THE H-PRINCIPLE, LECTURE 19: THE PROOF OF MCDUFF'S THEOREM, SECOND PART

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Last time: We saw that the restriction map $\mathrm{Conf}(D^n) \to \mathrm{Conf}(S^{n-1} \times [0, \delta])$ is not a fibration because we can't lift a path that starts on the "interior" boundary of $S^{n-1} \times \{\delta\}$ and then wanders inside the fattened sphere. The main problem is that we cannot "create" a point in the middle of the disk.

There are situations where this problem vanishes. Consider $U \subset V$ two compact manifolds. Suppose the boundary of U does not contain the boundary of V. Then we can lift a path that starts on the boundary of U, so long as the path sits on the boundary for a finite length of time $[0, \epsilon]$. We simply create a point on ∂V and traverse a path to ∂U in time ϵ. Alternatively, if every component of ∂U intersects ∂V, then we can also lift paths that sit on ∂U for finite time. (Note that this condition implies the ∂U does not surject onto ∂V.)

These conditions have a peculiar property: the path has to stay on ∂U for finite time. What is a path immediately leaves the boundary? In that case, we cannot lift the path because we would have to traverse through $V - U$ instantaneously. Nonetheless, we can *almost* lift paths since all fibers of the restriction map are homeomorphic.

To run our handlebody arguments, we just need to relax our conditions a little, as follows.

Definition 0.1. A map $\pi : E \to C$ is a *quasi-fibration* if the inclusion
$$Fib_x = \pi^{-1}(x) \hookrightarrow hFib_x = \{(e, \gamma) \,|\, \gamma : [0, 1] \to C, \gamma(0) = x, \gamma(1) = \pi(e)\}$$
is a weak homotopy equivalence for every $x \in C$.

We will see that our handlebody arguments work whenever restriction maps are *quasi*-fibrations, not just fibrations, and that $\mathrm{Conf}_X(-)$ has this property.

Remark 0.2. Homotopy-equivalent maps have homotopy-equivalent homotopy fibers ($hFib_x$, as above), and so for a fibration, the inclusion $Fib_x \hookrightarrow hFib_x$ is a homotopy equivalence.

Example 0.3. PICTURE OF THE EL

For us, the crucial property of quasi-fibrations is that we still get long exact sequences of homotopy groups.

Lemma 0.4. *For a quasi-fibration $\pi : E \to C$, for each point $e \in E$, we get a long exact sequence*
$$\cdots \to \pi_{n+1}(C, \pi(e)) \to \pi_n(Fib_{\pi(e)}, e) \to \pi_n(E, e) \to \pi_n(C, \pi(e)) \to \cdots$$

Proof. Every map factors into an inclusion followed by a fibration, using the path space construction. In our case, define
$$\tilde{\pi} : \tilde{E} = \{\gamma : [0, 1] \to E\} \to C$$
$$\gamma \to \gamma(1).$$
This is a fibration, and its fibers are precisely the homotopy fibers of $\pi : E \to C$. Since the inclusion $E \hookrightarrow \tilde{E}$ is a homotopy equivalence, the lemma follows. □

Warning: Quasi-fibrations do not always pull back!

We now state the main lemma, which implies our goal as an easy corollary.

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Lemma 0.5. *Given an inclusion $j : U \to V$, the restriction map*

$$\mathrm{Conf}_X(j) : \mathrm{Conf}_X(V) \to \mathrm{Conf}_X(U)$$

is a quasi-fibration if either of the following holds:

(1) *X is connected;*
(2) *each component of ∂U intersects ∂V, $\overline{V-U}$ is a closed submanifold of V, and $\partial U \cap \partial(\overline{V-U})$ is a closed submanifold of both ∂U and $\partial(\overline{V-U})$.*

Remark 0.6. The first condition clearly allows us to "create" points whenever we want. We can view any point in V as labelled by the basepoint of X and then simply vary the label in X to create the necessary point. Hence, we get path-lifting for paths that live on the boundary for finite time.

Remark 0.7. The second condition, albeit convoluted, holds for handlebodies and arises naturally from thinking carefully about handle attachment.

Corollary 0.8. *If* either *M is compact and open* or *X is connected, then*

$$\mathrm{Conf}_X(M) \to \Gamma(T_M^\infty \wedge_M X)$$

is a weak homotopy equivalence.

Proof of corollary. As in our earlier arguments (e.g., immersion or submersion), we pick a handle decomposition of M and apply induction with respect to the dimension of handles.

The base case was done last time. For a disk D^n, we saw

$$\mathrm{Conf}_X(D^n) \simeq \Sigma^n X = \mathrm{Map}(D^n, \Sigma^n X) = \Gamma(T_{D^n}^\infty \wedge_{D^n} X).$$

We now do the induction step. Let $V = U + \phi^k$, where ϕ is a handle of index k. (Note that for X is not connected, we only need to consider $k < n$.) Let $j : U \hookrightarrow V$ denote the inclusion; likewise $j' : S^{k-1} \times [0,\delta] \times D^{n-k} \hookrightarrow D^k \times D^{n-k}$.

Both $\mathrm{Conf}_X(-)$ and $\Gamma(-, T_M^\infty \wedge_M X)$ are sheaves, so we have pullback diagrams with respect to the handle attachment. For $\mathrm{Conf}_X(-)$ we have

$$\begin{tikzcd}
\mathrm{Conf}_X(V) \arrow[r] \arrow[d, "\mathrm{Conf}(j)"'] & \mathrm{Conf}_X(D^k \times D^{n-k}) \arrow[d, "\mathrm{Conf}(j')"] \\
\mathrm{Conf}_X(U) \arrow[r] & \mathrm{Conf}_X(S^{k-1} \times [0,\delta] \times D^{n-k})
\end{tikzcd}$$

where the two vertical maps $\mathrm{Conf}(j)$ and $\mathrm{Conf}(j')$ are quasi-fibrations, by the main lemma. For brevity, denote $\Gamma(-, T_M^\infty \wedge_M X)$ by $\Gamma(-)$. We then have

$$\begin{tikzcd}
\Gamma(V) \arrow[r] \arrow[d, "\Gamma(j)"'] & \Gamma(D^k \times D^{n-k}) \arrow[d, "\Gamma(j')"] \\
\Gamma(U) \arrow[r] & \Gamma(S^{k-1} \times [0,\delta] \times D^{n-k})
\end{tikzcd}$$

where the vertical maps are fibrations (since they come from a fiber bundle).

The scanning map $scan : \mathrm{Conf}_X(-) \to \Gamma(-)$ induces a map between the diagrams and, by hypothesis, we know it is a weak homotopy equivalence on all the inputs except V. To show that $scan : \mathrm{Conf}_X(V) \to \Gamma(V)$ is a weak homotopy equivalence, we will show that it induces an isomorphism on homotopy groups.

Observe that $hFib(\mathrm{Conf}(j)) \simeq hFib(\mathrm{Conf}(j'))$. As the scanning map is a weak homotopy equivalence on the right column, we find that

$$Fib(\mathrm{Conf}(j)) \simeq Fib(\Gamma(j)),$$

where we use the fact that fibers and homotopy fibers agree for quasifibrations (and hence fibrations).

We obtain a map of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n Fib(\mathrm{Conf}_X(j)) \arrow[r] \arrow[d, "\simeq"'] & \pi_n \mathrm{Conf}_X(V) \arrow[r] \arrow[d] & \pi_n \mathrm{Conf}_X(U) \arrow[r] \arrow[d, "\simeq"'] & \cdots \\
\cdots \arrow[r] & \pi_n Fib(\Gamma(j)) \arrow[r] & \pi_n \Gamma(V) \arrow[r] & \pi_n \Gamma(U) \arrow[r] & \cdots
\end{tikzcd}$$

and so the five lemma tells us that $\pi_n \mathrm{Conf}_X(V) \simeq \pi_n \Gamma(V)$. □

We now make a preliminary remark on the proof of the main lemma. If one blurs one's vision a little, a fibration is basically a fiber bundle. A quasi-fibration is then a map that looks like a fiber bundle over some *closed* subspaces of the base space. Thus, to check that a map is a quasi-fibration, one can search for a closed stratification of the base and verify the property stratum by stratum. In our case, the configuration space $\mathrm{Conf}_X(M)$ has a natural stratification by "number of points" in a configuration.

References

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[2] McDuff, Dusa. Configuration spaces. K-theory and operator algebras (Proc. Conf., Univ. Georgia, Athens, Ga., 1975), pp. 88–95. Lecture Notes in Math.,#Vol. 575, Springer, Berlin, 1977.