THE H-PRINCIPLE, LECTURE 14: HAEFLIGER'S THEOREM CLASSIFYING FOLIATIONS ON OPEN MANIFOLDS

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In this lecture we prove the following theorem:

Theorem 0.1 (Haefliger)**.** *If M is an open manifold, there is a bijection between*

(1) *codimension q foliations on M up to integral homotopy and*
(2) *homotopy classes of fibrewise surjective vector bundle maps $TM \to N\Gamma_q$.*

1. Defining the map

Recall that Γ_q is the topological groupoid of germs of local diffeomorphisms of $\mathbb{R}^q$, the topology being that of the étalé space of the discrete sheaf of local diffeomorphisms. The space $B\Gamma_q$ carries a Γ_q-structure $\mathfrak{U}$ with the following universal property: for any space X, $f \mapsto f^{-1}(\mathfrak{U})$ induces a bijection between homotopy classes of maps $[X, B\Gamma_q]$ and Γ_q-structures on X up to homotopy.

To define the map in Haefliger's theorem, we need to promote this bijection at least to an equivalence of groupoids, following the general theory of classifying spaces. To any Γ_q-structure $\mathfrak{F}$ on a space X is associated a map $f\colon X \to B\Gamma_q$ together with a homotopy $\mathfrak{H}$ from $f^{-1}(\mathfrak{U})$ to $\mathfrak{F}$. This data is canonically defined up to homotopy: given $(f_0, \mathfrak{H}_0)$ corresponding to $\mathfrak{F}_0$, $(f_1, \mathfrak{H}_1)$ corresponding to $\mathfrak{F}_1$, we can associate to any homotopy $\mathfrak{F}$ from $\mathfrak{F}_0$ to $\mathfrak{F}_1$ a homotopy f from f_0 to f_1 and a Γ_q-structure $\mathfrak{K}$ on $X \times [0,1]^2$ with restrictions given by

$$\begin{tikzcd} \cdot \arrow[r, dash, "\mathfrak{F}"] \arrow[d, dash, "\mathfrak{H}_0"'] & \cdot \arrow[d, dash, "\mathfrak{H}_1"] \\ \cdot \arrow[r, dash, "f^{-1}(\mathfrak{U})"'] \arrow[ur, phantom, "\mathfrak{K}"] & \cdot \end{tikzcd}$$

and the homotopy type of this data depends only on the homotopy type of $\mathfrak{F}$.

There is a map of topological groupoids $d\colon \Gamma_q \to GL_q$ given by $(x,g) \mapsto dg$. For any Γ_q-structure $\mathfrak{F}$ with classifying map f, we define its normal bundle $N_{\mathfrak{F}}$ to be the vector bundle classified by $Bd \circ f$. In other words, if $N\Gamma_q$ is the pullback by Bd of the universal bundle on BGL_q, then $N_{\mathfrak{F}} = f^*(N\Gamma_q)$. If f_0 and f_1 are part of different classifying data for $\mathfrak{F}$, then they are related by a canonical homotopy class of homotopy, so there exists a canonical homotopy class of isomorphisms $f_0^*(N\Gamma_q) \cong f_1^*(N\Gamma_q)$. By definition, the normal bundle is functorial in that $N_{r^{-1}(\mathfrak{F})} = r^*(N_{\mathfrak{F}})$ for r a continuous map.

Let now M be a manifold. A Γ_q-structure on M is called smooth if it is represented by a cocycle $\{U_\alpha, \phi_{\alpha\beta}\}$ in which the maps $\phi_\alpha\colon U_\alpha \to \mathbb{R}^q$ are smooth. If $\mathfrak{F}$ is a smooth Γ_q-structure, we can define a bundle map $TM \to N_{\mathfrak{F}}$ over the covering $\{U_\alpha\}$ as follows:

$$\begin{tikzcd} TU_\alpha \arrow[r, dashed] \arrow[rr, bend left, "d\phi_\alpha"] \arrow[d] & N_{\mathfrak{F}}|U_\alpha \arrow[r] \arrow[d] & T\mathbb{R}^q \arrow[d] \\ U_\alpha \arrow[r, equal] & U_\alpha \arrow[r, "\phi_\alpha"'] & \mathbb{R}^q, \end{tikzcd}$$

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where the right-hand square is a pullback by definition of $N_{\mathfrak{F}}$. By construction, this bundle map is natural in M: if $r\colon N \to M$ is a smooth map, then the Γ_q-structure $r^{-1}(\mathfrak{F})$ is represented by the cocycle $\{r^{-1}(U_\alpha), \phi_{\alpha\beta} \circ r\}$, so we have a commutative square

$$\begin{tikzcd} TN \arrow[r, "dr"] \arrow[d] & TM \arrow[d] \\ N_{r^{-1}(\mathfrak{F})} \arrow[r] & N_{\mathfrak{F}}. \end{tikzcd}$$

Our goal is now to classify Γ_q-foliations on M, the set of which we denote by $\mathrm{Fol}_q(M)$. These are the smooth Γ_q-structures on M whose associated bundle map $TM \to N_{\mathfrak{F}}$ is fibrewise surjective. Thus to any $\mathfrak{F} \in \mathrm{Fol}_q(M)$ we can associate an element

$$\begin{tikzcd} TM \arrow[r, two heads, "F"] \arrow[d] & N\Gamma_q \arrow[d] \\ M \arrow[r, "f"'] & B\Gamma_q \end{tikzcd}$$

in the space $\mathrm{Map}^{\mathrm{surj}}_{\mathrm{Vect}}(TM, N\Gamma_q)$ of fibrewise surjective bundle maps. We wish to show that this gives a well-defined map

$$\mathrm{Fol}_q(M) \to \pi_0 \, \mathrm{Map}^{\mathrm{surj}}_{\mathrm{Vect}}(TM, N\Gamma_q). \tag{1}$$

Suppose that $(f_0, \mathfrak{H}_0)$ and $(f_1, \mathfrak{H}_1)$ are two choices of classifying data for $\mathfrak{F}$, defining bundles maps (f_0, F_0) and (f_1, F_1), and let $(f, \mathfrak{K})$ be associated to the identity homotopy $p^{-1}(\mathfrak{F})$ of $\mathfrak{F}$, where $p\colon M \times [0,1] \to M$ is the projection. Then $N_{\mathfrak{K}}$ is a bundle on $M \times [0,1]^2$ that restricts to the normal bundles of $p^{-1}(\mathfrak{F})$ and $f^{-1}(\mathfrak{U})$ on the top and bottom sides. By homotopy invariance of vector bundles this gives an isomorphism $N_{p^{-1}(\mathfrak{F})} \cong N_{f^{-1}(\mathfrak{U})} = f^*(N\Gamma_q)$, and hence a path

$$\begin{tikzcd} TM \times [0,1] \arrow[r, two heads] \arrow[d] & N_{\mathfrak{F}} \times [0,1] \cong N_{p^{-1}(\mathfrak{F})} \cong f^*(N\Gamma_q) \arrow[r] \arrow[d] & N\Gamma_q \arrow[d] \\ M \times [0,1] \arrow[r, equal] & M \times [0,1] \arrow[r, "f"'] & B\Gamma_q \end{tikzcd}$$

in $\mathrm{Map}^{\mathrm{surj}}_{\mathrm{Vect}}(TM, N\Gamma_q)$ between (f_0, F_0) and (f_1, F_1).

Further, suppose that $\mathfrak{F}$ is an integrable homotopy between $\mathfrak{F}_0$ and $\mathfrak{F}_1$, that is, $\mathfrak{F}$ is a Γ_q-foliation on $M \times [0,1]$ whose restriction to $M \times \{t\}$ is a foliation for all t. If $f\colon M \times [0,1] \to B\Gamma_q$ is a classifying map for $\mathfrak{F}$, we can choose f_t as classifying maps for $\mathfrak{F}_t = i_t^{-1}(\mathfrak{F})$. Let (f, F) be the bundle map associated to f and (f_t, F_t) that associated to f_t. Then

$$\begin{tikzcd} TM \times [0,1] \arrow[r] \arrow[dr] & T(M \times [0,1]) \arrow[r, two heads, "F"] \arrow[d] & N\Gamma_q \arrow[d] \\ & M \times [0,1] \arrow[r, "f"'] & B\Gamma_q \end{tikzcd}$$

is a path from (f_0, F_0) to (f_1, F_1) in $\mathrm{Map}^{\mathrm{surj}}_{\mathrm{Vect}}(TM, N\Gamma_q)$, since for each t the restriction of the top map to $TM \times \{t\}$ is $F \circ di_t = F_t$ which is fibrewise surjective. So the map (1) descends to

$$\mathrm{Haef}\colon \mathrm{Fol}_q(M)/{\sim} \to \pi_0 \, \mathrm{Map}^{\mathrm{surj}}_{\mathrm{Vect}}(TM, N\Gamma_q)$$

where $\sim$ is the relation of integrable homotopy.

2. Proof of the theorem

Suppose that M and E are manifolds and that $\tilde{\mathfrak{F}}$ is a Γ_q-foliation on E. Then $\mathrm{Map}_{\pitchfork\tilde{\mathfrak{F}}}(M,E)$ denotes the space of smooth maps $s\colon M\to E$ that are transversal to $\tilde{\mathfrak{F}}$, i.e., such that the composite

$$TM \xrightarrow{ds} TE \overset{\tilde{F}}{\twoheadrightarrow} N\Gamma_q$$

is fiberwise surjective. This is just the bundle map associated to the smooth Γ_q-structure $s^{-1}(\tilde{\mathfrak{F}})$, so we have a map

$$\mathrm{Map}_{\pitchfork\tilde{\mathfrak{F}}}(M,E)\to \mathrm{Fol}_q(M),\quad s\mapsto s^{-1}(\tilde{\mathfrak{F}}). \tag{2}$$

What happens if we take homotopy classes? Suppose that

$$s\colon M\times[0,1]\to E$$

is a smooth homotopy such that $s_t\colon M\to E$ is transversal to $\tilde{\mathfrak{F}}$ for all $t\in[0,1]$. Then $s^{-1}(\tilde{\mathfrak{F}})$ is a homotopy between $s_0^{-1}(\tilde{\mathfrak{F}})$ and $s_1^{-1}(\tilde{\mathfrak{F}})$, and we claim that it is an integrable homotopy. The restriction of $s^{-1}(\tilde{\mathfrak{F}})$ to $M\times\{t\}$ is $s_t^{-1}(\tilde{\mathfrak{F}})$ which is a foliation by assumption, so it remains to show that $s^{-1}(\tilde{\mathfrak{F}})$ itself is a foliation, or equivalently that s is transversal to $\tilde{\mathfrak{F}}$. This follows from the the transversality of all the s_t, since the composition

$$TM \xrightarrow{di_t} T(M\times[0,1]) \xrightarrow{ds} TE \overset{\tilde{F}}{\twoheadrightarrow} N\Gamma_q$$

is fibrewise surjective and each fiber of $T(M\times[0,1])$ is accounted for in this way. Thus, the map (2) descends to

$$\pi_0\,\mathrm{Map}_{\pitchfork\tilde{\mathfrak{F}}}(M,E)\to \mathrm{Fol}_q(M)/{\sim}.$$

We can summarize the situation by the commutative square

$$\begin{tikzcd}
\pi_0\,\mathrm{Map}_{\pitchfork\tilde{\mathfrak{F}}}(M,E) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Fol}_q(M)/{\sim} \arrow[d, "\mathrm{Haef}"] \\
\pi_0\,\mathrm{Map}^{\mathrm{surj}}_{\mathrm{Vect}}(TM,N_{\tilde{\mathfrak{F}}}) \arrow[r] & \pi_0\,\mathrm{Map}^{\mathrm{surj}}_{\mathrm{Vect}}(TM,N\Gamma_q)
\end{tikzcd} \tag{3}$$

in which the left-hand map is a bijection when M is open, by the Gromov-Phillips theorem. This square will be used to prove both the surjectivity and the injectivity of Haef, by using appropriate pairs $(s,\tilde{\mathfrak{F}})$. These will be given by the following technical lemma.

Lemma 2.1. *Let $\mathfrak{F}$ be a Γ_q-structure on a manifold M. Then there exists a manifold E, a closed embedding $s\colon M\hookrightarrow E$, and a Γ_q-foliation $\tilde{\mathfrak{F}}$ on E such that $s^{-1}(\tilde{\mathfrak{F}})=\mathfrak{F}$.*

Proof. Let $\mathfrak{F}$ be represented by the cocycle $\{U_\alpha,\phi_{\alpha\beta}\colon U_{\alpha\beta}\to\Gamma_q\}$. By paracompactness we can assume that the cover $\{U_\alpha\}$ is locally finite. Consider the topological groupoid Γ_q^M of germs of diffeomorphisms of $M\times\mathbb{R}^q$ that are of locally the form $(x,v)\mapsto(x,\gamma(v))$ where γ is a local diffeomorphism of $\mathbb{R}^q$, i.e., all germs are of the form $\mathrm{id}\times g$ for some $g\in\Gamma_q$. The obvious map of sheaves induces an inclusion $i\colon\Gamma_q\hookrightarrow\Gamma_q^M$ that sends a germ g to $\mathrm{id}\times g$.

Observe that the diagram

$$\begin{tikzcd}
\Gamma_q \arrow[r, "i"] & \Gamma_q^M \arrow[d] \\
U_{\alpha\beta} \arrow[u, "\phi_{\alpha\beta}"] \arrow[r, "(\mathrm{id},\phi_\alpha)"'] & M\times\mathbb{R}^q.
\end{tikzcd}$$

is commutative. This means that $i\circ\phi_{\alpha\beta}$ is a section of Γ_q^M over the graph of $\phi_\alpha|_{U_{\alpha\beta}}$. Extending the germs of this section at every point to a neighborhood and then using a partition of unity on $U_{\alpha\beta}\times\mathbb{R}^q$, we can extend this section to a global section $U_{\alpha\beta}\times\mathbb{R}^q\to\Gamma_q^M$. Such a section corresponds to a diffeomorphism $\Phi_{\alpha\beta}$ defined on $U_{\alpha\beta} \times \mathbb{R}^q$, locally of the form $(x, v) \mapsto (x, \gamma(v))$, and whose germ at $(x, \phi_\alpha(x))$ is $\mathrm{id} \times \phi_{\alpha\beta}(x)$ for any $x \in U_{\alpha\beta}$.

For $x \in M$, denote by $V_x \subset M \times \mathbb{R}^q$ the set on which:

- $\Phi_{\alpha\alpha}$ is defined and the identity, whenever $x \in U_\alpha$;
- $\Phi_{\beta\alpha}^{-1}\Phi_{\alpha\beta}$ is defined and the identity, whenever $x \in U_{\alpha\beta}$;
- $\Phi_{\beta\gamma}\Phi_{\alpha\beta}$ and $\Phi_{\alpha\gamma}$ are defined and equal, whenever $x \in U_{\alpha\beta\gamma}$.

Now because all these identites hold true of the germs at $(x, \phi_\alpha(x))$ and our cover is locally finite, V_x is a neighborhood of $(x, \phi_\alpha(x))$. For each α, therefore, the set $\bigcup_{x \in U_\alpha} V_x \subset U_\alpha \times \mathbb{R}^q$ is a neighborhood of the graph of $\phi_\alpha \colon U_\alpha \to \mathbb{R}^q$; let E_α be an open subneighborhood.

Let

$$E = \coprod_\alpha E_\alpha / \sim$$

where $(\alpha, x, v) \sim (\beta, y, w)$ iff $x = y$ in $U_{\alpha\beta}$ and $\Phi_{\alpha\beta}(x, v) = (y, w)$. This is an equivalence relation because of the way we constructed the neighborhoods E_α. Clearly there is an induced smooth map $\pi \colon E \to M$, $[\alpha, x, v] \mapsto x$. To make sure that E is actually a manifold, we must still check that it is Hausdorff. So let $[\alpha, x, v], [\beta, y, w] \in E$ be distinct. If $x \neq y$, then the points are separated by π. Otherwise, $\Phi_{\alpha\beta}(x, v) \neq (y, w)$, so in this case the points are separated by the second component of the inclusion $E_\alpha \subset U_\alpha \times \mathbb{R}^q$.

Define $s \colon M \to E$ to be $x \mapsto [\alpha, x, \phi_\alpha(x)]$ for $x \in U_\alpha$. Finally, let $\tilde{\mathfrak{F}}$ be defined by the cocycle $\{E_\alpha, \tilde{\phi}_{\alpha\beta}\}$, where $\tilde{\phi}_{\alpha\beta} \colon E_{\alpha\beta} \to \Gamma_q$ sends $[\alpha, x, v]$ to the germ g such that $\mathrm{id} \times g$ is the germ of $\Phi_{\alpha\beta}$ at (x, v). It is then straightforward to check that $\tilde{\mathfrak{F}}$ is a Γ_q-structure and that $s^{-1}(\tilde{\mathfrak{F}}) = \mathfrak{F}$. Moreover, $\tilde{\phi}_\alpha \colon E_\alpha \to \mathbb{R}^q$ is just the composition of the open embedding $E_\alpha \subset U_\alpha \times \mathbb{R}^q$ and the projection to the second factor, hence is smooth and a submersion. □

Proof of surjectivity. Take a bundle map

$$\begin{tikzcd} TM \arrow[r, two heads, "F"] \arrow[d] & N\Gamma_q \arrow[d] \\ M \arrow[r, "f"'] & B\Gamma_q \end{tikzcd}$$

in the target, and let $\mathfrak{F} = f^{-1}(\mathfrak{U})$. By the lemma, there exists $s \colon M \to E$ and a foliation $\tilde{\mathfrak{F}}$ on E such that $s^{-1}(\tilde{\mathfrak{F}}) = \mathfrak{F}$. Let $\tilde{f} \colon E \to B\Gamma_q$ be a map classifying $\tilde{\mathfrak{F}}$. Because s is a cofibration, we can choose it so that $f = \tilde{f} \circ s$. Then the left-hand square in

$$\begin{tikzcd} TM \arrow[r, dashed] \arrow[d] \arrow[rr, bend left, "F"] & N_{\tilde{\mathfrak{F}}} \arrow[r] \arrow[d] & N\Gamma_q \arrow[d] \\ M \arrow[r, "s"'] & E \arrow[r, "\tilde{f}"'] & B\Gamma_q \end{tikzcd}$$

is an element of $\mathrm{Map}^{\mathrm{surj}}_{\mathrm{Vect}}(TM, N_{\tilde{\mathfrak{F}}})$ mapping to (f, F) by the bottom map in the square (3). This square completes the proof. □

Proof of injectivity. Let $\mathfrak{F}_0$ and $\mathfrak{F}_1$ be Γ_q-foliations on M and suppose that there is a path

$$\begin{tikzcd} TM \times [0,1] \arrow[r, two heads, "F"] \arrow[d] & N\Gamma_q \arrow[d] \\ M \times [0,1] \arrow[r, "f"'] & B\Gamma_q \end{tikzcd}$$

between the associated bundles maps (f_0, F_0) and (f_1, F_1). We can assume that $\mathfrak{F}_0 = f_0^{-1}(\mathfrak{U})$ and $\mathfrak{F}_1 = f_1^{-1}(\mathfrak{U})$. Then $\mathfrak{F} = f^{-1}(\mathfrak{U})$ is a homotopy from $\mathfrak{F}_0$ to $\mathfrak{F}_1$. By the lemma, there exists $s\colon M \times [0,1] \to E$ and a foliation $\tilde{\mathfrak{F}}$ on E such that $s^{-1}(\tilde{\mathfrak{F}}) = \mathfrak{F}$. Since $s_0^{-1}(\tilde{\mathfrak{F}}) = \mathfrak{F}_0$ and $s_1^{-1}(\tilde{\mathfrak{F}}) = \mathfrak{F}_1$ are foliations, s_0 and s_1 belong to $\mathrm{Map}_{\pitchfork \tilde{\mathfrak{F}}}(M, E)$, and they map to $\mathfrak{F}_0$ and $\mathfrak{F}_1$ by the top map of the square (3). Using this square it will suffice to show that the images of s_0 and s_1 in $\mathrm{Map}^{\mathrm{surj}}_{\mathrm{Vect}}(TM, N_{\tilde{\mathfrak{F}}})$ are in the same connected component. A path between them is given by the left-hand square in

$$\begin{tikzcd}
TM \times [0,1] \arrow[r, dashed] \arrow[d] \arrow[rr, bend left, "F"] & N_{\tilde{\mathfrak{F}}} \arrow[r] \arrow[d] & N\Gamma_q \arrow[d] \\
M \times [0,1] \arrow[r, "s"'] & E \arrow[r, "\tilde{f}"'] & B\Gamma_q,
\end{tikzcd}$$

since $F_0 = \tilde{F} \circ ds_0$ and $F_1 = \tilde{F} \circ ds_1$. □

This completes the proof of Haefliger's classification. To understand its meaning, let's pretend for a moment that $B\Gamma_q$ is a manifold and that $\mathfrak{U}$ is a Γ_q-foliation on it. Then we have a commutative triangle

$$\begin{tikzcd}
\pi_0 \mathrm{Map}_{\pitchfork \mathfrak{U}}(M, B\Gamma_q) \arrow[d] \arrow[dr, "\cong"] & \\
\mathrm{Fol}_q(M)/{\sim} \arrow[r, "\mathrm{Haef}"'] & \pi_0 \mathrm{Map}^{\mathrm{surj}}_{\mathrm{Vect}}(TM, N\Gamma_q)
\end{tikzcd}$$

in which the top map is an isomorphism by the theorem of Gromov and Phillips, and Haefliger's theorem is equivalent to the vertical map being an isomorphism. Morally speaking, therefore, Haefliger's theorem is the statement that $\mathfrak{U}$ is the universal Γ_q-foliation.

3. Some consequences

The following proposition gives Haefliger's classification a more homotopy-theoretic flavor.

Proposition 3.1. *If $\dim M = n$, $\pi_0 \mathrm{Map}^{\mathrm{surj}}_{\mathrm{Vect}}(TM, N\Gamma_q)$ is naturally identified with the set of lifts*

$$\begin{tikzcd}
 & B\Gamma_q \times BGL_{n-q} \arrow[d, "Bd \times \mathrm{id}"] \\
 & BGL_q \times BGL_{n-q} \arrow[d, "\oplus"] \\
M \arrow[r, "\tau"'] \arrow[uur, dashed] & BGL_n
\end{tikzcd}$$

up to homotopy, i.e., the set of maps $M \to B\Gamma_q \times BGL_{n-q}$ in the homotopy category of spaces over BGL_n. Equivalently, this is π_0 of the space of sections of the fibration $\tau^{-1}(B\Gamma_q \times BGL_{n-q})$ on M, where τ classifies the tangent bundle of M.

Proof. Such a lift is the same thing as a map $M \to B\Gamma_q \times BGL_{n-q}$ together with a homotopy between the two maps $M \to BGL_n$, up to homotopy. This is equivalent to a map $f\colon M \to B\Gamma_q$, a rank $n-q$ vector bundle K on M, and an isomorphism $TM \cong K \oplus f^*(N\Gamma_q)$, up to homotopy. Any homotopy class of such data has a representative in which the injection $K \hookrightarrow TM$ is the inclusion of a subbundle. Since a splitting of this inclusion is unique up to homotopy, this data is equivalent to a map $f\colon M \to B\Gamma_q$ together with a fibrewise surjective map $TM \to f^*(N\Gamma_q)$, up to homotopy. This is exactly $\pi_0 \mathrm{Map}^{\mathrm{surj}}_{\mathrm{Vect}}(TM, N\Gamma_q)$. □

Corollary 3.2. *If M is contractible, there exists a unique Γ_q-foliation on M up to integrable homotopy.*

Proof. There is a unique lift in Proposition 3.1, since $B\Gamma_q \times BGL_{n-q}$ and BGL_n are connected. $\square$

Corollary 3.3. *If M is open and parallelizable, then there exists at least one Γ_q-foliation on M.*

Proof. The map $M \to BGL_n$ factors through a point, so it can be lifted by the previous corollary. $\square$

Define $B\overline{\Gamma}_q$ by the homotopy fiber sequence

$$B\overline{\Gamma}_q \longrightarrow B\Gamma_q \xrightarrow{Bd} BGL_q.$$

In other words, $B\overline{\Gamma}_q$ classifies Γ_q-structures with trivialized normal bundle.

Proposition 3.4. *$Bd\colon B\Gamma_q \to BGL_q$ is q-connected.*

Proof. Let $0 \leq k \leq q-1$. There exists a unique codimension q foliation on the open manifold $S^k \times \mathbb{R}^{q-k}$, since it has dimension q. By Proposition 3.1, there exists a unique lift

$$\begin{tikzcd} & B\Gamma_q \arrow[d] \\ S^k \times \mathbb{R}^{q-k} \arrow[r] \arrow[ur, dashed] & BGL_q \end{tikzcd}$$

up to homotopy. But $S^k \times \mathbb{R}^{q-k}$ has trivial tangent bundle, so the bottom map is null-homotopic and therefore these lifts identify with homotopy classes of maps $S^k \to B\overline{\Gamma}_q$. $\square$

Corollary 3.5. *Let M be open and k-truncated. If $q \geq k$, then any rank $n-q$ subbundle of TM is the tangent bundle of a Γ_q-foliation, which is unique up to integrable homotopy if $q > k$.*

References

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