Y$ is an equivalence by (A6.1). Hence we may assume X is simplicial. Then X^p is also a simplicial complex and X^p is a $\mathbb{Z}/p$-CW-complex.

REMARK: Quillen proves a more general theorem on $(\mathbb{Z}/p)^n$-actions. The statement is the same, but with y a certain non-nilpotent element in $H^{2(p^n-1)}B(\mathbb{Z}/p)^n$. The crucial property of y is that y restricts to zero on BH for any proper subgroup H. Using only this fact, it is a nice exercise to extend the proof of (A6.2) to this more general situation.

A7. Homology of Extended Powers.

Let G be a discrete group acting on a space X. We assume that X is a CW-complex and that the action is cellular in the sense that it preserves skeleta and maps "open" cells homeomorphically one to another. For example, the action on any G-CW-complex (see Appendix 6) is cellular, but this is a weaker notion. Let $C.X$ denote the cellular chain complex with coefficients in a fixed commutative ring K. Then $C.X$ is a complex of KG-modules.

(A7.1) PROPOSITION. *$H_*(EG \times_G X)$ is naturally isomorphic to the homology of the chain complex $C.EG \otimes_{KG} C.X$.*

PROOF: $EG \times X$, with the diagonal action, is a free G-CW-complex, and the projection $EG \times X \to EG \times_G X$ is a covering map. The proposition follows easily from this. □

This suggests an algebraic generalization. Let $C.$ be any chain complex of KG-modules, and let $P. \to K$ be a projective resolution of the trivial module. We define the ***equivariant homology*** $H^G_* C.$ by $H^G_* C. = H_*(P. \otimes_{KG} C.)$. The usual argument shows this is independent of the choice of $P.$, up to canonical isomorphism. Note that by definition $H^G_* C.$ is the total homology of a double complex $C_{p,q} = P_p \otimes_{KG} C_q$. Hence we automatically get two spectral sequences converging to $H^G_* C$.

(A7.2) PROPOSITION. *There is a spectral sequence*

$$E^2_{p,q} = H_p(G, H_qC) \Rightarrow H^G_* C.$$

PROOF: Use the spectral sequence associated to the filtration by columns. Thus $E^1_{p,q}$ is the homology of the complex $P_p \otimes_{KG} C.$ (p fixed). Since P_p is projective and hence flat, $P_p \otimes_{KG} (-)$ commutes with homology. So $E^1_{p,q} = P_p \otimes_{KG} H_qC$. Then the homology of $P. \otimes_{KG} H_qC$ (q fixed) is by definition $H_p(G, H_qC)$. □

REMARK: In the topological setting this is the Serre spectral sequence of $EG \times_G X \to BG$. The other spectral sequence is the Leray spectral sequence of the map (not a fibration!) $EG \times_G X \to X/G$. We don't need this second spectral sequence here, but it is an instructive exercise to determine its E^1-term (take X a G-CW-complex for simplicity).

We obtain an analogue of A6.1.

(A7.3) PROPOSITION. *Suppose $f. : C. \to C'.$ is a map of KG-chain complexes such that $H_*(f)$ is an isomorphism. Then $H^G_*(f) : H^G_*(C.) \to H^G_*(C'.)$ is also an isomorphism.*

PROOF: There is an induced map of spectral sequences as in A7.2, $f^r : E^r_{p,q} \to E'^r_{p,q}$. By assumption f^2 is an isomorphism, so f^r is also an isomorphism for all r. Taking $r = \infty$ shows that $H^G_* f$ is a map of filtered chain complexes which is an isomorphism on the associated graded objects. Since the filtration is increasing (or because the filtration is finite in each degree), this implies $H^G_*(f)$ is itself an isomorphism. □

Call $C.$ *split* if for all n the short exact sequence (cycles mod boundaries)

$$0 \to B_n \to Z_n \to H_nC \to 0$$

splits as KG-modules.

(A7.4) PROPOSITION. *If $C.$ is split, then*

$$H^G_n(C.) = \bigoplus_{p+q=n} H_p(G; H_qC.).$$

PROOF: Regard $H.C$ as a chain complex with boundary maps identically zero. By assumption there exists a map of KG-chain complexes $H.C \to C.$ which induces an isomorphism on homology. By A7.3, we get an isomorphism $H^G_*(H.C) \to H^G_*C$. But by inspection $H^G_*(H.C)$ is just $H_*(G, H_*C)$. □

In fact it is rare for such a chain complex to be split. However there is one important situation where this always happens.

(A7.5) PROPOSITION. *Suppose K is a field, $G \subseteq \Sigma_n$ and $C.$ has the form $\otimes^n A.$ for some complex of K-modules $A.$, with G acting by permutation (with the appropriate signs). Then $C.$ is split.*

PROOF: By the Künneth theorem, there is a natural isomorphism of KG-modules $\otimes^n H_* A \to H_* C.$. Now the complex $A.$ is trivially split as K-modules. Fix splitting maps $H_k A. \to Z_k A.$ For each $I = (i_1, \ldots, i_n)$ with $i_1 + \cdots + i_n = m$ we get maps

$$H_{i_1} A \otimes \cdots \otimes H_{i_n} A \to Z_{i_1} A \otimes \cdots \otimes Z_{i_n} A \subseteq Z_m C.$$

Taking a direct sum over all such I yields a map $H_m C \to Z_m C$ which is clearly a map of KG-modules and provides the desired splitting. □

This leads to our main result:

(A7.6) THEOREM. *Suppose K is a field $G \subset \Sigma_n$ and G acts on X^n by permuting the factors. Then for all $m \geq 0$ there is a natural isomorphism,*

$$H_m(EG \times_G X^n) = \bigoplus_{i+j=m} H_i(G; H_j X^n).$$

PROOF: The cellular chain complex of X^n is isomorphic to $\otimes^n C_* X$ as KG-modules. The theorem follows by combining (1), (4), and (5) above. □

(A7.7) COROLLARY. *The Serre spectral sequence of $EG \times_G X^n \to BG$ collapses.*

The corollary is already implicit in A7.4. However it also follows from the statement (as opposed to the proof) of A7.6. For one can immediately reduce to the case X a finite complex, and then a dimension count forces the collapse.

The result used in §6 was the special case $n = p$, $G = \mathbb{Z}/p$. The spaces $EG \times_G X^n$ are sometimes called extended powers. They play a central role in the construction of Steenrod operations in cohomology, and of Dyer-Lashof operations in the homology of iterated loop spaces. Theorem A7.6 is a key input for these constructions.

A8. Miscellaneous Representation Theory.

Recall that any representation of a finite abelian group G over an algebraically closed field is a direct sum of one-dimensional representations. The following generalization was used in §6.

(A8.1) PROPOSITION. *Let G be a finite abelian group with exponent n (i.e. $nG = 0$). Let K be a field containing all n-th roots of unity, and with char $K \nmid n$. Then any representation of G over K is a direct sum of one-dimensional representations.*

PROOF: Since char $K \not| |G|$, Maschke's theorem shows KG is semisimple and every representation is a direct sum of irreducible representations. The Artin-Wedderburn theorem implies that KG is a product of matrix rings and hence a product of fields since KG is commutative. Thus $KG = \prod F_i$ where the F_i are fields and are also precisely the irreducible representations of G. But since $KG \to F_i$ is onto and G has exponent n, F_i is generated as a K-algebra by n-th roots of unity. So $F_i = K$ for all i and every irreducible representation is one-dimensional. □

Let G be any group, L/K a finite field extension of degree n. There is a forgetful map $\text{Rep}(G,L) \xrightarrow{r} \text{Rep}(G,K)$ and an extension of scalars map $\text{Rep}(G,K) \xrightarrow{e} \text{Rep}(G,L)$. Here $\text{Rep}(-,-)$ denotes the monoid under direct sum of isomorphism classes of representations; $r(V)$ is just V regarded as KG-module and $e(V) = L \otimes_K V$. It is clear that if $V \in \text{Rep}(G,K)$, then $re(V) = nV$. Our next goal is a formula for $er(V)$, $V \in \text{Rep}(G,L)$, in the case when L is Galois over K.

Let Σ denote the Galois group of L/K. For each $\sigma \in \Sigma$ we can define a "twisted" version V^σ of V as follows: V^σ has the same K-module and G-module structure, but the L-module structure is twisted by $\sigma \cdot v = \sigma(\alpha)v$. If we choose a basis for V and regard the representation as a homomorphism $\rho : G \to GL_m(L)$, this construction is equivalent to leaving the L-module structure on L^m alone but twisting the matrix coordinates $\rho(g)_{ij}$ by σ^{-1}.

(A8.2) PROPOSITION. *Let L be a finite Galois extension of K, with group Σ. Then for any group G and $V \in \text{Rep}(G,L)$, $er(V) = \bigoplus_{\sigma\in\Sigma} V^\sigma$.*

We need the following very useful lemma. The K-algebra $L \otimes_K L$ is also an L-bimodule—it has the obvious commuting module structures $\alpha \cdot (a \otimes b) = \alpha a \otimes b$ and $(a \otimes b)\alpha = a \otimes b\alpha$. We also have an L-bimodule structure on $\prod_{\sigma\in\Sigma} L : \alpha \cdot (a_\sigma) = (\sigma a_\sigma)$ and $(a_\sigma) \cdot \alpha = (a_\sigma \sigma(\alpha))$.

(A8.3) LEMMA. *The map $\varphi : L \otimes_K L \to \prod_{\sigma\in\Sigma} L$ given by $\varphi(a \otimes b) = \prod(a\sigma(b))_\sigma$ is an isomorphism of L-bimodules and of K-algebras.*

PROOF: The only nonobvious point is that φ is an isomorphism. By field theory $L \cong K[x]/f(x)$ for some polynomial f that splits in f. Thus if $\alpha \in L$ is one root of f, $f(x) = \prod_\sigma(x - \sigma\alpha)$ in $L[x]$.

Now consider the composite φ'

$$L \otimes_K L = L \otimes_K (K[x]/f(x)) = L[x]/f(x) \xrightarrow{\eta} \prod_\sigma L$$

where $\eta(g(x)) = (g(\sigma(\alpha))_\sigma$. By the Chinese Remainder Theorem η is an isomorphism (note η is in effect the map $L[x]/\prod(x-\sigma\alpha) \to \prod(L[x]/(x-\sigma\alpha))$, so φ' is an isomorphism. Finally $\varphi' = \varphi$, since both maps are bimodule maps and K-algebra maps. □

PROOF OF A8.2:

$$L \otimes_K V = (L \otimes_K L) \otimes_L V = (\bigoplus_\sigma L) \otimes_L V = \bigoplus_\sigma V^\sigma.$$

where we get V^σ because in $\bigoplus_\sigma L$ the right L-module structure is the twisted one. □

(A8.4) COROLLARY. *Let V be an L-vector space of dimension n, $\alpha \in L^*$. Regard scalar multiplication by α as an element of $\operatorname{Aut}_K V$. Then $1 \otimes \alpha \in \operatorname{Aut}_L(L \otimes_K V)$ is diagonalizable with eigenvalues $\sigma\alpha$, $\sigma \in \Sigma$, each with multiplicity n.*

REMARK: In particular we get an important result of basic Galois theory: regarding $\sigma \in \operatorname{End}_K L$, $\det \alpha = \prod \sigma\alpha$ and trace $\alpha = \Sigma\sigma\alpha$.

Our final topic is the Brauer induction theorem, and its application to the theorem of Green used in §2. The main reference here is the marvelous book [Serre]. We first need some elementary facts about induced representations. The reader should note the close analogy with the transfer (A1).

Let G be a finite group, F a field. The representation ring $\mathcal{R}_F G$ is $\mathbb{Z}\operatorname{Rep}(G,F)/\sim$, where $\operatorname{Rep}(G,F)$ is the set of isomorphism classes of finite-dimensional FG-modules and we set $[V] \sim [V'] + [V'']$ whenever there is a short exact sequence

$$0 \to V' \to V \to V'' \to 0.$$

If $\operatorname{char} F \nmid |G|$, all such sequences split and $\mathcal{R}_F G$ is just the group completion of the monoid $\operatorname{Rep}(G, F)$, but this is not true in general. The ring structure is tensor product of representations, with G acting diagonally. A homomorphism $\varphi : H \to G$ induces a ring homomorphism $\varphi^* = \mathcal{R}_F G \to \mathcal{R}_F H$ in the obvious way. In the case of an inclusion $i : H \subset G$ we also write $i(G, H)$ for i^*.

If $H \subset G$ and W is an FH-module, the *induced* module is $FG \otimes_{FH} W$. Since FG is a free and hence flat right FH-module, it follows that induction is well-defined on $\mathcal{R}_F H$ and yields a homomorphism of groups $\operatorname{Ind}(H, G) : \mathcal{R}_F H \to \mathcal{R}_F G$.

(A8.5) PROPOSITION. *$\operatorname{Ind}(H, G) \circ i(G, H)$ is multiplication by the permutation module $[F(G/H)]$ on $\mathcal{R}_F G$.*

This property won't be used here but is included for completeness. The proof is an easy exercise.

Let K be another subgroup and consider a typical double coset $H \times H$ in G. If W is an FH-module we get a representation W_x of $K_x = K \cap xHx^{-1}$ by pulling back along the monomorphism $K_x \to H$ given by conjugation by x^{-1}.

(A8.6) PROPOSITION. *$i(G, K) \circ \mathrm{Ind}(H, G)(W) \cong \bigoplus_x \mathrm{Ind}(K_x, K)(W_x)$ where x ranges over a set of $K - H$ double coset representatives.*

The proof is straightforward—see [Serre], p. 58.

$\mathrm{Ind}(H, G)$ is not a ring homomorphism, but rather is a homomorphism of $\mathcal{R}_F G$-modules (a similar assertion holds for the transfer, incidentally):

(A8.7) PROPOSITION. *If W is an FH-module, $H \overset{i}{\subset} G$, and V is an FG-module, $\mathrm{Ind}(H, G)(i^* V \otimes W) = V \otimes \mathrm{Ind}(H, G)(W)$.*

Again the proof is an easy check.

(A8.8) PROPOSITION. *Suppose $F = \mathbb{C}$, W is an FH-module and $V = \mathbb{C}G \otimes_{\mathbb{C}H} W$. Then*

$$\chi_V(x) = \frac{1}{|H|} \sum \chi_W(g^{-1}xg)$$

where the sum is over all $g \in G$ such that $g^{-1}xg \in H$.

PROOF: See [Serre], p. 30. One could also approach this using (A8.6). □

(A8.9) REMARK: If we replace χ_W by an arbitrary class function on H in (A8.8), we still get a class function on G. Hence one may use A8.8 to *define* $\mathrm{Ind}(H, G)$ on class functions. Alternatively, since the ring of class functions $C(G)$ is generated by the characters (as $\mathbb{C}$-vector space), one could simply extend $\mathrm{Ind}(H, G)$. In any event, both $i(G, H)$ and $\mathrm{Ind}(H, G)$ extend to class functions, and the analogue of A8.7 remains valid.

(A8.10) LEMMA. *Suppose $\operatorname{char} F = p$ and $G = H \times K$ with H a p-group. Then every irreducible FG-module V is trivial as H-module. Hence $V = \pi^* W$ for some FK-module W, where $\pi : G \to K$ is the projection.*

PROOF: Since $\operatorname{char} F = p$ and H is a p-group, $V^H \neq 0$. But V^H is a G-submodule, so $V^H = V$ since V irreducible. □

A representation of G is called *monomial* if it is induced from a one-dimensional representation: $V = FG \otimes_{FH} W$ for some H, with $\dim W = 1$.

(A8.11) PROPOSITION. *If F is algebraically closed and G is nilpotent, every irreducible FG-module V is monomial.*

PROOF: If char $F \nmid |G|$, the proof of Theorem 16 on p. 66 of [Serre] goes through unchanged. Now suppose char $F = p$ and $p \mid |G|$. Since G is nilpotent, G is the direct product of its r-Sylow subgroups for various primes r. In particular $G = G_p \times G'$ where G_p is the p-Sylow subgroup and $p \nmid |G'|$. Then (A8.11) follows from the lemma and the case already proved.

We say that G is *r-elementary*, r a prime, if G is the product of an r-group and a cyclic group of order prime to r. G is *elementary* if G is r-elementary for some r. Note that an elementary group is nilpotent.

(A8.12) THEOREM (BRAUER INDUCTION). *Let G be a finite group, F an algebraically closed field, and let X denote the set of elementary subgroups of G. Then*

$$\text{Ind} : \bigoplus_{H \in X} \mathcal{R}_F H \to \mathcal{R}_F G$$

is surjective.

We are only going to use the case $F = \mathbb{C}$. Note the theorem says that every representation of G over $\mathbb{C}$ is a $\mathbb{Z}$-linear combination of representations induced from elementary subgroups. However it may not be possible to make the coefficients positive—see [Serre], Exercise 10.5, p. 79.

(A8.13) COROLLARY. *In $\mathcal{R}_F G$, every representation is a $\mathbb{Z}$-linear combination of monomial representations.*

PROOF: Use A8.11 and A8.12. □

(A8.14) COROLLARY ($F = \mathbb{C}$). *Let f be a class function on G. Then f is a virtual character if and only if the restriction of f to each elementary subgroup H is a virtual character.*

PROOF: By Brauer induction we can write $1 \in \mathcal{R}_{\mathbb{C}} G \subset C(G)$ as $1 = \Sigma\, \text{Ind}(H_\alpha, G)\chi_\alpha$ where the H_α are elementary and χ_α is a virtual character of H_α. So

$$f = f \cdot 1 = \Sigma f \cdot (\text{Ind}(H_\alpha, G \mid \chi_\alpha) = \Sigma\, \text{Ind}(H_\alpha, G)(f_\alpha\, |_{H_\alpha} \cdot \chi_\alpha) \quad \text{(cf. Remark A8.9).}$$

The sum on the right is a virtual character by assumption. □

We now turn to Green's theorem. Let V be a representation of G over $\bar{\mathbb{F}}_q$, with Brauer character $\chi = \chi_V$.[5]

[5]Our "Brauer character" is not what Serre calls a Brauer character on p. 147 of [Serre]. Our χ is analogous to the class function f' defined in section 18.4 of [Serre] (Theorem 43(i) of that section is a more general version of Green's theorem).

(A8.15) THEOREM (GREEN). χ_V *is a virtual complex character of* G.

Before starting the proof we comment on the behaviour of χ_V with respect to induction. Suppose V is a monomial representation, say $V = \bar{\mathbb{F}}_q G \otimes_{\bar{\mathbb{F}}_q H} W$ with $\dim W = 1$. Then $W = \bar{\mathbb{F}}_q$ with H acting via some homomorphism $\lambda : H \to \bar{\mathbb{F}}_q^\times$. Let $\widetilde{\lambda} = i\lambda$ where $i : \bar{\mathbb{F}}_q^\times \subset \mathbb{C}^\times$ is our fixed embedding. Then $\widetilde{\lambda}$ defines a representation $\widetilde{W}$ of H over $\mathbb{C}$ with $\dim \widetilde{W} = 1$, and $\chi_{\widetilde{W}} = \chi_W$. Let $\widetilde{V} = \mathbb{C}G \otimes_{\mathbb{C}H} \widetilde{W}$. Then it is not true in general that $\chi_{\widetilde{V}} = \chi_V$. For example if G is a p-group and $H = \{1\}$, $\widetilde{V}$ is the regular representation but χ_V is the character of the trivial representation of dimension $|G|$.

PROOF OF A8.15:

Case 1: G is a nilpotent group of order prime to p. We can assume V is irreducible, and hence monomial by (A8.11). In this case we actually have $\chi_V = \chi_{\widetilde{V}}$ with $\widetilde{V}$ as above. To see this, it suffices to show $\chi_V(g) = \chi_{\widetilde{V}}(g)$ for each fixed g, so we restrict to the cyclic subgroup $C = \langle g \rangle$. By A8.6 both V and $\widetilde{V}$ split in parallel fashion as direct sums of C-modules U, $\widetilde{U}$ respectively, where U is a monomial representation $\bar{\mathbb{F}}_q C \otimes_{\bar{\mathbb{F}}_q D} \bar{\mathbb{F}}_q$. Here $D = \langle g^d \rangle$ for some d and D acts via $\lambda : D \to \bar{\mathbb{F}}_q^\times$. It suffices to show $\chi_U(g) = \chi_{\widetilde{U}}(g)$. Let $\alpha = \lambda(g^d)$. Then the characteristic polynomial of g on U is $T^d - \alpha$. Since d is prime to p (by our assumption on G), there are d distinct d-th roots of α in $\bar{\mathbb{F}}_q^\times$, say $\alpha_1, \ldots, \alpha_d$. Then $i(\alpha_1), \ldots, i(\alpha_d)$ are precisely the eigenvalues of g acting on $\widetilde{U}$, so $\chi_U(g) = \chi_{\widetilde{U}}(g)$.

Case 2: G is an arbitrary nilpotent group. Again we may assume V irreducible. Let $G = G_p \times G'$ as in A8. . Since Brauer characters obviously commute with pullbacks π^*, Case 2 follows from Case 1 and Lemma A8.11.

Case 3: G is arbitrary. By Case 2, the restriction of χ_V to each elementary subgroup H is a virtual character of H. So by Corollary A8.14, χ_V is a virtual character of G. □

REMARK: Case 1 has a more elegant proof if one has available the Witt rings $W\mathbb{F}_q$, which are complete local rings of characteristic zero with residue field $\mathbb{F}_q$. Then for any G of order prime to p, nilpotent or not, representations over $\mathbb{F}_q$ can always be lifted to $W\mathbb{F}_q$. This yields a shorter proof along the lines of Theorem 43, p. 153 in [Serre].

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