THE H-PRINCIPLE, LECTURE 12: FOLIATIONS AND HAEFLIGER STRUCTURES

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Last time: We defined a foliation of a manifold M^n as a family of codimension-q submanifolds $\{M_\mathfrak{f} \subset M\}$ such that every point was contained in a small ball with a diffeomorphism to $\mathbb{R}^n$ such that the image of the components of the leaves were of the form $\{x\} \times \mathbb{R}^{n-q}$. Although this is a very concrete description, it would be nicer to have a purely local description so as to more directly apply topological techniques.

1. More background on foliations

We introduce an alternative definition.

Definition 1.1. A codimension-q foliation $\mathfrak{F}$ on an n-manifold M is an atlas $\{\phi_\alpha : U_\alpha \xrightarrow{\cong} \mathbb{R}^n\}$ such that the composite map $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ has the form $(x, y) \in \mathbb{R}^q \times \mathbb{R}^{n-1} \mapsto (\psi_1(x), \psi_2(x, y))$. In other words, the first q coordinates in the source determined the first q coordinates in the target.

Note that we construct the family of leaves $\{M_\mathfrak{f} \subset M\}$ as follows. Start with some point p in some chart U_α and let $\phi_\alpha(p) = (x, y) \in \mathbb{R}^q \times \mathbb{R}^{n-q}$. Then p lives on $\phi_\alpha^{-1}(\{x\} \times \mathbb{R}^{n-1})$, which is part of the leaf for p. Now consider the points where this codimension-q submanifold intersects other charts and repeat this process (take preimage of the codimension-q slice for the point), gluing these submanifolds together by the transition functions. You will construct a codimension-q submanifold, which is a leaf.

Remark 1.2. The role of the atlas is totally analogous to its role in defining manifolds. There is a natural notion of "equivalence of foliations" by the induced collection of leaves, and we identify two atlases if they induce the same collection of leaves.

Remark 1.3. We could also define a foliation $\mathfrak{F}$ as a subsheaf $\mathcal{O}_M^\mathfrak{F}$ of the sheaf of smooth functions $\mathcal{O}_M$ such that there exists an atlas so that on each chart U_α, the vector space $\mathcal{O}_M^\mathfrak{F}(U_\alpha)$ consists of functions that are locally constant along leaves.

We now address the question: What kind of structure results from a foliation?

Observe that $\mathfrak{F} = \{M_\mathfrak{f} \subset M\}$ defines a subspace $T\mathfrak{F}$ of the tangent bundle TM by saying that a point v is in some fiber $T_p\mathfrak{F} \subset T_pM$ if $v \in T_pM_\mathfrak{f}$ for the leaf containing $p \in M_\mathfrak{f}$. This space consists of the "tangents along leaves." As should be obvious, $T\mathfrak{F}$ is a subbundle such that $T\mathfrak{F}|_{M_\mathfrak{f}} \cong TM_\mathfrak{f}$.

Example 1.4. A fiber bundle $\pi : M \to N$ is equipped with a foliation by fibers of π. The "tangents along the fibers" is the relative tangent bundle $T_{M|N}$, which is the kernel of the differential of the projection map $d\pi : TM \to \pi^*TN$.

This construction $T\mathfrak{F}$ should suggest that we get another canonical bundle from a foliation, namely the normal bundle $N\mathfrak{F}$, which is the cokernel of the inclusion $T\mathfrak{F} \hookrightarrow TM$. To make this construction more concrete, pick a Riemannian metric on M so that we can identify $N\mathfrak{F}$ with the subbundle of TM that is complementary to $T\mathfrak{F}$.

These two bundles from a foliation provide us with some great structure. Here's a simple consequence of this easy work.

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Example 1.5. Because the tangent bundle of S^2 has no 1-dimensional subbundles, we see that there is no codimension-1 foliation of S^2!

We now recall a powerful theorem that gives yet another characterization of foliations.

Theorem 1.6 (Frobenius)**.** *Let V be a subbundle of the tangent bundle TM. Then $V \cong T\mathfrak{F}$ for a foliation $\mathfrak{F}$ if and only if the space $\Gamma^{\mathrm{sm}}(V)$ of smooth sections of V is a Lie subalgebra of vector fields on M, i.e., if $\Gamma^{\mathrm{sm}}(V) \subset \Gamma^{\mathrm{sm}}(TM)$ is closed under the Lie bracket.*

This theorem lets us relate constructions in geometry to algebra. For instance, Frobenius' theorem tells us that giving a foliation (a geometric object) corresponds to a Lie subalgebra of the tangent sheaf. Similarly, the very nice foliation arising from a submersion $\pi : M \to N$ (by fibers) induces a Lie ideal in the tangent sheaf of M, and hence sections of the normal bundle $\Gamma^{\mathrm{sm}}(N\mathfrak{F})$ form a Lie algebra.

On the other hand, the algebraic viewpoint ought to let us translate into homotopy theory. For instance, just the structure of a subbundle V of the tangent bundle TM corresponds to factoring the classifying map $g_{TM} : M \to BO(n)$ through a classifying map $g_{V\oplus V^\perp} : M \to BO(q) \times BO(n-q)$. We'd like to find similar descriptions of the Lie subalgebras and Lie ideals of the tangent sheaf.

Our new question is thus: What structure in homotopy or bundle theory corresponds to a foliation?

2. Haefliger structures

We are headed to constructing a kind of classifying space for codimension-q foliations, but the construction is slightly more involved than the construction of the classifying space BG for G-bundles. First, we need a topological *groupoid*, not just a group.

Definition 2.1. Γ_q is the topological groupoid of germs of local diffeomorphisms of $\mathbb{R}^n$. That is, the space of objects of Γ_q is $\mathbb{R}^q$, and the space of morphisms $\mathrm{Map}_{\Gamma_q}(x, y)$ is a quotient of the discrete group $\mathrm{Diff}(\mathbb{R}^q)^\delta$ (where the "δ" indicates the discrete topology), where two maps $f \sim g$ are identified if they agree on a sufficiently small open neighborhood of x. The space of morphisms (which we also denote Γ_q), is thus the space of local diffeomorphisms of $\mathbb{R}^n$, with topology induced from $\mathbb{R}^n \times \mathrm{Diff}(\mathbb{R}^q)^\delta$.

We now define what it means to have a kind of bundle with structure group Γ_q.

Definition 2.2. A (Haefliger) Γ_q structure on an n-manifold consists of a cover $\{U_\alpha\}$ with transition maps $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \Gamma_q$ satisfying the cocycle condition. A Γ_q foliation of M is a Γ_q structure for which the maps $\phi_{\alpha\alpha} : U_\alpha \to \mathbb{R}^q$ are submersions.

Why did this strange space appear? Notice that a codimension-q foliation on M^n gives a Γ_q foliation on M. This assertion follows from unwinding our definition of foliation in terms of charts: given $\phi_\alpha : U_\alpha \to \mathbb{R}^q \times \mathbb{R}^{n-q}$, the projection onto the $\mathbb{R}^q$ factor is a submersion, and the transition maps $\phi_\beta^{-1} \circ \phi_\alpha$ give the germ of a diffeomorphism of $\mathbb{R}^q$ at each point $x \in U_\alpha \cap U_\beta$. By refining the cover $\{U_\alpha\}$ of a Γ_q foliation, you can go backward to obtain a codimension-q foliation.

This definition breaks down the structure of a foliation into two parts (Γ_q structures and submersions), both of which seem more homotopy-theoretic than the original definition. We have the following homotopy-theoretic notion of equivalence of foliations:

Definition 2.3. A homotopy of codimension-q foliations between $\mathfrak{F}_0$ and $\mathfrak{F}_1$ on M consists of a codimension-q foliation $\mathfrak{F}$ on $M \times [0, 1]$ for which $\mathfrak{F}|_{M\times\{0\}} \cong \mathfrak{F}_0$ and $\mathfrak{F}|_{M\times\{1\}} \cong \mathfrak{F}_1$.

Definition 2.4. A homotopy of foliations is *integrable* if $\mathfrak{F}$ is transverse to $M \times \{t\}$ for all $t \in [0, 1]$. In particular, $\mathfrak{F}|_{M\times\{t\}}$ defines a foliation on $M \times \{t\}$ for each t.

Γ_q structures are determined by the homotopy type of M: Namely, $H^1(M, \Gamma_q)$ is equivalent to $[M, B\Gamma_q]$, where $B\Gamma_q$ is the classifying space of Γ_q (for this, see [1] or [3]). Further, by the Phillips-Gromov theorem on submersions, for open manifolds M submersions out of M are equivalent to surjective bundle maps out of TM; this is a homotopy invariant of the map $g_{TM} : M \to BO(n)$. Thus, at least for open manifolds M, one might optimistically think that you could classify foliations on M (up to integrable homotopy) using algebraic topology, and only using the homotopy type of M and of the map $M \to BO(n)$.

Next time we'll discuss just such a theorem, due to Haefliger, using the h-principle.

References

[1] Haefliger, André. Homotopy and integrability. Lecture Notes in Math., vol. 197, Springer-Verlag, Berlin and New York, 1971, pp. 133-163.

[2] Lawson, Blaine. Foliations. Bull. Amer. Math. Soc. 80 (1974), 369–418.

[3] Segal, Graeme. Classifying spaces and spectral sequences. Inst. Hautes Études Sci. Publ. Math. No. 34 1968 105112.