THE H-PRINCIPLE, LECTURE 11: THE H-PRINCIPLE FOR DIFFERENTIAL RELATIONS

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1. Gromov’s generalization of the Hirsch-Smale theorem

During our extended treatment of the proof of \( \operatorname{Imm}(\mathcal{M}, \mathcal{N}) \simeq \operatorname{Imm}(\mathcal{M}, \mathcal{N}) \), we made frequent comment as to the generality of our reasoning and how little use we made of the fact that sheaf \( \mathcal{I} \) on the source manifold \( \mathcal{M} \) was in fact the sheaf \( \mathcal{I} = \operatorname{Imm}(-, \mathcal{N}) \) of immersions into \( \mathcal{N} \). The salient features we used were the following:

- \( \mathcal{I}(K) \subset \operatorname{Map}_{\text{sm}}(K, \mathcal{N}) \) is an open subspace, for \( K \subset \mathcal{M} \) compact, and \( \operatorname{Map}_{\text{sm}}(-, \mathcal{N}) \) is a flexible sheaf (i.e., \( \operatorname{Map}_{\text{sm}}(K, \mathcal{N}) \to \operatorname{Map}_{\text{sm}}(K_0, \mathcal{N}) \) is a Serre fibration for \( K_0 \hookrightarrow K \) a closed inclusion);
- The group \( \text{Diff}(\mathcal{M}) \) of diffeomorphisms of \( \mathcal{M} \) acts on \( \operatorname{Map}_{\text{sm}}(\mathcal{M}, \mathcal{N}) \), and it preserves the subspace \( \operatorname{Imm}(\mathcal{M}, \mathcal{N}) \). Further, by embedding \( \mathcal{M} \hookrightarrow V \) as the zero section of a bundle, then \( \text{Diff}(V, \partial V) \) acts transitively on a small enough neighborhood of a point \( f \in \mathcal{I}(\mathcal{M}) \subset \operatorname{Map}_{\text{sm}}(\mathcal{M}, \mathcal{N}) \).

The proof of the weak homotopy equivalence \( \operatorname{Imm}(\mathcal{M}, \mathcal{N}) \to \operatorname{Imm}(\mathcal{M}, \mathcal{N}) \) then proceeds by induction on handle decomposition of \( \mathcal{M} \). The key part of the inductive step is to show that the restriction map \( \mathcal{I}(U + \varphi q) \to \mathcal{I}(U) \) is a Serre fibration, where \( U + \varphi q \cong U \sqcup_{S^{q-1} \times D^{n-q}} D^q \times D^{n-q} \) is \( U \) with a handle of index \( q \) attached. The necessary lifting could be constructed by maneuvering collar neighborhoods, using the features above, but required that the index \( q \) is less than \( n \) (and thus the proof only worked when \( \mathcal{M} \) is an open manifold).

These features enable, mutatis mutandis, the proof of the following generalization: Let \( E \to \mathcal{M} \) be a smooth fiber bundle with an action of \( \text{Diff}(\mathcal{M}) \) lifting the canonical action of \( \text{Diff}(\mathcal{M}) \) on \( \mathcal{M} \). Let \( E^{(k)} \to \mathcal{M} \) be the bundle of \( k \)-jets of \( E \), which inherits an action of \( \text{Diff}(\mathcal{M}) \), and let \( \mathcal{R} \subset E^{(k)} \) be a subspace (i.e., \( \mathcal{R} \) is a differential relation). Gromov proved the following h-principle:

**Theorem 1.1 (Gromov).** If \( \mathcal{M} \) is an open manifold and the relation \( \mathcal{R} \subset E^{(k)} \) is an open subspace which is preserved by the action of \( \text{Diff}(\mathcal{M}) \) on \( E^{(k)} \), then the \( k \)-jet prolongation map

\[
\tilde{j}^{(k)} : \operatorname{Sol}_\mathcal{R}(\mathcal{M}) \longrightarrow \Gamma(\mathcal{R})
\]

is a weak homotopy equivalence, where the space of solutions \( \operatorname{Sol}_\mathcal{R}(\mathcal{M}) \) consists of those smooth sections of \( E \) whose \( k \)-jet lies in \( \mathcal{R} \).

To summarize: diffeomorphism invariant open differential relations adhere to the h-principle on open manifolds.

This theorem is ready to be put to use. We just need to find some examples, which will turn out to be plentiful.

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1 More precisely: Choose an element \( \tilde{f} \in \mathcal{I}(V) \) which restricts to \( f \in \mathcal{I}(\mathcal{M}) \). Then, there exists a neighborhood \( U_f \) of \( f \) and a map \( U_f \to \text{Diff}(V, \partial V) \) such that the composite \( U_f \to \text{Diff}(V, \partial V) \to \mathcal{I}(V) \to \mathcal{I}(\mathcal{M}) \) is the inclusion of \( U_f \) in \( \mathcal{I}(\mathcal{M}) \), where the map \( \text{Diff}(V, \partial V) \to \mathcal{I}(V) \) is the composite \( \text{Diff}(V, \partial V) \times \{\tilde{f}\} \to \text{Diff}(V, \partial V) \times \mathcal{I}(V) \to \mathcal{I}(V) \).
2. Submersions

**Definition 2.1.** A smooth map \( f : M \to N \) is a submersion if the derivative map \( d_x f : T_x M \to T_{f(x)} N \) is surjective for every point \( x \) in \( M \). A formal submersion \( F \) is a bundle map \( T_M \to T_N \) which is surjective on each fiber. The space \( \text{Subm}(M,N) \) has the compact-open \( C^\infty \) topology, and \( \text{Subm}^f(M,N) \) has the compact-open topology.

The map \( \text{Subm}(M,N) \to \text{Subm}^f(M,N) \) is an example of a 1-jet prolongation map, where \( E \) is the product \( M \times N \). It is easy to see that the relation \( \mathcal{R} \subset E^{(1)} \) defining submersions is open and diffeomorphism invariant. We obtain the following theorem, the first published proof of which is in [4]:

**Theorem 2.2 (Phillips).** *If \( M \) is an open manifold, then the map*

\[
\text{Subm}(M,N) \to \text{Subm}^f(M,N)
\]

*is a weak homotopy equivalence.*

**Proof.** Apply Gromov’s theorem. \( \square \)

**Remark 2.3.** Submersions are a very familiar concept when the source \( M \) is closed: In this case, Ehresmann’s theorem states that the map \( M \to N \) is a fiber bundle, which is perhaps the most interesting kind of submersion. However, this is exactly the case in which Gromov’s theorem does not apply: There may exist formal submersions of \( M \) to \( N \) even when \( M \) does not fiber over \( N \) (for instance, if \( M \) is parallelizable and \( N \) is parallelizable and noncompact). However, even in the case of \( M \) closed, a formal submersion of \( M \) on \( N \) is clearly a good first step to obtain an actual submersion. Is there a systematic way of taking successive steps? We will discuss such issues later in this course. (Hint: Goodwillie calculus.)

3. Foliations

We now begin the next major topic of focus in our course, the theory of foliations. Before giving a formal definition, it is helpful to have an example and a rough picture. In geology, foliation of a rock consists of a division the rock into layers; in metalwork, to foliate a metal is to cut it into thin leaves. The same holds true in topology, but where “manifold” replaces “rock” and “metal.”

**Example 3.1.** One foliation of the space \( \mathbb{R}^n \) consists of the family of all submanifolds \( \{x_1, \ldots, x_q\} \times \mathbb{R}^{n-q} \), where \( \{x_1, \ldots, x_q\} \) is a point of \( \mathbb{R}^q \). This is a codimension \( q \) foliation. Note that every point of \( \mathbb{R}^n \) lies in exactly one of these submanifolds. The submanifolds \( \{x_1, \ldots, x_q\} \times \mathbb{R}^{n-q} \) are the leaves of the foliation.

**Definition 3.2 (Intuitive version).** A codimension \( q \) foliation \( \mathcal{F} \) of an \( n \)-manifold \( M \) consists of a collection of \( (n-q) \)-dimensional manifolds with disjoint inclusions into \( M \), \( \{M_f \subset M\} \), and which looks locally like the above foliation of \( \mathbb{R}^n \).\(^2\)

We have a wealth of examples of foliations coming from fiber bundles:

**Example 3.3.** Let \( \pi : M \to N \) be a smooth fiber bundle, with fibers \( M_x := \pi^{-1}\{x\} \). Then the collection of fibers \( \mathcal{F} := \{M_x \subset M | x \in N\} \) foliates \( M \). The codimension of the foliation is equal to the dimension of \( N \).

This source of examples behaves, in some sense, too well. Namely, the following are true for a foliation of a bundle by its fibers bundle but do not hold in general:

- All of the fibers \( M_x \) are diffeomorphic;

\(^2\)I.e., every point \( x \) has a neighborhood \( U \) and a diffeomorphism \( g : U \to \mathbb{R}^n \) such that the inverse image of each leaf of \( \mathbb{R}^n \) is a component of a leaf of \( \mathcal{F} \) restricted to \( U: g^{-1}(\{x_1, \ldots, x_q\} \times \mathbb{R}^{n-q}) \subset M_f \cap U \), for some \( M_f \in \mathcal{F} \).
The leaf space of this foliation (i.e., the quotient $M/\sim$, where $x \sim y$ if they are contained in the same leaf) is actually a smooth manifold (namely, the base of the bundle), rather than some space with a gnarly topology;

If $M$ is compact, then the fibers $M_x$ are compact.

Let us conclude with examples to show that general foliations are more interesting and need not be so tidy.

**Example 3.4** (The Kronecker foliation). Consider the torus $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$. Let $a$ be any irrational real number, and define the subspace $M_t \subset T^2$ to be the image of the line $y = ax + t$. Then the collection $\mathcal{R} = \{M_t \subset T^2\}$ foliates $T^2$. Note that each leaf of the foliation is dense in $T^2$, and the topology of the leaf space is badly-behaved. Also, the leaves are noncompact, diffeomorphic to $\mathbb{R}$, although $T^2$ is compact.

**Example 3.5** (Reeb foliation). We construct a foliation of $S^{k+1}$ that has only a single compact leaf. First, we construct a foliation of $\mathbb{R} \times D^k$. Thinking of $D^k$ as the unit disk in $\mathbb{R}^k$, choose a function $f(x)$ on the interior of $D^k$ such $f(x)$ tends to $\infty$ as $|x|$ tends to 1. (E.g., $f(x) = \frac{1}{1 - |x|}$ works.) Define the foliation $\mathcal{R}$ as having leaves $M_t \subset \mathbb{R} \times D^k$, where $M_t$ is the graph of the function $f(x) + t$, $t \in \mathbb{R}$, and one final leaf given by the boundary $\partial \mathcal{R} \times D^k \subset \mathbb{R} \times D^k$. Since the foliation is invariant under translation in the $\mathbb{R}$ direction, we can use the projection map $\pi : \mathbb{R} \times D^k \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}^k$ to define a foliation $\mathcal{R}$ on $S^1 \times D^k$, consisting of the manifolds $\pi(M_t) \subset S^1 \times D^k$ and the boundary $S^1 \times \partial D^k \subset S^1 \times D^k$. (Note $\pi(M_t) = \pi(M_{t+1})$.) I lifted the following pretty picture of this foliation from Wikipedia, which shows the case $k = 2$:

Now, using the decomposition of $S^{k+1}$ as a union

$$S^{k+1} \cong S^1 \times D^k_0 \coprod_{S^1 \times S^{k-1}} S^1 \times D^k_1$$

the Reeb foliation of $S^{k+1}$ is then defined by the union of the above foliations $\mathcal{R}_0 \cup \mathcal{R}_1$ on each subspace $S^1 \times D^k$. The foliation has one leaf diffeomorphic to $S^1 \times S^{k-1}$ and all others diffeomorphic to $\mathbb{R}^k$.

**References**


