THE H-PRINCIPLE, LECTURE 16: CONFIGURATION SPACES, WITH ANNIHILATION AND WITH LABELS

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In the previous several lectures, we were studying the collection of codimension-q foliations on a manifold M up to the equivalence relation given by integrable homotopy. Our most significant tool was Gromov's h-principle. We could have organized this study by constructing a space of foliatons $\mathrm{Fol}_q(M)$, so that $\pi_0 \mathrm{Fol}_q(M)$ was the set of foliations modulo integrable homotopy. (The kth homotopy group of $\mathrm{Fol}_q(M)$ would have encoded codimension-q foliations on $D^k \times M$, with some integrability and boundary conditions.)

However, this space $\mathrm{Fol}_q(M)$ is not the space of solutions to a differential relation (at least, so far as I can see). To make use of the h-principle for differential relations, we needed a trick. Namely, we used that maps $\mathrm{Map}_{\pitchfork\mathfrak{F}}(M, E)$ transversal to a foliation *did* form a space of solutions to a differential relation. Still, even with this clever trick, in Haefliger's theorem we were only able to describe π_0 of the space $\mathrm{Fol}_q(M)$ rather than the homotopy type of the whole space. It seems clearly desirable to have a version of the h-principle that can be directly applied to the sheaf of folations Fol_q.

To buttress to this suggestion, we now consider another very interesting example of a sheaf of spaces on a manifold not obviously approachable by differential relations, this time defined by *configuration spaces*.

Definition 0.1. The space $\mathrm{Conf}_i(M)$ of configurations of i **ordered** points in M consists of the subspace of M^i of i-tuples $\{(x_1, \ldots, x_n)| \ x_i \neq x_j \text{ for } i \neq j\}$. The quotient by the free action of the symmetric group $\mathrm{Conf}_i(M)_{\Sigma_i}$ is the configuration space of i **unordered** points in M.

These spaces are both interesting and accessible. Before going forward, we give some examples and basic facts.

Example 0.2. Let $M = \mathbb{R}^1$. Choosing a component $f \in \pi_0 \mathrm{Conf}_i(\mathbb{R}^1)$ defines an isomorphism $\pi_0 \mathrm{Conf}_i(\mathbb{R}^1) \cong \Sigma_i$. The components of $\mathrm{Conf}_i(\mathbb{R}^1)$ are contractible, so we obtain the homotopy equivalence $\Sigma_i \simeq \mathrm{Conf}_i(\mathbb{R}^1)$.

Example 0.3. For $M = \mathbb{R}^2$ and a choice of basepoint $f \in \mathrm{Conf}_i(\mathbb{R}^2)$, we can interpret a based map $g : S^1 \to \mathrm{Conf}_i(\mathbb{R}^2)$ as a map $g \times \mathrm{id} : [0, 1] \to \mathbb{R}^2 \times \mathbb{R}$, which can be seen to be a braid. Deforming the map g likewise leads to equivalent braids, and there is thereby an isomorphism $\pi_1 \mathrm{Conf}_i(\mathbb{R}^2) \cong P_i$ with the pure braid group on i strands. Likewise, $\pi_1 \mathrm{Conf}_i(\mathbb{R}^2)_{\Sigma_i}$ is isomorphic to B_i, the full braid group. The higher homotopy groups of these spaces vanish: $\mathrm{Conf}_i(\mathbb{R}^2) \simeq K(P_i, 1)$ and $\mathrm{Conf}_i(\mathbb{R}^2)_{\Sigma_i} \simeq K(B_i, 1)$. First proved in [**?**].

Remark 0.4. One can understand the space $\mathrm{Conf}_i(M)$ inductively on i, because the the map $\mathrm{Conf}_{i+1}(M) \to \mathrm{Conf}_i(M)$, defined by forgetting the last point, is a fiber bundle, with fiber over the point $\{x_1, \ldots, x_i\}$ given by $M - \{x_1, \ldots, x_i\}$, i.e., M punctured i times. In the case of $\mathbb{R}^n$, the homology of $\mathrm{Conf}_i(\mathbb{R}^n)$ is readily computable using these fiber bundles. Note, in particular, that $\mathrm{Conf}_i(\mathbb{R}^n)$ is $(n - 2)$-connected.

Example 0.5. For $M = \mathbb{R}^\infty$, the spaces $\mathrm{Conf}_i(\mathbb{R}^\infty)$ are contractible, since $\mathrm{Conf}_i(\mathbb{R}^\infty)$ is equivalent to the sequential colimit $\varinjlim_{k\to\infty} \mathrm{Conf}_i(\mathbb{R}^k)$, the connectivity of which tends to infinity by the preceding remark. As a consequence, we have the homotopy equivalence $\mathrm{Conf}_i(\mathbb{R}^\infty)_{\Sigma_i} \simeq K(\Sigma_i, 1)$.

Date: Lecture February 18, 2010. Last edited on February 19, 2010.

These spaces $\mathrm{Conf}_i(M)$ are, of course, great, but there are infinitely many of them, and one might wish for some interesting way of combing them to get a single, even more interesting space. Also, the assignment $\mathrm{Conf}_i(-)_{\Sigma_i}$ clearly defines a covariant functor of open subspaces of M, $\mathcal{U}_M \to$ Spaces, assigning the inclusion $\mathrm{Conf}_i(U)_{\Sigma_i} \to \mathrm{Conf}_i(V)_{\Sigma_i}$ for every inclusion $U \subset V$ in M. We, however, have been concerned with contravariant functors, i.e., presheaves and sheaves. This is the domain of the h-principle. The following construction addresses both these perceived shortcomings:

If M has a boundary, or a specified subspace, then there is an interesting way of gluing all of these configuration spaces together.

Definition 0.6. Let M be a manifold with a closed submanifold M_0. The space $\mathrm{Conf}(M, M_0)$ of configurations of unordered points of M annihilated in M_0. That is, it is the disjoint union

$$\mathrm{Conf}(M, M_0) = (\coprod_i \mathrm{Conf}_i(M)_{\Sigma_i})/\sim$$

modulo the equivalence relation in which the elements $[x_1, \ldots, x_n]$ and $[x_2, \ldots, x_n]$ are identified if x_1 lies in M_0.

We have a generalization of this notion by allowing for a pointed space of labels X. This might seem like a lot of definition, but this is an excellent construction and will prove to be well worth it.

Definition 0.7. Let M be a manifold with a closed submanifold M_0 and let X be a pointed space. $\mathrm{Conf}_X(M, M_0)$ is the space of configurations of unordered points of M labeled by points of X and annihilated in M_0. That is, it is the disjoint union

$$\mathrm{Conf}(M, M_0) = (\coprod_i \mathrm{Conf}_i(M) \times_{\Sigma_i} X^i)/\sim$$

where Σ_i acts diagonally on the we identify

$$[(m_1, \ldots, m_i), (x_1, \ldots, x_i)] \sim [(m_2, \ldots, m_i), (x_2, \ldots, x_i)]$$

if either $m_1 \in M_0$ or $x_1 = *$ the basepoint of X.

Remark 0.8. Observe that this specializes to the previous construction by using the space $X = S^0$, i.e., $\mathrm{Conf}(M, M_0) \cong \mathrm{Conf}_{S^0}(M, M_0)$.

Notation 0.9. If no subspace is specified, we will assume that the subspace refers to the boundary of M (or the empty set, for M without boundary) and we will further abbreviate $\mathrm{Conf}_X(M) := \mathrm{Conf}_X(M, \partial M)$.

Note that the space $\mathrm{Conf}_X(M, M_0)$ has a natural basepoint given by the equivalence class of $[m_0, *]$, for $m_0 \in M_0$ and $* \in X$ the basepoint of X.

The space $\mathrm{Conf}_X(M)$ is an interesting invariant not only of the homeomorphism type of M, but also of the homotopy type of X, as the following example shows:

Example 0.10. Consider the case of $M = \mathbb{R}$. Then the space $\mathrm{Conf}_i(\mathbb{R}) \times_{\Sigma_i} X^i$ retracts onto X^i, and we obtain a homotopy equivalence $\mathrm{Conf}_X(\mathbb{R}) \simeq JX$, where the James construction, JX, is the free topological monoid on the based space X with the identity element given by the base point of X.

Now, observe that if $f : M \hookrightarrow N$ is a **closed** embedding into any manifold N, then we obtain a restriction map

$$\mathrm{Conf}_X(N) \longrightarrow \mathrm{Conf}_X(M)$$

defined by annihilating points outside of the image of M. That is, we map the point $[f(m_1), \ldots f(m_i), x_1, \ldots, x_i]$ to $[m_1, \ldots m_i, x_1, \ldots, x_i]$, and if a point y_1 is not image of f, then we send $[y_1, \ldots, y_i, x_1, \ldots, x_i]$ to the same point as we do $[y_2, \ldots y_i, x_2, \ldots, x_i]$. This algorithm defines a map, which can be seen to be continuous using the essential hypothesis that f was a closed embedding.

Definition 0.11. The presheaf $\mathfrak{Conf}_X$ on M assigns to an open subspace $U \subset M$ the value

$$\mathfrak{Conf}_X(U) := \mathrm{Conf}_X(\overline{U}, \overline{U} - \mathring{U})$$

where $\overline{U}$ is the closure of U in M and $\mathring{U}$ is the interior of U, i.e., the intersection of U with the interior of M.

If U is nicely embedded in the interior of M, then the expression $\overline{U} - \mathring{U} \cong \partial\overline{U}$ is just the boundary of $\overline{U}$. The expression is a little more complicated because, (a) U might occupy part of the boundary of M, (b) the closure of U might be badly behaved.

In other words, for a nicely embedded manifold U for which the closure $\overline{U}$ is a manifold with boundary, then we have

$$\mathfrak{Conf}_X(U) = \mathrm{Conf}_X(\overline{U}).$$

If the closure of U in M is badly behave, I encourage you not to think about it, because we can always make local-to-global arguments using open covers whose closures behave nicely, as above. If you must think about it, well, use the definition above.[1]

The following is easy to verify:

Proposition 0.12. $\mathfrak{Conf}_X$ *is a sheaf on* M*.*

We now have a sheaf of spaces not given by solving any differential relations, and this begs for an h-principle to apply to it. Next time we will formulate this h-principle for microflexible sheaves.

References

[1] Fox, R.H., Neuwirth, L., The braid groups. Math. Scand. 10, (1962), 119-126.

[2] Bödigheimer, C.-F. Stable splittings of mapping spaces. Algebraic topology (Seattle, Wash., 1985), 174187, Lecture Notes in Math., 1286, Springer, Berlin, 1987.

[1] This, and some of the other examples in this course suggest giving 2nd thought to Leray's original definition of a sheaf, which used closed, rather than open, subspaces. Using the category of closed, codimension zero submanifolds of M, rather than the category $\mathcal{U}_M$ of opens in M, works a bit nicer for many purposes.