# THE H-PRINCIPLE, LECTURE 11: THE H-PRINCIPLE FOR DIFFERENTIAL RELATIONS

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## 1. Gromov's generalization of the Hirsch-Smale theorem

During our extended treatment of the proof of $\mathrm{Imm}(M,N) \simeq \mathrm{Imm}^{\mathrm{f}}(M,N)$, we made frequent comment as to the generality of our reasoning and how little use we made of the fact that sheaf $\mathcal{I}$ on the source manifold $M$ was in fact the sheaf $\mathcal{I} = \mathrm{Imm}(-,N)$ of immersions into $N$. The salient features we used were the following:

- $\mathcal{I}(K) \subset \mathrm{Map}^{\mathrm{sm}}(K,N)$ is an **open** subspace, for $K \subset M$ compact, and $\mathrm{Map}^{\mathrm{sm}}(-,N)$ is a **flexible** sheaf (i.e., $\mathrm{Map}^{\mathrm{sm}}(K,N) \to \mathrm{Map}^{\mathrm{sm}}(K_0,N)$ is a Serre fibration for $K_0 \hookrightarrow K$ a closed inclusion);
- The group $\mathrm{Diff}(M)$ of diffeomorphisms of $M$ acts on $\mathrm{Map}^{\mathrm{sm}}(M,N)$, and it preserves the subspace $\mathrm{Imm}(M,N)$. Further, by embedding $M \hookrightarrow V$ as the zero section of a bundle, then $\mathrm{Diff}(V,\partial V)$ acts transitively on a small enough neighborhood of a point $f \in \mathcal{I}(M) \subset \mathrm{Map}^{\mathrm{sm}}(M,N)$.[1]

The proof of the weak homotopy equivalence $\mathrm{Imm}(M,N) \to \mathrm{Imm}^{\mathrm{f}}(M,N)$ then proceeds by induction on handle decomposition of $M$. The key part of the inductive step is to show that the restriction map $\mathcal{I}(U+\varphi^q) \to \mathcal{I}(U)$ is a Serre fibration, where $U + \phi^q \cong U \sqcup_{S^{q-1}\times D^{n-q}} D^q \times D^{n-q}$ is $U$ with a handle of index $q$ attached. The necessary lifting could be constructed by maneuvering collar neighborhoods, using the features above, but required that the index $q$ is less than $n$ (and thus the proof only worked when $M$ is an open manifold).

These features enable, mutatis mutandis, the proof of the following generalization: Let $E \to M$ be a smooth fiber bundle with an action of $\mathrm{Diff}(M)$ lifting the canonical action of $\mathrm{Diff}(M)$ on $M$. Let $E^{(k)} \to M$ be the bundle of $k$-jets of $E$, which inherits an action of $\mathrm{Diff}(M)$, and let $\mathcal{R} \subset E^{(k)}$ be a subspace (i.e., $\mathcal{R}$ is a differential relation). Gromov proved the following h-principle:

**Theorem 1.1** (Gromov)**.** *If $M$ is an open manifold and the relation $\mathcal{R} \subset E^{(k)}$ is an open subspace which is preserved by the action of $\mathrm{Diff}(M)$ on $E^{(k)}$, then the $k$-jet prolongation map*

$$j^{(k)} : \mathrm{Sol}_{\mathcal{R}}(M) \longrightarrow \Gamma(\mathcal{R})$$

*is a weak homotopy equivalence, where the space of solutions $\mathrm{Sol}_{\mathcal{R}}(M)$ consists of those smooth sections of $E$ whose $k$-jet lies in $\mathcal{R}$.*

To summarize: diffeomorphism invariant open differential relations adhere to the h-principle on open manifolds.

This theorem is ready to be put to use. We just need to find some examples, which will turn out to be plentiful.

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[1] More precisely: Choose an element $\widetilde{f} \in \mathcal{I}(V)$ which restricts to $f \in \mathcal{I}(M)$. Then, there exists a neighborhood $U_f$ of $f$ and a map $U_f \to \mathrm{Diff}(V,\partial V)$ such that the composite $U_f \to \mathrm{Diff}(V,\partial V) \xrightarrow{\widetilde{f}} \mathcal{I}(V) \to \mathcal{I}(M)$ is the inclusion of $U_f$ in $\mathcal{I}(M)$, where the map $\mathrm{Diff}(V,\partial V) \to \mathcal{I}(V)$ is the composite $\mathrm{Diff}(V,\partial V) \times \{\widetilde{f}\} \to \mathrm{Diff}(V,\partial V) \times \mathcal{I}(V) \to \mathcal{I}(V)$.

## 2. Submersions

**Definition 2.1.** A smooth map $f : M \to N$ is a submersion if the derivative map $d_x f : T_x M \to T_{f(x)} N$ is surjective for every point $x$ in $M$. A formal submersion $F$ is a bundle map $T_M \to T_N$ which is surjective on each fiber. The space $\mathrm{Subm}(M, N)$ has the compact-open $C^\infty$ topology, and $\mathrm{Subm}^{\mathrm{f}}(M, N)$ has the compact-open topology.

The map $\mathrm{Subm}(M, N) \to \mathrm{Subm}^{\mathrm{f}}(M, N)$ is an example of a 1-jet prolongation map, where $E$ is the product $M \times N$. It is easy to see that the relation $\mathcal{R} \subset E^{(1)}$ defining submersions is open and diffeomorphism invariant. We obtain the following theorem, the first published proof of which is in [4]:

**Theorem 2.2** (Phillips)**.** *If $M$ is an open manifold, then the map*

$$\mathrm{Subm}(M, N) \longrightarrow \mathrm{Subm}^{\mathrm{f}}(M, N)$$

*is a weak homotopy equivalence.*

*Proof.* Apply Gromov's theorem. □

*Remark* 2.3. Submersions are a very familiar concept when the source $M$ is closed: In this case, Ehresmann's theorem states that the map $M \to N$ is a fiber bundle, which is perhaps the most interesting kind of submersion. However, this is exactly the case in which Gromov's theorem does *not* apply: There may exist formal submersions of $M$ to $N$ even when $M$ does not fiber over $N$ (for instance, if $M$ is parallelizable and $N$ is parallelizable and noncompact). However, even in the case of $M$ closed, a formal submersion of $M$ on $N$ is clearly a good first step to obtain an actual submersion. Is there a systematic way of taking successive steps? We will discuss such issues later in this course. (Hint: Goodwillie calculus.)

## 3. Foliations

We now begin the next major topic of focus in our course, the theory of foliations. Before giving a formal definition, it is helpful to have an example and an rough picture. In geology, foliation of a rock consists of a division the rock into layers; in metalwork, to foliate a metal is to cut it into thin leaves. The same holds true in topology, but where "manifold" replaces "rock" and "metal."

*Example* 3.1. One foliation of the space $\mathbb{R}^n$ consists of the family of all submanifolds $\{x_1, \dots, x_q\} \times \mathbb{R}^{n-q}$, where $\{x_1, \dots, x_q\}$ is a point of $\mathbb{R}^q$. This is a codimension $q$ foliation. Note that every point of $\mathbb{R}^n$ lies in exactly one of these submanifolds. The submanifolds $\{x_1, \dots, x_q\} \times \mathbb{R}^{n-q}$ are the *leaves* of the foliation.

**Definition 3.2** (Intuitive version)**.** A codimension $q$ foliation $\mathfrak{F}$ of an $n$-manifold $M$ consists of a collection of $(n-q)$-dimensional manifolds with disjoint inclusions into $M$, $\{M_{\mathfrak{f}} \subset M\}$, and which looks locally like the above foliation of $\mathbb{R}^n$.[2]

We have a wealth of examples of foliations coming from fiber bundles:

*Example* 3.3. Let $\pi : M \to N$ be a smooth fiber bundle, with fibers $M_x := \pi^{-1}\{x\}$. Then the collection of fibers $\mathfrak{F} := \{M_x \subset M | x \in N\}$ foliates $M$. The codimension of the foliation is equal to the dimension of $N$.

This source of examples behaves, in some sense, too well. Namely, the following are true for a foliation of a bundle by its fibers bundle but do **not** hold in general:

- All of the fibers $M_x$ are diffeomorphic;

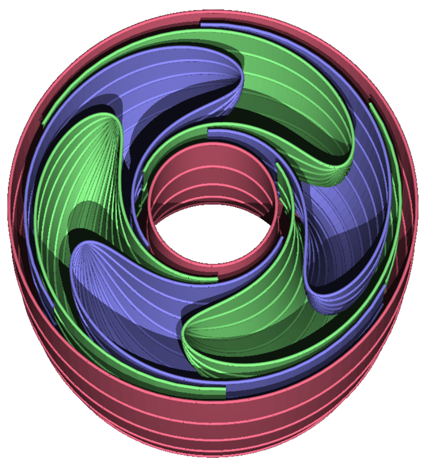
[2] I.e., every point $x$ has a neighborhood $U$ and a diffeomorphism $g : U \to \mathbb{R}^n$ such that the inverse image of each leaf of $\mathbb{R}^n$ is a component of a leaf of $\mathfrak{F}$ restricted to $U$: $g^{-1}(\{x_1, \dots, x_q\} \times \mathbb{R}^{n-q}) \subset M_{\mathfrak{f}} \cap U$, for some $M_{\mathfrak{f}} \in \mathfrak{F}$.

- The leaf space of this foliation (i.e., the quotient $M/\sim$, where $x \sim y$ if they are contained in the same leaf) is actually a smooth manifold (namely, the base of the bundle), rather than some space with a gnarly topology;
- If $M$ is compact, then the fibers $M_x$ are compact.

Let us conclude with examples to show that general foliations are more interesting and need not be so tidy.

*Example* 3.4 (The Kronecker foliation). Consider the torus $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$. Let $a$ be any irrational real number, and define the subspace $M_t \subset T^2$ to be the image of the line $y = ax + t$. Then the collection $\mathfrak{K} = \{M_t \subset T^2\}$ foliates $T^2$. Note that each leaf of the foliation is dense in $T^2$, and the topology of the leaf space is badly-behaved. Also, the leaves are noncompact, diffeomorphic to $\mathbb{R}^1$, although $T^2$ is compact.

*Example* 3.5 (Reeb foliation). We construct a foliation of $S^{k+1}$ that has only a single compact leaf. First, we construct a foliation of $\mathbb{R} \times D^k$. Thinking of $D^k$ as the unit disk in $\mathbb{R}^k$, choose a function $f(x)$ on the interior of $D^k$ such $f(x)$ tends to $\infty$ as $|x|$ tends to 1. (E.g., $f(x) = \dfrac{1}{1-|x|}$ works.) Define the foliation $\widetilde{\mathfrak{R}}$ as having leaves $M_t \subset \mathbb{R} \times D^k$, where $M_t$ is the graph of the function $f(x)+t$, $t \in \mathbb{R}$, and one final leaf given by the boundary $\mathbb{R} \times \partial D^k \subset \mathbb{R} \times D^k$. Since the foliation is invariant under translation in the $\mathbb{R}$ direction, we can use the projection map $\pi : \mathbb{R} \times D^k \to \mathbb{R}/\mathbb{Z} \times D^k$ to define a foliation $\mathfrak{R}$ on $S^1 \times D^k$, consisting of the manifolds $\pi(M_t) \subset S^1 \times D^k$ and the boundary $S^1 \times \partial D^k \subset S^1 \times D^k$. (Note $\pi(M_t) = \pi(M_{t+1})$.) I lifted the following pretty picture of this foliation from Wikipedia, which shows the case $k = 2$:



Now, using the decomposition of $S^{k+1}$ as a union

$$S^{k+1} \cong S^1 \times D_0^k \coprod_{S^1 \times S^{k-1}} S^1 \times D_1^k$$

the Reeb foliation of $S^{k+1}$ is then defined by the union of the above foliations $\mathfrak{R}_0 \cup \mathfrak{R}_1$ on each subspace $S^1 \times D_i^k$. The foliation has one leaf diffeomorphic to $S^1 \times S^{k-1}$ and all others diffeomorphic to $\mathbb{R}^k$.

## References


[1] Gromov, Mikhael. Partial differential relations. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 9. Springer-Verlag, Berlin, 1986. x+363 pp.

[2] Haefliger, André. Lectures on the theorem of Gromov. Proceedings of Liverpool Singularities Symposium, II (1969/1970), pp. 128–141. Lecture Notes in Math., Vol. 209, Springer, Berlin, 1971.

[3] Lawson, Blaine. Foliations. Bull. Amer. Math. Soc. 80 (1974), 369–418.

[4] Phillips, Anthony. Submersions of open manifolds. Topology 6 1967 171206.